Universal Antimagic Graphs

LAGOS2017

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Definition (B-labeling)

Given a set of non-negative integers B, *a* B*-labeling of a graph* $G = (V, E)$ *is a bijective function* $f : E \to B$ *such that the sums*

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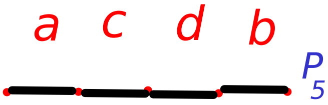
Definition (Universal Antimagic Graphs)

A graph G with m edges is universal antimagic if it has a B-labeling, for each set B of m positive integers.

Some Universal Antimagic graphs

1. A path on n vertices P_n.



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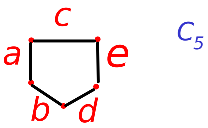

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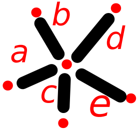

$K_{1,5}$

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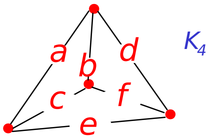
4. A complete graph K_n.



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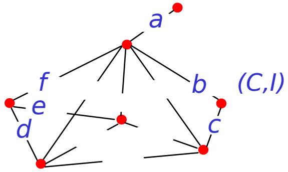

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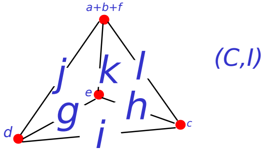

Definition (Antimagic Graphs)

G with m edges is (only) antimagic if it has a $[m]$-labeling, where

$$[m] = \{1, \ldots, m\}.$$

Conjecture (Hartsfield and Ringel 1990)

Every connected graph G is antimagic unless G is an edge (K_2).

Remark

Open, even for trees.

Probabilistic and Algebraic tools

A connected graph $G = (V, E)$, $G \neq K_2$, with n vertices is antimagic when it satisfies some of the following properties.

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   Uses Combinatorial Nullstellensatz.

Results based on arithmetic properties of $[m]$

The following graphs are antimagic

1. Product of regular graphs with paths or cycles, regular bipartite graphs, regular graphs of odd degree and regular graphs.
   Cheng 2008, Wang and Hsiao 2008 DM, Cranston 2008, Wang-Zhu 2013 JGT, Cranston, Liang, Zhu 2015 JGT, Bérczi, Bernáth, Vizer 2015 EIJC.

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2. Trees with at most one vertex of degree two.
Kaplan, Lev, Roddity 2009, JGT.
3. Graphs admitting a regular dominating subgraph.
Sliva 2012, IPL.

Results relaying on order properties of $[|E|]$

A connected graph $G = (V, E)$, $G \neq K_2$, with n vertices is antimagic when

1. G is a complete partite graph.
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4. G contains a subgraph $K_{p,n-p}$ with $p \geq 2$ and $n \geq 5$.
   M. and Zamora 2016, DAM. (LAGOS' 2015).

Universal antimagic can be seen as the list version of antimagic.

(M. and Zamora, 2017)

For $p, q \geq 1$, $p + q \geq 3$, any graph G containing $K_{p,q}$ as a spanning subgraph is UA.

In this talk we present the cases $p \leq 2$.

(M. and Zamora, 2017)

Any linear forest whose connected components have odd length at least three is UA.

If time allows it, we discuss case with lengths three or five.

($p = 1$)

Every graph G with n vertices, m edges and an universal vertex v_1 is UA.

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- Order the vertices of $G - v_1$ as $v_2, \dots, v_n$ according to their partial sums with v_2 having the largest value. Sums are not necessarily distinct.
- Assign the i-th largest value of B to vv_{i+1}, $i = 1 \dots, n-1$.

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- Total sums at $v_2, \ldots, v_n$ are all distincts.
- Total sum at v_1 is larger than any other sum.

(M. and Zamora, 2017)

A graph $G = (V, E)$ with m edges and $2 + q$ vertices containing $K_{2,q}$ as spanning subgraph is UA.

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- Assing values a_i to edge ax_i and b_i to edge bx_i, for $i = 1, \ldots, q$.

Schematic view of ℓ_0

	x_1	x_2	$\cdots$	x_q
Σ_a	a_1	a_2	$\cdots$	a_q
Σ_b	b_1	b_2	$\cdots$	b_q
	w_1	w_2	$\cdots$	w_q

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Claim : ℓ_0 is a B-labeling unless $w_1 + a_1 + b_1 \in \{\Sigma_b, \Sigma_a\}$.

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- $w_2 < b_q(q-2)$.

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- $w_2 < b_q(q-2)$.
- $w_2 + a_2 + b_2 < b_q(q-2) + b_1 + b_2 \leq \Sigma_b < \Sigma_a$.

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- $w_q + a_q + b_q < \cdots < w_2 + a_2 + b_2$.
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- $w_2 + a_2 + b_2 < b_q(q-2) + b_1 + b_2 \leq \Sigma_b < \Sigma_a$.

When $w_1 + a_1 + b_1 \in \{\Sigma_b, \Sigma_a\}$ we get a *collision.*

Column flips reduces collisions.

A new candidate ℓ_1 is obtained by flipping a_1 with b_1 in ℓ_0.

	x_1	x_2	$\cdots$	x_q
Σ'_a	b_1	a_2	$\cdots$	a_q
Σ'_b	a_1	b_2	$\cdots$	b_q
	w_1	w_2	$\cdots$	w_q

$\Sigma'_a = \Sigma_a + b_1 - a_1$ and $\Sigma'_b = \Sigma_b + a_1 - b_1$.

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- $w_1 + a_1 + b_1 \notin \{\Sigma'_a, \Sigma'_b\}$.

Column flips...

A new candidate ℓ_2 is obtained by flipping a_2 with b_2 in ℓ_0.

	x_1	x_2	$\cdots$	x_q
Σ''_a	a_1	b_2	$\cdots$	a_q
Σ''_b	b_1	a_2	$\cdots$	b_q
	w_1	w_2	$\cdots$	w_q

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  $\Rightarrow \Sigma''_a - \Sigma''_b > 0$.

Odd Linear forest

(M. and Zamora, 2017)

A linear forest whose connected components have odd size at least three is UA.

(M. and Zamora, 2017)

A disconnected graph whose connected components are paths of length two is not UA.