Bispindle in strongly connected digraphs with large chromatic number

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Joint work with N. Cohen, F. Havet and R. Lopes

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Digraphs and chromatic number

Definition

- The *chromatic number*, $\chi(D)$, of a digraph D is the chromatic number of its underlying graph.
- A directed path (dipath) is a path where all the arcs are oriented in the same direction.

Question

- What are the subgraphs of a digraph with large chromatic number?
- For a digraph D, can we bound the chromatic number of D-free digraphs?

Known results

Theorem (Gallai-Hasse-Roy-Vitaver, 60s)

$\chi(D) = k \Rightarrow$ *D has a directed path with* k *vertices.*

This can only be extended to acyclic digraphs:

Theorem (Erdős)

There are digraphs with arbitrarily large girth and chromatic number.

Conjecture (Burr, 1980)

Every digraph of chromatic number $2k - 2$ *contains all oriented tree of order* k

Best bound is $k^2/2$ by Addario-berry et al.

Subdivision

Definition

A *subdivision* of a digraph D is obtained by replacing arcs by dipaths.

Theorem (Cohen, Havet, L. and Nisse)

For any oriented cycle C, there exists digraphs with arbitrarily large chromatic number and without a subdivision of C.

The question is different in strong digraphs:

Theorem (Bondy, 1976)

D is strong, $\chi(D) = k \Rightarrow D$ has a directed cycle with at least k vertices.

Blocks

Definition

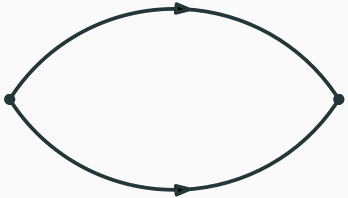
In an oriented path or cycle, a *block* is a maximal directed sub-path.

Theorem (Cohen, Havet, L. and Nisse)

Let C be a oriented cycle with two blocks. There exists $f(C)$ s.t: If D is a strong digraph with $\chi(D) \geq f(C)$, then D contains a subdivision of C.

Disjoint path between a pair of vertices

A cycle on two blocks can be seen as two disjoint dipaths between a pair of vertices.

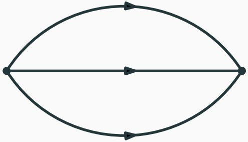


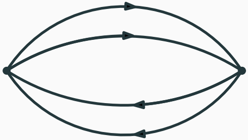
Question

Can we generalise this question for more dipaths?

Proposition

There exists strong digraphs with arbitrarily large chromatic number avoiding the following digraphs as subdivision:

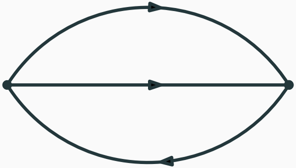




$B(k,k;k)$

Definition

Let $B(k_1,k_2;k_3)$ be the union of three dipaths, two of length k_1 and k_2 in one direction, and one of length k_3 in the other.



Theorem (Cohen, Havet, L. and Lopes)

For every k, there exists $f(k)$ such that every strong digraph D with $\chi(D) \geq f(k)$ contains a subdivision of $B(k,1;k)$.

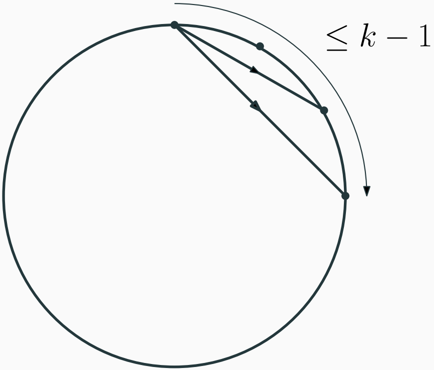
I will present the proof for $B(k,1;1)$.

Hamiltonian case

Theorem

Any Hamiltonian digraph on $n > k+3$ vertices without a $B(k,1;1)$ is $2k-1$ degenerate.

The maximum out degree of a vertex is $k-1$.



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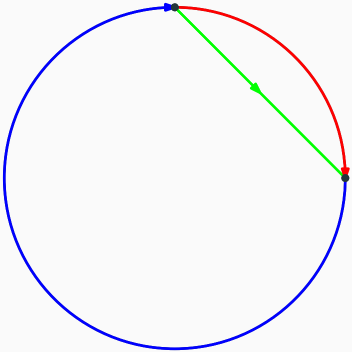


First Idea

If D is a $B(k,1;1)$-free strong digraph with large chromatic number:

- Bondy's Theorem implies the existence of a large cycle.
- Large cycles have a simple structure.

So the idea is to contract long cycles such that:

- We control the chromatic number of the contracted structure.
- There are no long cycle remaining.

Overall strategy

Lemma

Let D be a digraph, $D_1 \dots D_l$ be disjoint subdigraphs of D and D' the digraph obtained by contracting each D_i into one vertex d_i. Then $\chi(D) \leq \chi(D') \cdot \max\{\chi(D_i) \mid i \in [l]\}$.

Proof.

Let c' be a colouring of D' and c_i be colouring of the D_i. Define c a colouring of D as follows:

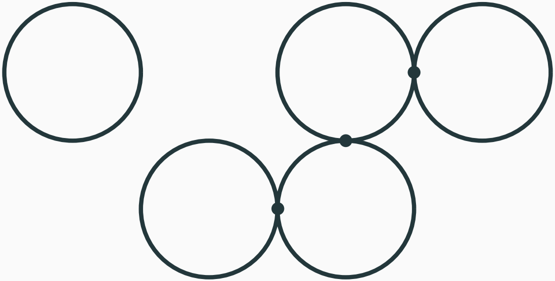
- if $x \in D_i$, $c(x) = (c'(d_i), c_i(x))$.
- else $c(x) = (c'(x), 1)$.

□

Nice collection of cycles

We say a set of cycles $\mathcal{C}$ of D is *nice* if :

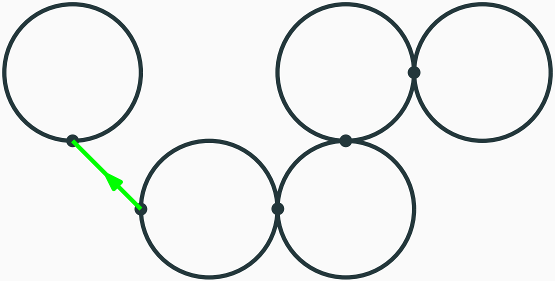
- All cycles of $\mathcal{C}$ are longer than $2k$.
- Any two distinct cycles C_1, C_2 of $\mathcal{C}$ intersect in at most one vertex.



Components of $\mathcal{C}$

Definition

- Two cycles of $\mathcal{C}$ are *adjacent* if they intersect.
- A *component* of $\mathcal{C}$ is a set of cycle forming a connected component in the adjacency graph of the cycle of $\mathcal{C}$.



Maximal nice collection

Let D be a strong digraph without $B(k,1;1)$.

Take $\mathcal{C}$ a maximal collection of cycles and call D' the digraph obtained by contracting each component into one vertex.

We will prove the two following lemmas:

Lemma 1

For each component S of $\mathcal{C}$ $\chi(D[S]) \leq 2k$

Lemma 2

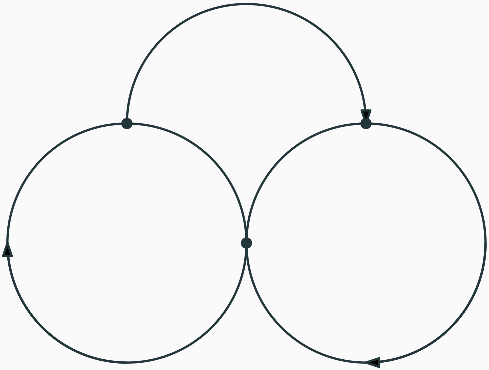
$\chi(D') \leq 2k$

Which will prove that $\chi(D) \leq 4k^2$.

Headphone Lemma

Lemma

Let C_1 C_2 be two cycles of the same component of $\mathcal{C}$, then there is no dipath from C_1 to C_2 outside of $\mathcal{C}$.



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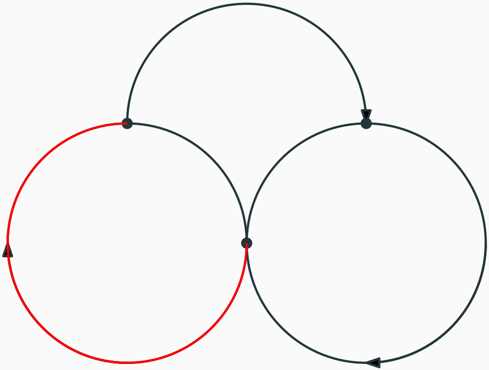

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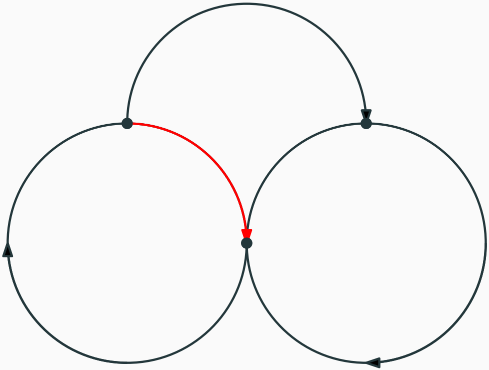

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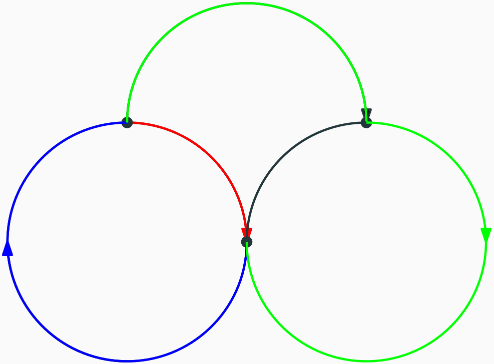

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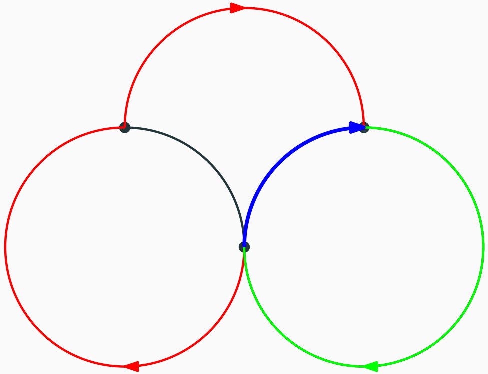
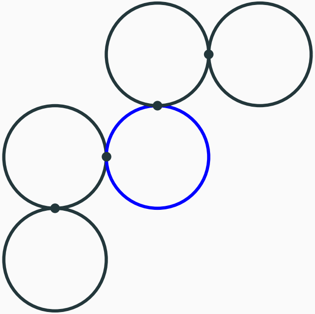


Lemma

Let $\mathcal{C}$ be a nice collection of cycles and S a component, then $\chi(D[S]) \leq 2k$

Proof.

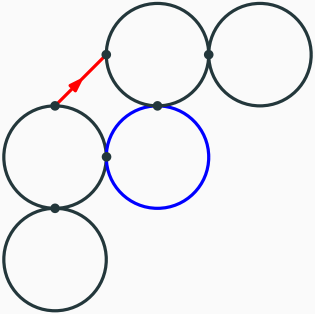
By induction on the number of cycles in S. □



Colouring components (2/4)

Proposition

There is no arcs between the new components.



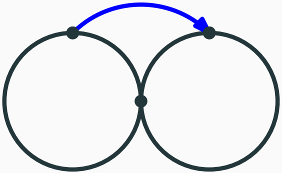
This would contradict the headphone lemma.

Colouring components (3/4)

Proposition

Each new component intersects the blue cycle in one vertex.

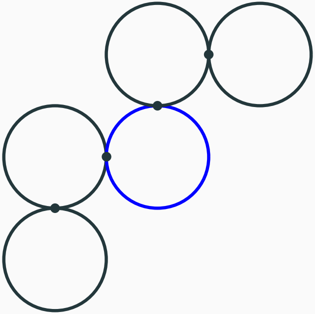
- Each cycle intersects the blue cycle in at most one vertex.
- Two cycles intersecting the blue cycle contradicts the headphone Lemma.



Colouring components (4/4)

We can prove the result:

- Colour the blue cycle by degeneracy.
- Apply induction on each new component, where one vertex is already coloured.



Colouring D'

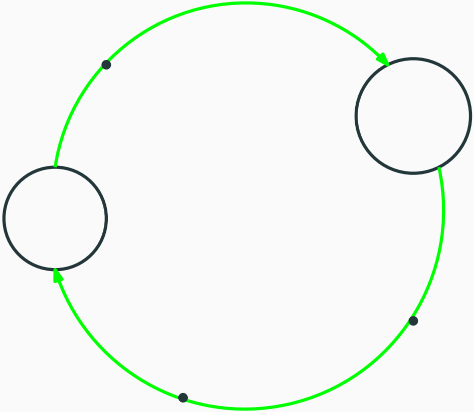
Proposition

- *D' is strongly connected*
- *D' has no cycle longer than $2k-1$*

Bondy's result then implies that $\chi(D') \leq 2k$.

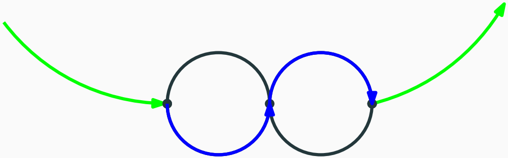
Long cycle in D' (1/4)

Suppose there exists a cycle C' longer than $2k$ in D'



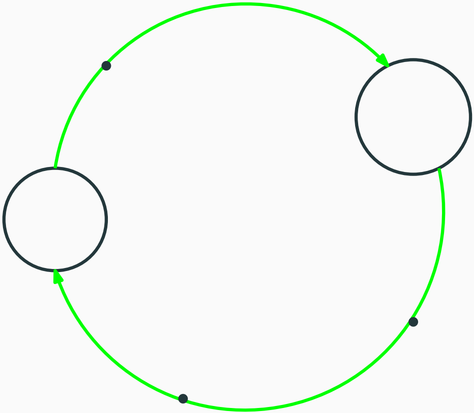
Long cycle in D' (2/4)

Each contracted vertex can be replaced by a path



Long cycle in D' (3/4)

This means we obtain a cycle C of D.



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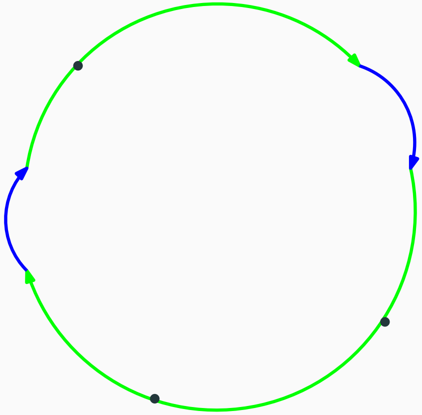


Long cycle in D' (4/4)

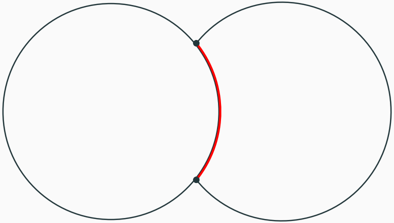
Finally, it is easy to show that:

- C is longer than $2k$.
- It intersect each other cycle of $\mathcal{C}$ in at most 1 vertex.

Which contradicts the maximality of $\mathcal{C}$!

$B(k, 1; k)$

The ideas are similar, but we allow cycles to intersect on a short path.



- Proving that the contracted digraph is without any long cycle is easy.
- Bounding the chromatic number of the components is way more difficult.

Thank you!