Ramsey for complete graphs with a dropped edge or a triangle

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Join work with J. Chappelon and J. Ramírez-Alfonsín

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Ramsey Theory

Typical problems in Ramsey Theory

Determine or estimate the maximum or minimum possible size of a collection of finite objects (e.g., graphs, sets, vectors, numbers) satisfying certain restrictions.

General Philosophy of Ramsey Theory

Every large system contains a well organized subsystem

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General Philosophy of Ramsey Theory

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Question

How many elements of some structure must there be to guarantee that a particular property will hold?

Ramsey Theory

One fundamental theorem of Ramsey Theory is Van der Waerden's Theorem.

Van der Waerden's Theorem, 1927.

For any given positive integers t and n, there exists some integer W such that if the integers $1, 2, ..., W$ are colored with t colors, then there are at least n integers in arithmetic progression all of the same color.

The smallest number W is the Van der Waerden number $W(t, n)$.

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The smallest number W is the Van der Waerden number $W(t, n)$.
Example: $t = 2$ and $n = 3$.

w(2,3)>8

1 **2** **3** **4** **5** **6** **7** **8**

Ramsey Theory

In 1935, P. Erdős and G. Szekeres posed the following question:

Happy Ending problem, 1935.

For any positive integer n, there exists an integer $N(n)$ such that for any set containing $N(n)$ points in the plane, it is possible to select n points forming a convex polygon

It is known that $N(3) = 3$, $N(4) = 5$, $N(5) = 9$ and $N(6) = 17$.

Erdős and G. Szekeres conjecture

$N(n) = 2^{n-2} + 1$

Classical Ramsey Theorem

Let K_r be the complete graph with r vertices.

Ramsey Theorem

For any integers $n, m > 0$ there exists an integer r such that any 2-edge-coloring (red-blue) of K_r contains either a K_n red or a K_m blue.

Question

What is the smallest number $r = r(n, m)$ such that any 2-edge-coloring (red-blue) of K_r contains a red K_n or a blue K_m ?

Known values of $r(n, m)$

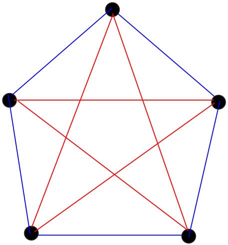
What is known?

- $r(n, 2) = n$.
- $r(3, 3) = 6$.
- $r(3, 4) = 9$, in 1955.
- $r(3, 5) = 14$, in 1955.
- $r(3, 6) = 18$, in 1964.
- $r(3, 7) = 23$, in 1968.
- $r(3, 8) = 28$, in 1992.
- $r(3, 9) = 36$, in 1982.
- $r(4, 4) = 18$, in 1955.
- $r(4, 5) = 25$, in 1995.

Known values of $r(n, m)$

$r(3, 3) = 6$

Proof. $r(3, 3) > 5$.



Known values of $r(n, m)$

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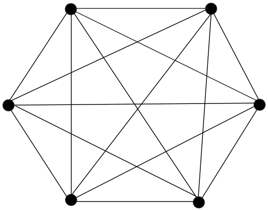
Proof. Consider any 2-edge-coloring of K_6.



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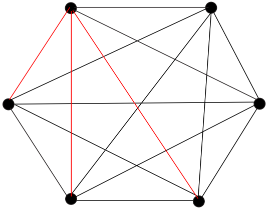

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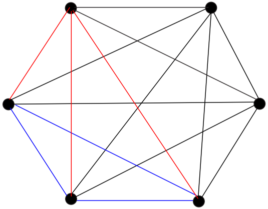

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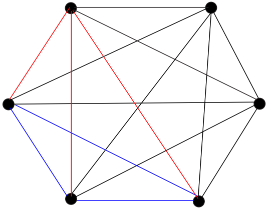

It is known the value $r(n,n)$ for $n \leq 4$.

Theorem [Exoo, 1989. Angeltveit and McKay, 2017]

$43 \leq r(5,5) \leq 48$.

Bounds for Ramsey Numbers

The best recursive upper bound for $r(n, m)$ is the following.

Theorem [Greenwood and Gleason, 1955].

$r(n, m) \leq r(n-1, m) + r(n, m-1)$ with strict inequality when both terms on the right hand side are even.

Bounds for Ramsey Numbers

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In 2010, Conlon obtained the best to date upper bound for the diagonal case:

Theorem [Conlon, 2010].

$r(n+1, n+1) \leq \binom{2n}{n} n^{-c \log n / \log \log n}$.

In 1947, Erdős gave a simple probabilistic proof of the following lower bound for $r(n, n)$.

Theorem [Erdős, 1947].

$cn2^{n/2} \leq r(n, n)$ (Spencer improved the constant c to $\sqrt{2}/e$).

Generalizations to any Graph

The problem $r(G_1, G_2)$

For any graphs G_1 and G_2, what is the smallest number $r = r(G_1, G_2)$ such that any 2-edge-coloring (red-blue) of K_r contains a red G_1 or a blue G_2.

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It is known all the ramsey numbers $r(G, H)$ for all graphs G and H on 4 vertices. For graphs with 5 vertices without isolates, Hendry presented in 1989 a table of $r(G, H)$ except 7 entries.

Graphs with at most 5 vertices

Five of the open entries have been solved:

- $r(K_5, K_4 + e) = r(5,4) = 25$ [McKay and Radziszowski, 1995]
- $r(K_5, K_5 - K_{1,2}) = 25$ [Boza, 2011]
- $r(K_5, B_3) = 20$ [Kung-Kuen, Babak and Radziszowski, 2004]
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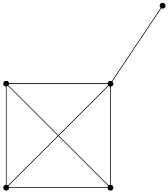
$K_4 + e =$



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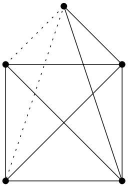
$K_5 - K_{1,2} =$



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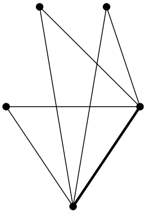
$B_3 =$



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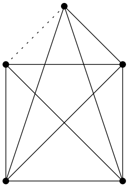
$K_5 - e =$



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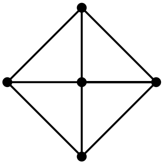
$W_5 =$ 

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There still two open cases for graphs with 5 vertices:

- $30 \leq r(K_5, K_5 - e) \leq 34$ [Exoo, 1992]
- $43 \leq r(5,5) \leq 48$ [Exoo, 1989. Angeltveit and McKay, 2017]

Cycles, Paths and Trees

For $m \leq n$, it is known:

Paths. [Gerencser and Gyarfas, 1967]

$r(P_m, P_n) = m + n/2$

Cycles. [Karolyi and Rosta, 2001]

- $r(C_m, C_n) = 2n - 1$ for m odd.
- $r(C_m, C_n) = n - 1 + m/2$ for m and m even.
- $r(C_m, C_n) = max\{n - 1 + m/2, 2m - 1\}$ for m even, n odd.

Conjecture Trees. [Burr and Erdős, 1976]

$r(T_m, T_n) \leq n + m - 2$

Dropping one edge from complete graph

Known values for $r(K_m - e, K_n - e)$ and $r(K_m, K_n - e)$.

G_1 \ G_2	K_3-e	K_4-e	K_5-e	K_6-e	K_7-e	K_8-e	K_9-e	K_{10}-e
K_3-e	3	5	7	9	11	13	15	17
K_3	5	7	11	17	21	25	31	?
K_4-e	5	10	13	17	28	?	?	
K_4	7	11	19	?	?			
K_5-e	7	13	22	?				
K_5	9	16	?					
K_6-e	9	?						
K_6	11	?						
K_7-e	11	?						

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K_5	9	16	?					
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K_7-e	11	?						

$$r(K_3 - e, K_n) = r(K_3 - e, K_{n+1} - e) = 2n - 1.$$

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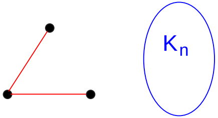
$r(K_3\text{-e}, K_n) = 2n-1$



Dropping one edge from complete graph

Ramsey for complete graphs with a dropped edge or a triangle

$G_1 \backslash G_2$	K_3-e	K_4-e	K_5-e	K_6-e	K_7-e	K_8-e	K_9-e	K_{10}-e
K_3-e	3	5	7	9	11	13	15	17
K_3	5	7	11	17	21	25	31	?
K_4-e	5	10	13	17	28	?	?	
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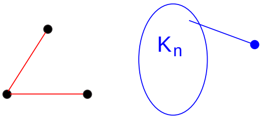
$$r(K_3\text{-e}, K_{n+1} - K_{1,n-1}) = 2n-1$$



Dropping one edge from complete graph

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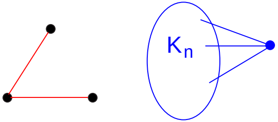

Dropping one edge from complete graph

Question.

For which s the equality $r(G, K_{n+1} - K_{1,s}) = r(G, K_n)$ holds?

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r(4,4)=18

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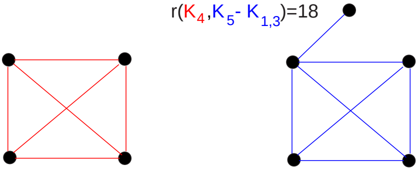

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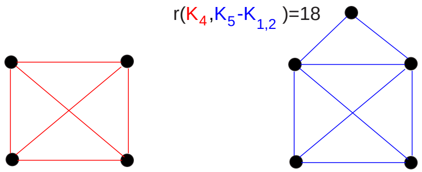

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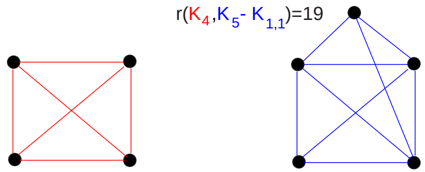

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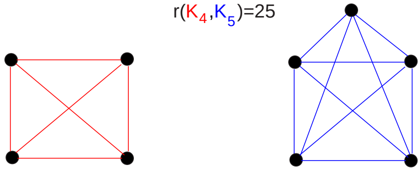
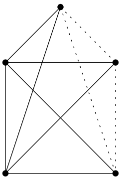


Ramsey numbers for graphs with a dropped clique

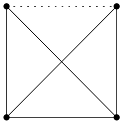
We are interested in study the Ramsey numbers of $r(G_1, G_2)$ when G_1, G_2 are graphs with a dropped clique.

Definition

Let $K_{[n,k]}$ be the complete graph on n vertices from which a set of edges, induced by a clique of order k, has been dropped.



$K_{[5,3]}$



$K_{[4,2]}$

We denote $r(K_{[n_1,k_1]}, K_{[n_2,k_2]})$ as $r([n_1, k_1], [n_2, k_2])$.

Ramsey for complete graphs with a dropped edge or a triangle

New results

- $$r([n,2],[4,3]) = \begin{cases} 2 & \text{for } n = 2 \\ 5 & \text{for } n = 3 \\ 3n-5 & \text{for } n \geq 4 \end{cases}$$
- $$r([n,3],[4,3]) = \begin{cases} 3 & \text{for } n = 3 \\ 6 & \text{for } n = 4 \\ 8 & \text{for } n = 5 \\ 11 & \text{for } n = 6 \\ 3n-8 & \text{for } n \geq 7 \end{cases}$$
- $$r([n,2],[5,3]) = \begin{cases} = 2 & \text{for } n = 2 \\ = 7 & \text{for } n = 3 \\ \leq 3\binom{n+1}{2} - 5n + 4 & \text{for } n \geq 4 \end{cases}$$

Ramsey for complete graphs with a dropped edge or a triangle

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Ramsey for complete graphs with a dropped edge or a triangle

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Proof.

Ramsey for complete graphs with a dropped edge or a triangle

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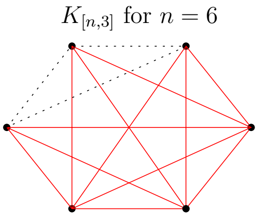

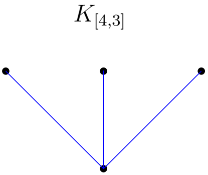

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Proof (Lower bound).

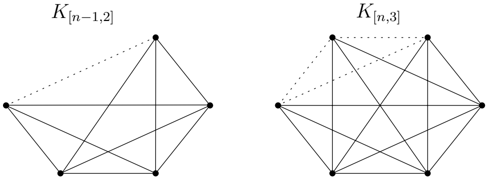
Ramsey for complete graphs with a dropped edge or a triangle

Theorem

$$r([n,3],[4,3]) = \begin{cases} 3 & \text{for } n = 3 \\ 6 & \text{for } n = 4 \\ 8 & \text{for } n = 5 \\ 11 & \text{for } n = 6 \\ 3n-8 & \text{for } n \geq 7 \end{cases}$$

Proof (Lower bound).

We note that $K_{[n-1,2]}$ is a subgraph of $K_{[n,3]}$:



Example for $n = 6$

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$$r([n,3],[4,3]) \geq r([n-1,2],[4,3]).$$

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By our previous result, we know that $r([n-1,2],[4,3]) = 3(n-1) - 5 = 3n - 8$.

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By our previous result, we know that $r([n-1,2],[4,3]) = 3(n-1) - 5 = 3n - 8$. For this, we use some ideas used to study the Ramsey number of $K_n - e$ and some results given by Chvátal in 1977.
Hence

$$r([n,3],[4,3]) \geq 3n - 8$$

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Proof (Upper bound).

We use induction and adapt the recursive formula for this kind of graphs. The proof it's very technical.

Ramsey for complete graphs with a dropped edge or a triangle

Lemma

Let $r \geq 2$ and let $k_1, \ldots, k_r$ and $t_1, \ldots, t_r$ be positive integers with $k_i \geq t_i + 1 \geq 2$ for all i. Then,

$$\begin{aligned} R([k_1, t_1], \ldots, [k_r, t_r]) \leq\ & R([k_1 - 1, t_1], [k_2, t_2], \ldots, [k_r, t_r]) \\ & + R([k_1, t_1], [k_2 - 1, t_2], \ldots, [k_r, t_r]) \\ & \vdots \\ & + R([k_1, t_1], [k_2, t_2], \ldots, [k_r - 1, t_r]) \\ & (r - 2). \end{aligned}$$

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Proof.

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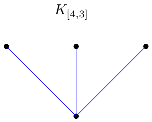
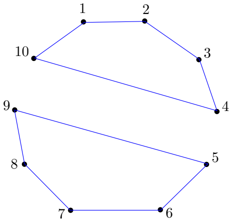
Consider the following red-blue edge-coloring of K_{10}:





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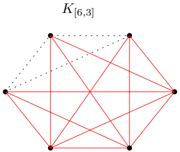

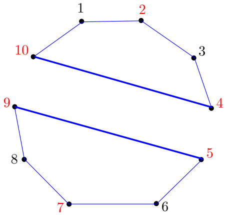

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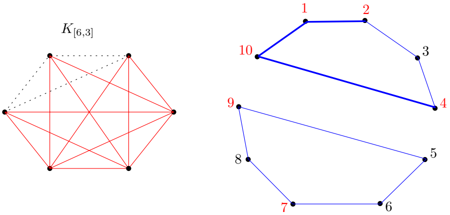

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Proof (Lower bound).

Hence $r([6,3],[4,3]) \geq 11$.

Ramsey for complete graphs with a dropped edge or a triangle

Theorem

$$r([n,3],[4,3]) = \begin{cases} 3 & \text{for } n = 3 \\ 6 & \text{for } n = 4 \\ 8 & \text{for } n = 5 \\ 11 & \text{for } n = 6 \\ 3n-8 & \text{for } n \geq 7 \end{cases}$$

Proof (Upper bound).

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Proof (Upper bound).

Consider any red-blue edge-coloring of $G = K_{11}$.

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$$r([n,3],[4,3]) = \begin{cases} 3 & \text{for } n = 3 \\ 6 & \text{for } n = 4 \\ 8 & \text{for } n = 5 \\ 11 & \text{for } n = 6 \\ 3n-8 & \text{for } n \geq 7 \end{cases}$$

Proof (Upper bound).

Consider any red-blue edge-coloring of $G = K_{11}$. If G does not contain a blue $K_{[4,3]}$ it follows that $d_B(v) \leq 2$ for all vertices $v \in V(G)$.

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$$r([n,3],[4,3]) = \begin{cases} 3 & \text{for } n = 3 \\ 6 & \text{for } n = 4 \\ 8 & \text{for } n = 5 \\ 11 & \text{for } n = 6 \\ 3n-8 & \text{for } n \geq 7 \end{cases}$$

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Consider any red-blue edge-coloring of $G = K_{11}$. If G does not contain a blue $K_{[4,3]}$ it follows that $d_B(v) \leq 2$ for all vertices $v \in V(G)$. Hence

$$d_R(v) \geq 8$$

for all vertices $v \in V(G)$.

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Proof (Upper bound).

Let $v \in V(G)$ and $\{v_1, \ldots, v_8\} \subseteq N(v)$ such that the edges vv_i are red for every $i = 1, \ldots, 8$.

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Let $v \in V(G)$ and $\{v_1, \ldots, v_8\} \subseteq N(v)$ such that the edges vv_i are red for every $i = 1, \ldots, 8$.
Since $r([5,3],[4,3]) = 8$ it follows that $\{v_1, \ldots, v_8\}$ contains a red $K_{[5,3]}$.

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Proof (Upper bound).

Let $v \in V(G)$ and $\{v_1, \ldots, v_8\} \subseteq N(v)$ such that the edges vv_i are red for every $i = 1, \ldots, 8$.

Since $r([5,3],[4,3]) = 8$ it follows that $\{v_1, \ldots, v_8\}$ contains a red $K_{[5,3]}$. Suppose that the vertices of this red $K_{[5,3]}$ are $\{v_1, \ldots, v_5\}$. Then the subgraph induced by $\{v, v_1, \ldots, v_5\}$ contains a red $K_{[6,3]}$. Therefore $r([6,3],[4,3]) \leq 11$.

Future Work

(a) Exact value for $r([n,4],[4,3])$?

Ramsey for complete graphs with a dropped edge or a triangle

Future Work

(a) Exact value for $r([n, 4], [4, 3])$?
(b) Exact value for $r([n, 2], [5, 3])$?

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Y. Li, C.C. Rousseau (1996) and B. Sudakov (2005) studied the ramsey numbers $r(K_n, B_m)$.

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for all positive integers n.

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Is it true that $r([n,1],[5,3]) = r([n,2],[5,3])$?

Ramsey for complete graphs with a dropped edge or a triangle

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Future Work

(a) Exact value for $r([n,4],[4,3])$?
(b) Exact value for $r([n,2],[5,3])$?
(c) Good upper bound for $r([n_1,k_1],[n_2,k_2])$?

MERCI!!