On the (di)graphs with (directed) proper connection number two

Guillaume Ducoffe [1,3], **Ruxandra Marinescu-Ghemeci** [2,4], Alexandru Popa [2,3]

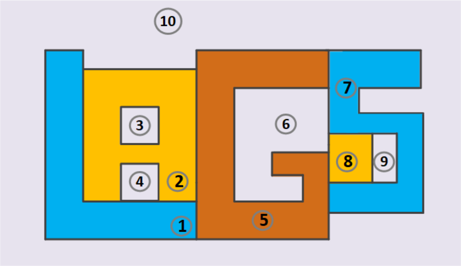
[1]Université Côte d'Azur, Inria, CNRS, I3S, France
[2]University of Bucharest, Romania
[3]National Institute for Research and Development in Informatics, Romania
[4]ICUB Research Institute of the University of Bucharest, Romania

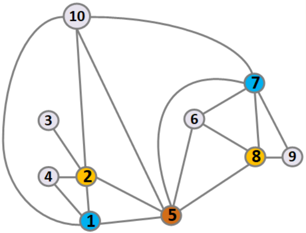
LAGOS 2017

Proper colorings

Well known...

- four color map problem [Appel and Haken (1989)]
- chromatic number problem - NP-complete
- for vertices / edges
- many applications - avoid conflicts
    - register allocation
    - scheduling problems
    - interference, security in communication networks





Proper path colorings

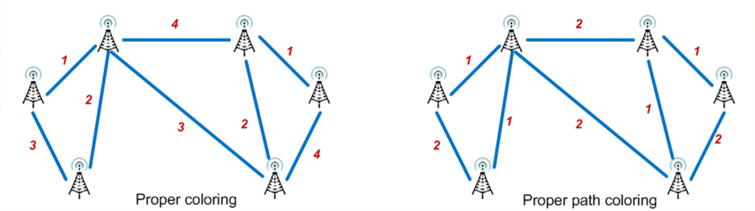
Relaxed constrains

- Not necessary to impose constrains on the colors for all pairs of adjacent edges/vertices in order avoid conflicts, but:
  - **assure paths between any pair of vertices on which communication is safe**
  - Possible advantages: less colors, algorithms

Proper path colorings

Relaxed constrains

- Not necessary to impose constrains on the colors for all pairs of adjacent edges/vertices in order avoid conflicts, but:
  - **assure paths between any pair of vertices on which communication is safe**
  - Possible advantages: less colors, algorithms
- **Proper path coloring** - proper connection number
  - Borozan et al. (2012)



Proper path colorings

Definition (Borozan et al. (2012); Andrews et al. (2016))

$c : E(G) \longrightarrow \{1, 2, \ldots, k\}$

- **proper path** P: $c|_{E(P)}$ proper edge-coloring
- **proper connection number** of G: $pc_e(G)$
- similar - **proper vertex-connection number** of G: $pc_v(G)$
- **proper connection for strong digraphs:** for every ordered pair u,v of vertices exists a proper (di)path from u to v.

Proper path colorings

Definition (Borozan et al. (2012); Andrews et al. (2016))

$c : E(G) \longrightarrow \{1, 2, \ldots, k\}$

- **proper path** P: $c|_{E(P)}$ proper edge-coloring
- **proper connection number** of G: $pc_e(G)$
- similar - **proper vertex-connection number** of G: $pc_v(G)$
- **proper connection for strong digraphs:** for every ordered pair u,v of vertices exists a proper (di)path from u to v.

Proper path colorings

Definition (Borozan et al. (2012); Andrews et al. (2016))

$c : E(G) \longrightarrow \{1, 2, \ldots, k\}$

- **proper path** P: $c|_{E(P)}$ proper edge-coloring
- **proper connection number** of G: $pc_e(G)$
- similar - **proper vertex-connection number** of G: $pc_v(G)$
- **proper connection for strong digraphs:** for every ordered pair u,v of vertices exists a proper (di)path from u to v.

Related to **rainbow coloring** - [Chartrand et al. (2008)]

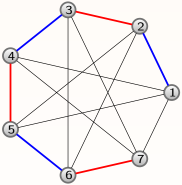
- computing the rainbow connection number is NP-hard and not FPT for any fixed $k \geq 2$ [Chakraborty et al. (2011)]

Known results

Survey - [Li and Magnant (2015)]

Combinatorial results:

- Existence problems [Andrews et al. (2016)]
  - difference between chromatic number and pc can be arbitrary large



- Connection to structure properties of graph
  - minimum degree, domination, connectivity [Li et al. (2015)]

Known results

Combinatorial results:

- Extremal graphs [Laforge et al. (2016)]
  - graphs with $pc(G) = m - 1, m - 2$
  - graphs with $pc = 2$ - **no complete characterization or algorithmic results**
    - 3-connected, 2-connected with diameter 2, bipartite bridgeless

Known results

Combinatorial results:

- Extremal graphs [Laforge et al. (2016)]
  - graphs with $pc(G) = m - 1, m - 2$
  - graphs with $pc = 2$ - **no complete characterization or algorithmic results**
    - 3-connected, 2-connected with diameter 2, bipartite bridgeless
- Classes of graphs
  - random graphs [Gu et al. (2016)]
    - Almost all graphs have proper connection number 2
  - **bipartite graphs** [Borozan et al. (2012); Huang et al. (2015, 2016)]
    - **sufficient conditions** to have proper connection number 2

Known results

Combinatorial results:

- Extremal graphs [Laforge et al. (2016)]
  - graphs with $pc(G) = m - 1, m - 2$
  - graphs with $pc = 2$ - **no complete characterization or algorithmic results**
    - 3-connected, 2-connected with diameter 2, bipartite bridgeless
- Classes of graphs
  - random graphs [Gu et al. (2016)]
    - Almost all graphs have proper connection number 2
  - **bipartite graphs** [Borozan et al. (2012); Huang et al. (2015, 2016)]
    - **sufficient conditions** to have proper connection number 2
- Digraphs [Magnant et al. (2016)]
  - $\overrightarrow{pc_e}(D) \leq 3$
  - **Conjecture**: A strong digraph with no even dicycle has $\overrightarrow{pc_e}(D) = 3$.

Known results

Combinatorial results:

- Extremal graphs [Laforge et al. (2016)]
  - graphs with $pc(G) = m - 1, m - 2$
  - graphs with $pc = 2$ - **no complete characterization or algorithmic results**
    - 3-connected, 2-connected with diameter 2, bipartite bridgeless
- Classes of graphs
  - random graphs [Gu et al. (2016)]
    - Almost all graphs have proper connection number 2
  - **bipartite graphs** [Borozan et al. (2012); Huang et al. (2015, 2016)]
    - **sufficient conditions** to have proper connection number 2
- Digraphs [Magnant et al. (2016)]
  - $\overrightarrow{pc_e}(D) \leq 3$
  - **Conjecture**: A strong digraph with no even dicycle has $\overrightarrow{pc_e}(D) = 3$.
- **Proper vertex-connection** for undirected graphs - trivial

Our results

- **No algorithmic results**
- Bipartite graphs with $pc = 2$ - **no complete characterization or algorithms**

Polynomial-time recognition **algorithms** for bounded-treewidth graphs and bipartite graphs with $pc = 2$

Characterization of bipartite graphs with $pc = 2$

Our results

- **Digraphs**
  - $\overrightarrow{pc_e}(D) \leq 3$

> Deciding whether $\overrightarrow{pc_e}(D) \leq 2$ is NP-complete.
> - Reduction from Positive NAE-SAT

  - **Conjecture**: A strong digraph with no even dicycle has $\overrightarrow{pc_e}(D) = 3$

> There exists an infinite family of digraphs with no even dicycles that also have properly connected 2-colorings.

Our results

- **Digraphs**
  - $\overrightarrow{pc_e}(D) \leq 3$

    Deciding whether $\overrightarrow{pc_e}(D) \leq 2$ is NP-complete.

    - Reduction from Positive NAE-SAT

  - **Conjecture**: A strong digraph with no even dicycle has $\overrightarrow{pc_e}(D) = 3$

    There exists an infinite family of digraphs with no even dicycles that also have properly connected 2-colorings.

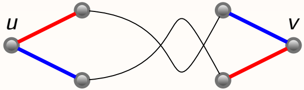
- **Proper vertex-connection** for undirected graphs - trivial

  - Initiate study of proper vertex-connection for digraphs
  - $\overrightarrow{pc_v}(D) \leq 3$
  - Deciding whether $\overrightarrow{pc_v}(D) \leq 2$ is NP-complete
    - Reduction from 3-SAT

Bipartite graphs

Lemma (Huang et al. (2015))

If G is a connected bipartite bridgeless graph, then $pc_e(G) \leq 2$. Furthermore, such a coloring can be produced with the strong property.

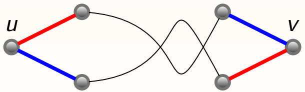


Coloring with strong property - ear decomposition

Bipartite graphs

Lemma (Huang et al. (2015))

If G is a connected bipartite bridgeless graph, then $pc_e(G) \leq 2$. Furthermore, such a coloring can be produced with the strong property.



Coloring with strong property - ear decomposition

Remark

- $pc_e(G) \geq b(G) =$ *maximum number of bridges incident in a vertex [Andrews et al. (2016)]*
- $pc_e(G) \leq 2 \Longrightarrow b(G) \leq 2$

Bipartite graphs

? Any bipartite graph with $b(G) \leq 2$ has $pc_e(G) \leq 2$?

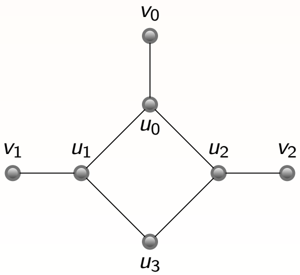
Bipartite graphs

? Any bipartite graph with $b(G) \leq 2$ has $pc_e(G) \leq 2$?

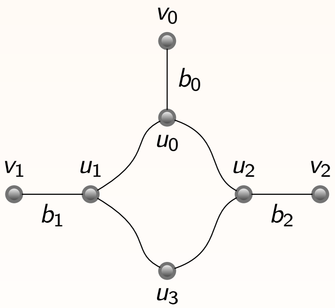
No

Lemma

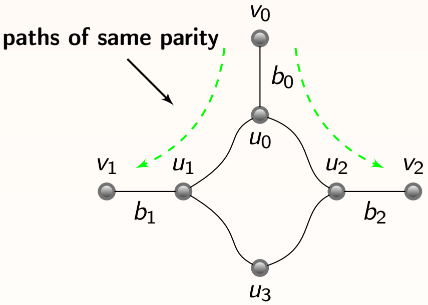
Let $G = (V, E)$ be a connected graph, B be a bridge-block of G that is bipartite. If B is incident to at least three bridges then $pc_e(G) \geq 3$.



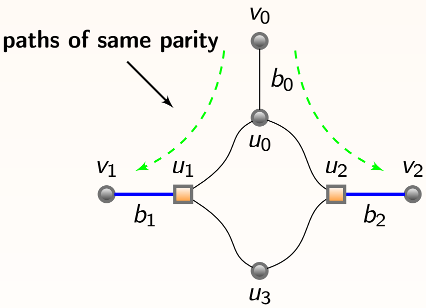
Bipartite graphs



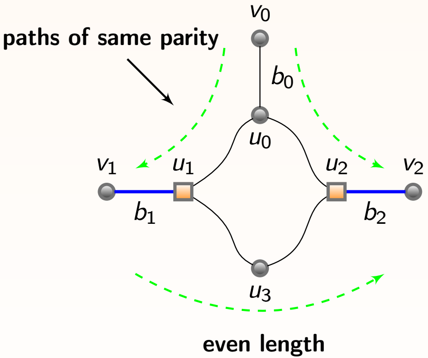
Bipartite graphs



Bipartite graphs



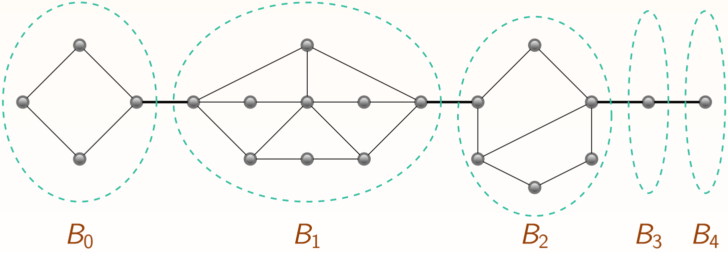
Bipartite graphs



Bipartite graphs

Theorem

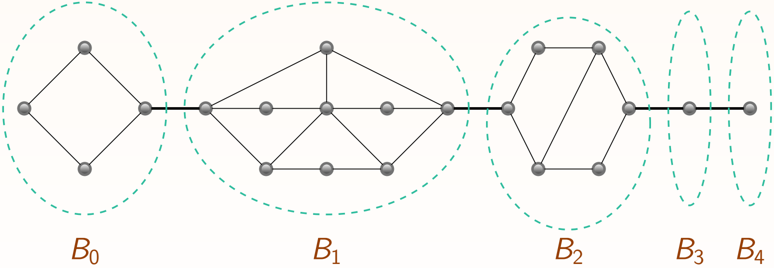
Let $G = (V, E)$ be a connected bipartite graph. We have $pc_e(G) \leq 2$ if and only if the bridge-block tree of G is a path. Furthermore, if $pc_e(G) \leq 2$, then such a coloring can be computed in linear-time.



Bipartite graphs

If bridge-block tree of G is a path $\Longrightarrow$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

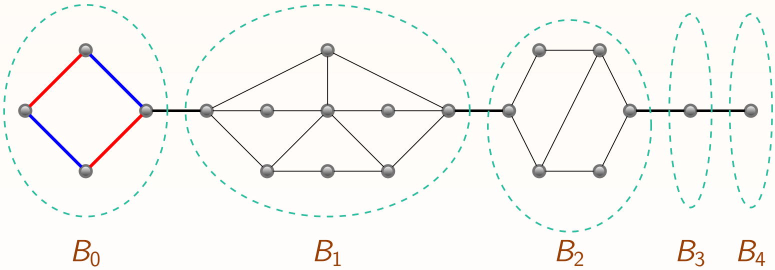
- Color blocks in this order (with strong property)



Bipartite graphs

If bridge-block tree of G is a path $\Longrightarrow$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

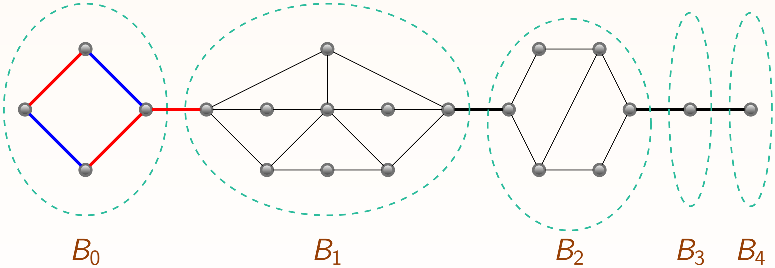
- Color B_0 (with strong property)



Bipartite graphs

If bridge-block tree of G is a path $\Longrightarrow$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

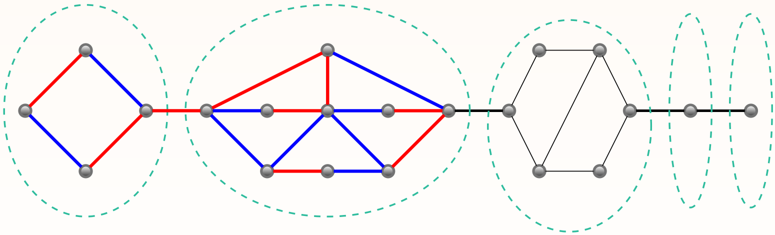
- Color bridge between B_0 and B_1 arbitrary



Bipartite graphs

If bridge-block tree of G is a path $\Longrightarrow$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

- Color B_1 (with strong property)

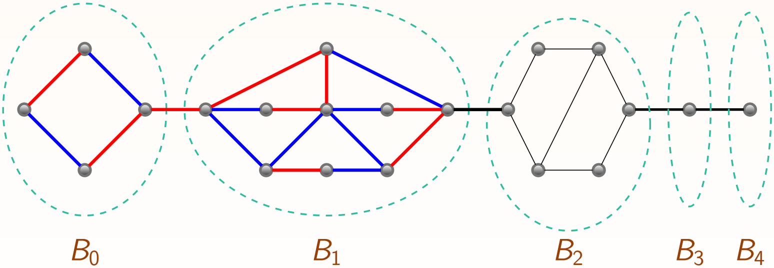


B_0 B_1 B_2 B_3 B_4

Bipartite graphs

If bridge-block tree of G is a path $\Longrightarrow$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

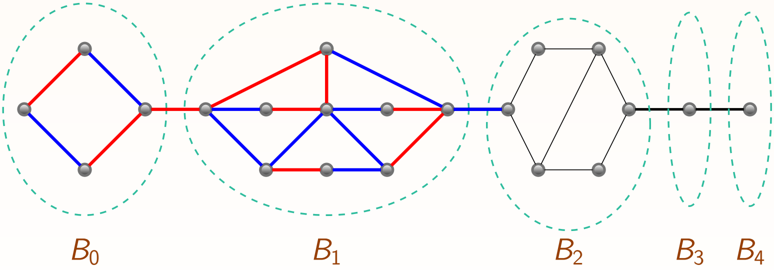
- Color bridge between B_i and B_{i+1} according to color of bridge between B_{i-1} and B_i and parity of paths length



Bipartite graphs

If bridge-block tree of G is a path $\Longrightarrow$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

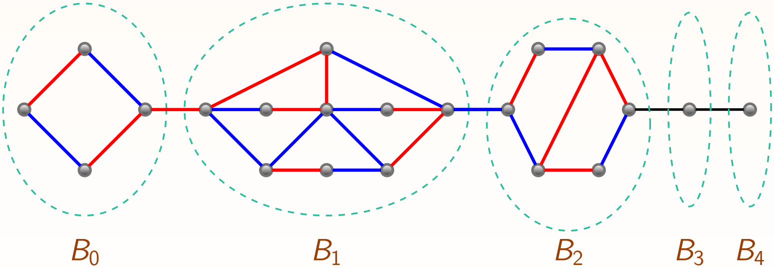
- Color bridge between B_i and B_{i+1} according to color of bridge between B_{i-1} and B_i and parity of paths length



Bipartite graphs

If bridge-block tree of G is a path $\implies$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

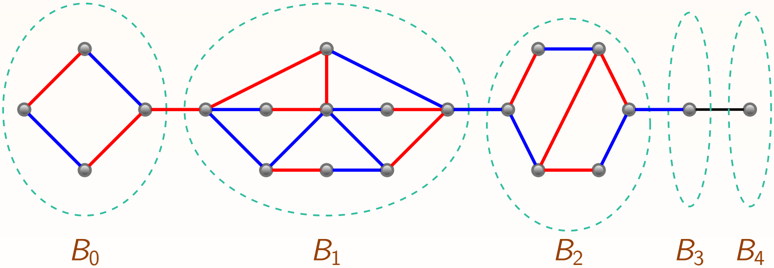
- Color bridge between B_i and B_{i+1} according to color of bridge between B_{i-1} and B_i and parity of paths length



Bipartite graphs

If bridge-block tree of G is a path $\Longrightarrow$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

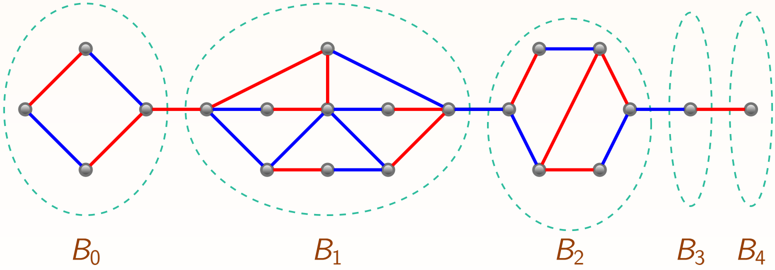
- Color bridge between B_i and B_{i+1} according to color of bridge between B_{i-1} and B_i and parity of paths length



Bipartite graphs

If bridge-block tree of G is a path $\Longrightarrow$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

- Color bridge between B_i and B_{i+1} according to color of bridge between B_{i-1} and B_i and parity of paths length



Conclusions

- Complete characterization of bipartite graphs with $pc = 2$
- First algorithmic and complexity results on proper connection
- **Open problems**
  - NP-completeness results and algorithms for undirected case and related type of colorings
    - characterize graphs with $pc \leq 2$
    - proper connection number of bipartite graphs
  - Conditions for strong digraphs to have proper edge/vertex connection number 2 or 3.

E. Andrews, C. Lumduanhom, E. Laforge, and P. Zhang. On proper-path colorings in graphs. *J. Comb. Math. and Comb. Comp.*, 97:189–207, 2016.

K. Appel and W. Haken. *Every Planar Map is Four-Colorable*. Contemporary Mathematics, 1989.

V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Montero, and Z. Tuza. Proper connection of graphs. *Discrete Mathematics*, 312(17): 2550–2560, 2012.

S. Chakraborty, E. Fischer, A. Matsliah, and R. Yuster. Hardness and algorithms for rainbow connection. *J. of Combinatorial Optimization*, 21(3):330–347, 2011.

G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang. Rainbow connection in graphs. *Mathematica Bohemica*, 133(1):85–98, 2008.

R. Gu, X. Li, and Z. Qin. Proper connection number of random graphs. *Theoretical Computer Science*, 609:336–343, 2016.

F. Huang, X. Li, and S. Wang. Proper connection number and 2-proper connection number of a graph. *arXiv preprint arXiv:1507.01426*, 2015.

F. Huang, X. Li, Z. Qin, and C. Magnant. Minimum degree condition for proper connection number 2. *Theoretical Computer Science*, 2016.

E. Laforge, C. Lumduanhom, and P. Zhang. Characterizations of graphs having large proper connection numbers. *Discuss. Math. Graph Theory*, 36(2): 439–453, 2016.

X. Li and C. Magnant. Properly colored notions of connectivity-a dynamic survey. *Theory and Applications of Graphs*, 0(1):2, 2015.

X. Li, M. Wei, and J. Yue. Proper connection number and connected dominating sets. *Theoretical Computer Science*, 607:480–487, 2015.

C. Magnant, P. R. Morley, S. Porter, P. S. Nowbandegani, and H. Wang. Directed proper connection of graphs. *МАТЕМАТИЧКИ ВЕСНИК*, 68(1):58–65, 2016.

W. McCuaig. Even dicycles. *J. of Graph Theory*, 35(1):46–68, 2015.

Thank You!

Strongly 2-connected digraphs with no even dicycle

Proposition (McCuaig (2015))

There is only one strongly 2-connected digraph with no even dicycle (up to an isomorphism), digraph D_7.

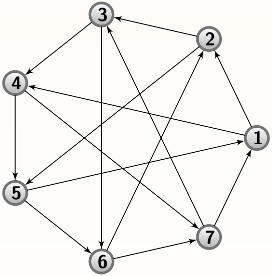
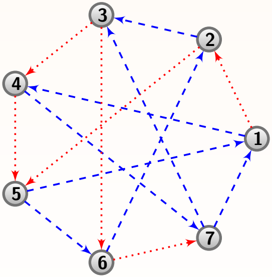


Figure: D_7.

Strongly 2-connected digraphs with no even dicycle

Lemma

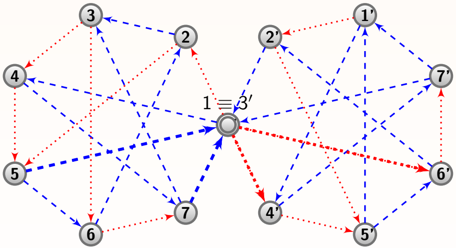
$\overrightarrow{pc_e}(D_7) = 2.$



Strongly connected digraphs with no even dicycle

Theorem

There is an infinite family of strongly connected digraphs with no even dicycle having proper connection number equal to 2



NP-completness result

Theorem

Deciding whether $\overrightarrow{pc_e}(D) \leq 2$ for a given digraph D is NP-complete.

Proof.

- NP-hard: reduction from Positive NAE-SAT

Problem (Positive NAE-SAT)

Input: *A propositional formula Φ in conjunctive normal form, with unnegated variables.*

Question: *Does there exist a truth assignment satisfying Φ in which no clause has all its literals valued 1 ?*

□

Reduction example

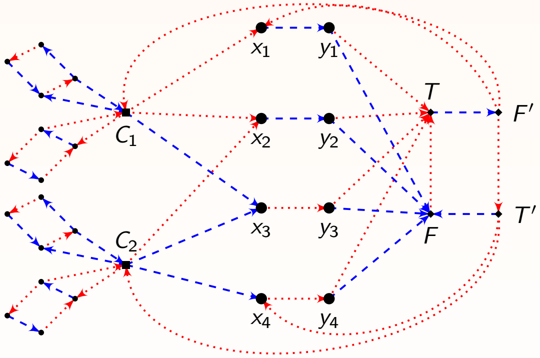


Figure: D_Φ for $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee x_4)$

Clause gadget

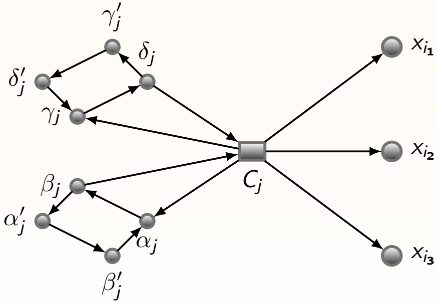


Figure: Gadget representing the clause $C_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$.

Clause gadget

- unique paths: $[\alpha_j, \beta_j, C_j, \gamma_j, \delta_j]$ and $[\gamma_j, \delta_j, C_j, \alpha_j, \beta_j]$

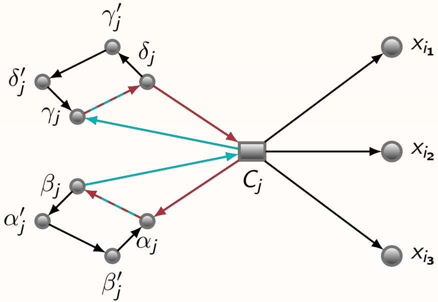


Figure: Gadget representing the clause $C_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$.

Clause gadget

- unique paths: $[\alpha_j, \beta_j, C_j, \gamma_j, \delta_j]$ and $[\gamma_j, \delta_j, C_j, \alpha_j, \beta_j] \Longrightarrow$ arcs with same color

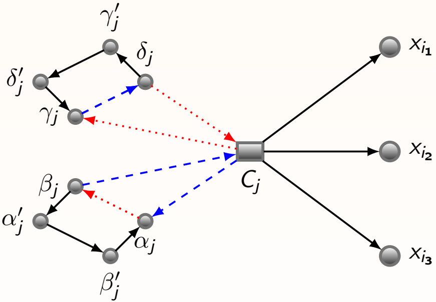


Figure: Gadget representing the clause $C_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$.

Proper vertex connection number of digraphs

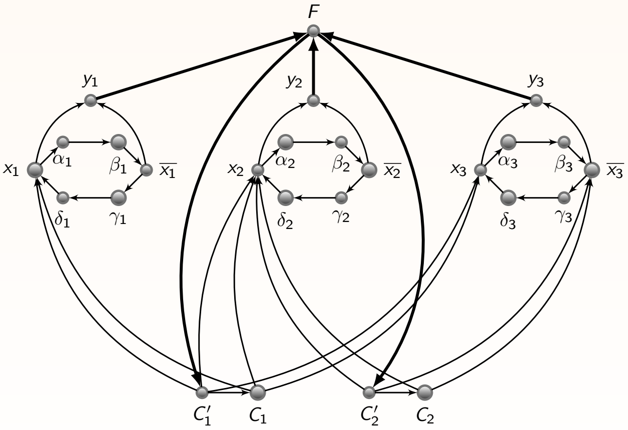


Figure: D_Φ for $\Phi = (x_1 \vee x_2 \vee x_3) \;\wedge\; (x_2 \vee \overline{x_3})$