Strong intractability of generalized convex recoloring problems

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joint work with Yoshiko Wakabayashi

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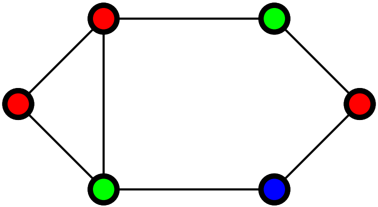
Total coloring



Total coloring of a graph G (simple and connected):

Convex total coloring

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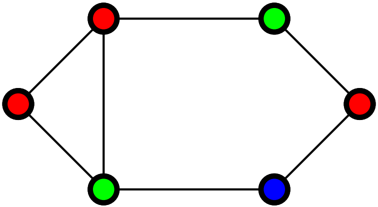

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- $C : V(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set with k colors

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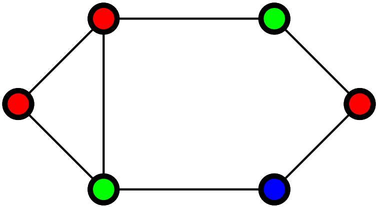

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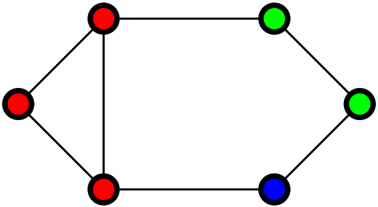

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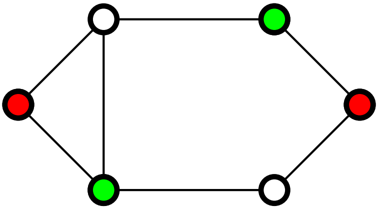

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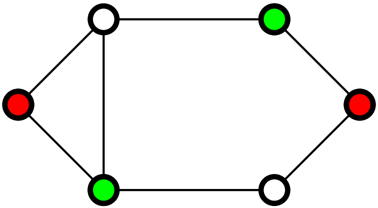

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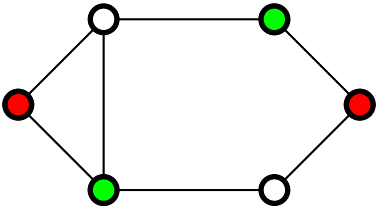

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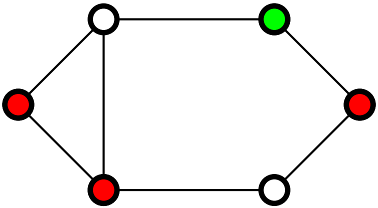

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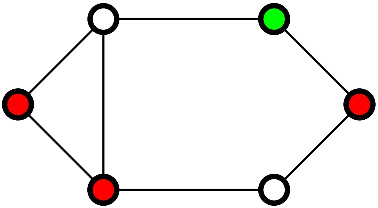

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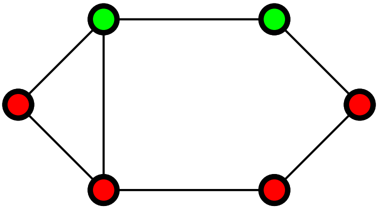

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The convex recoloring problem

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- $R(C') := \{v \in V(G) \mid C(v) \neq \emptyset \text{ and } C(v) \neq C'(v)\}$

Known results

Complexity

Approximation algorithms

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Complexity

- $\mathcal{NP}$-hard on paths even if each color appears at most twice [Kanj and Kratsch, 2009]
- $\mathcal{NP}$-hard on k-colored grids for each $k \geq 2$ [Campêlo et al., 2014]

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Approximation algorithms

- Ratio 2 for paths [Moran and Snir, 2007]
- Ratio $(2 + \varepsilon)$ for trees [Bar-Yehuda et al., 2008]
- Ratio $\frac{3}{2}$ for general graphs in which each color appears at most twice [Bar-Yehuda et al., 2016]

Known results

Approximation threshold

FPT algorithms (param. k = number of color changes)

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- $\mathcal{O}(\log n)$ on n-vertex bipartite graphs [Campêlo et al., 2014]

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FPT algorithms (param. k = number of color changes)

- Kernel of size $\mathcal{O}(k^2)$ on trees [Bodlaender et al., 2011]
- $\mathcal{O}(E(G)) + 2^{\mathcal{O}(k \log k)}$ on graphs in which each color appears at most twice [Bar-Yehuda et al., 2016]

Applications

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- Study of perfect phylogenetic trees [Moran and Snir, 2008]
- Routing problems and transportation networks [Kammer and Tholey, 2012]

Connected Coloring Completion (CCC) [Chor et al., 2007]

Input:

- Graph G and partial coloring $C\colon V(G) \to \mathcal{C} \cup \{\emptyset\}$

Question:

- Is it possible to extend the initial coloring C to a total convex coloring ? is C a **convex** partial coloring?

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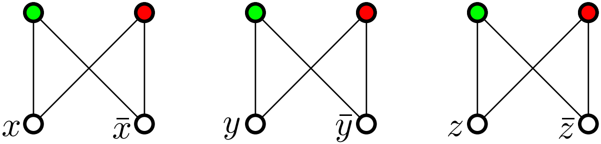
Theorem [Chor et al., 2007]

CCC is $\mathcal{NP}$-complete even on bipartite graphs with only two colors

Reduction from 3SAT to CCC by [Chor et al., 2007]

Given a 3CNF formula $\mathcal{F}$ (e.g. $(x \vee \bar{y} \vee z) \wedge (\bar{x} \vee y \vee z)$)



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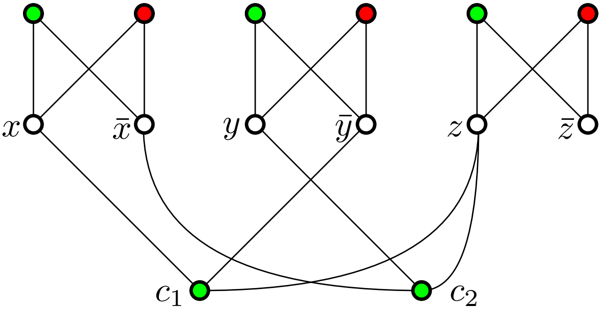

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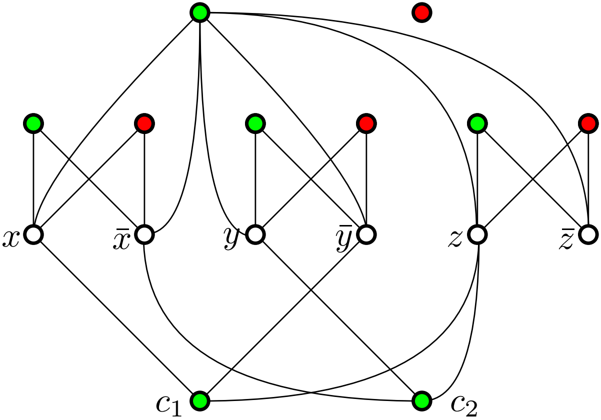

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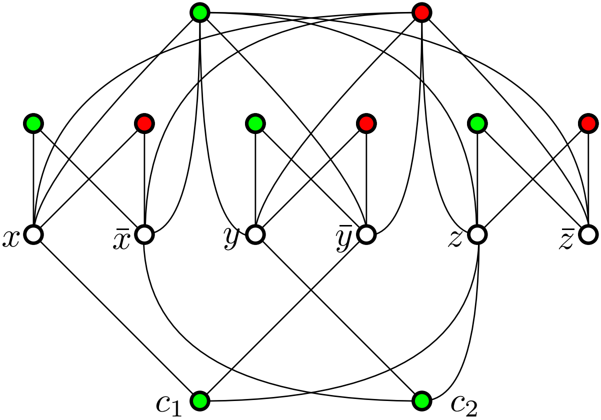
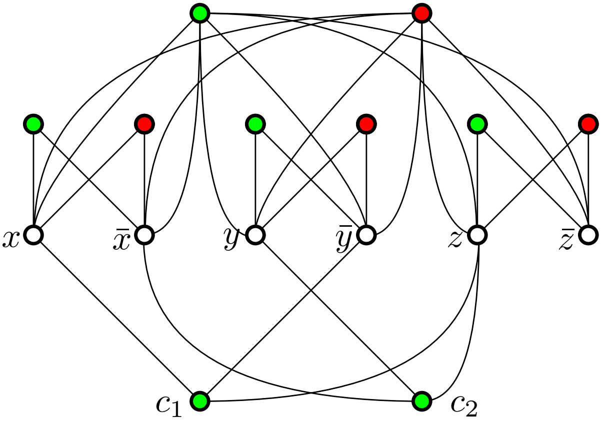


Reduction from 3SAT to CCC by [Chor et al., 2007]

Given a 3CNF formula $\mathcal{F}$ (e.g. $(x \vee \bar{y} \vee z) \wedge (\bar{x} \vee y \vee z)$)
$\mathcal{F}$ is satisfiable **iff** the coloring is convex



Inapproximability of convex recoloring (unweighted version)

Theorem

For every $\varepsilon > 0$, there is no $n^{1-\varepsilon}$-approximation algorithm for the Convex Recoloring problem on bipartite graphs, unless $\mathcal{P} = \mathcal{NP}$.

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Idea of proof

- Suppose the existence of a polynomial-time $n^{1-\varepsilon}$-approximation algorithm $\mathcal{A}$

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Idea of proof

- Suppose the existence of a polynomial-time $n^{1-\varepsilon}$-approximation algorithm $\mathcal{A}$
- Using $\mathcal{A}$, we decide CCC in polynomial time

Generalized convex recoloring

r-Convex coloring

- C is r-**convex** if $G[C^{-1}(c)]$ has at most r components $\forall\, c \in \mathcal{C}$

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The r-convex recoloring problem

Input:

- Partially colored graph (G, C) and weights $w\colon V(G) \to \mathbb{Q}_{\geq}$

Objective:

- Find an r-convex recoloring $C'\colon V \to \mathcal{C} \cup \{\emptyset\}$ of G
- Minimize

$$\sum_{v \in R(C')} w(v)$$

Generalized Connected Coloring Completion

The r-CCC problem

Input:

- Graph G and partial coloring $C\colon V(G) \rightarrow \mathcal{C} \cup \{\emptyset\}$

Question:

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Generalized Connected Coloring Completion

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Lemma

For every r and $k \geq 2$, r-CCC is $\mathcal{NP}$-complete on k-colored bipartite graphs.

Inapproximability of r-**convex** recoloring

Theorem (unweighted version)

For every r and $k \geq 2$, and positive $\varepsilon < 1$, there is no $n^{1-\varepsilon}$-approximation for the (unweighted) r-convex recoloring problem on k-colored n-vertex bipartite graphs, unless $\mathcal{P} = \mathcal{NP}$.

Inapproximability of r-**convex** recoloring

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For every r and $k \geq 2$, and positive $\varepsilon < 1$, there is no $n^{1-\varepsilon}$-approximation for the (unweighted) r-convex recoloring problem on k-colored n-vertex bipartite graphs, unless $\mathcal{P} = \mathcal{NP}$.

Theorem (weighted version)

For every r and $k \geq 2$, there is no $2^{\mathrm{poly}(n)}$-approximation for the r-convex recoloring problem on k-colored n-vertex bipartite graphs, unless $\mathcal{P} = \mathcal{NP}$.

Parameterized intractability

Theorem

For every r and $k \geq 2$, r-CR (unweighted) problem parameterized by the number of colors changes is $\mathcal{W}[2]$-hard on k-colored bipartite graphs.

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FPT-reduction from **Bounded CNF Satisfiability problem** ($\mathcal{W}[2]$-complete [Dantchev et al., 2011])

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FPT-reduction from **Bounded CNF Satisfiability problem** ($\mathcal{W}[2]$-complete [Dantchev et al., 2011])

Input: A CNF formula F and a positive integer p
Parameter: p
Question: Does there exist a satisfying truth assignment for F that assigns true to at most p variables?

That's all folks

Thank you for your attention! :-)

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