Random Hyperbolic Graphs

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Part I: Motivation and model specification

Random hyperbolic graphs (RHGs): Introduction

- Introduced by Krioukov, Papadopoulos, Kitsak, Vahdat, Boguña [Phys. Rev. '10]
- Appeal: Replicate characteristic properties observed in "real world networks" or "complex networks"

Example of networks:	*Power grid* *Internet* *Social networks* *Biological interaction networks* *...*
Typical properties:	*Sparse* *Heterogeneous* *Locally dense (exhibit clustering phenomena)* *Small world* *Navigable* *Scale free (with exponent between 2 and 3)* *...*

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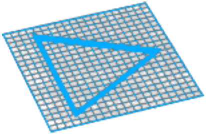
Susceptible to mathematical analysis!

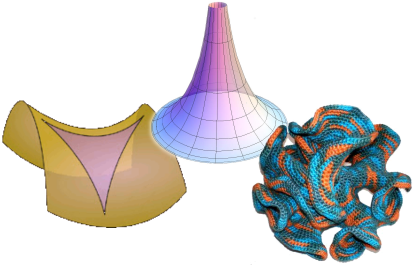
Informal definition of RHGs model

Like random geometric graphs but where the underlying space instead of being Euclidean is Hyperbolic.

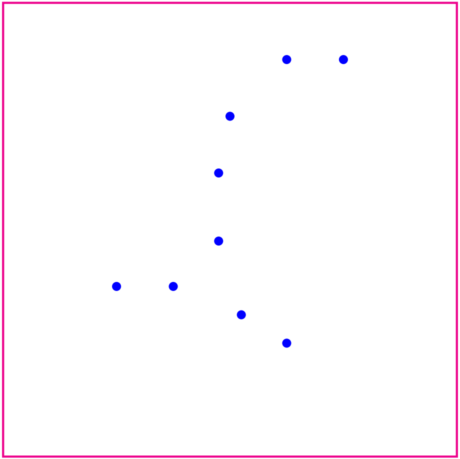
Hyperbolic plane $\mathbb{H}^2$

Euclidean plane $\mathbb{R}^2$





Geometric graphs



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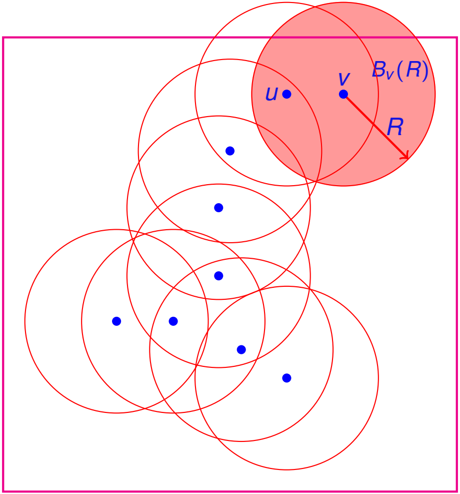
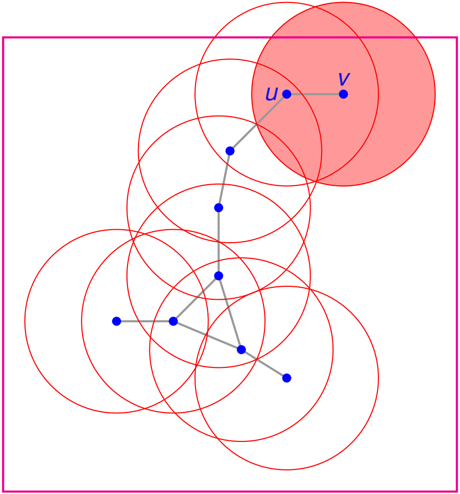


Geometric graphs



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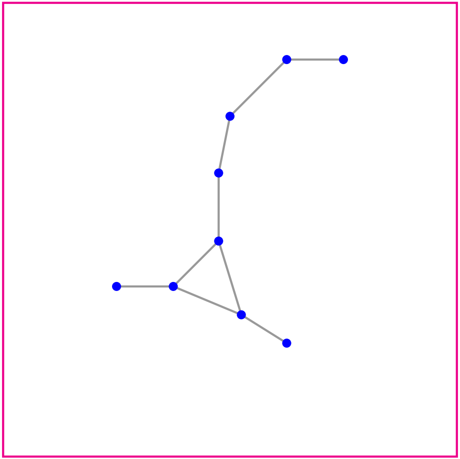
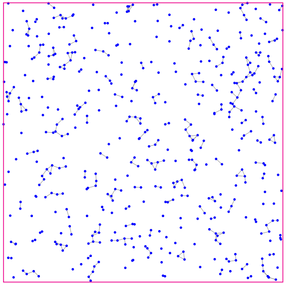
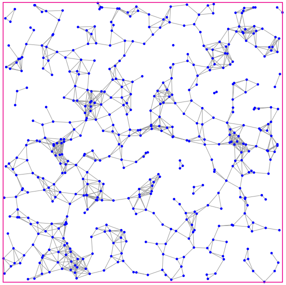


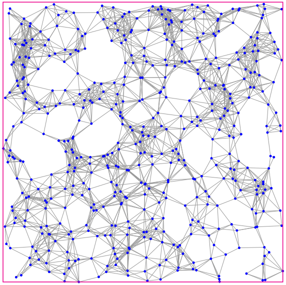
Examples of random geometric graphs



$R = 0.03$



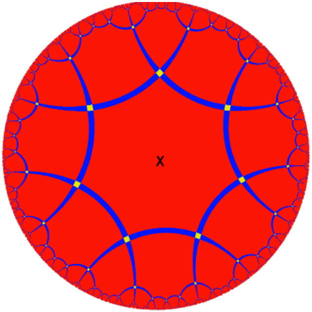
$R = 0.06$



$R = 0.09$

$n = 500$ points

Poincaré disk model of $\mathbb{H}^2$



[Rendered with KaleidoTile by J. Weeks]

- $\mathbb{H}^2$ is represented as an open disk D.
- Blue curves are geodesics (arcs of circles perpendicularly incident to D).
- Each heptagon has the same area.
- Points in ∂D are at infinite distance from X.
- Points at (Euclidean) distance y from X are at hyperbolic distance r from X where

$$r = \ln \frac{1+y}{1-y}.$$

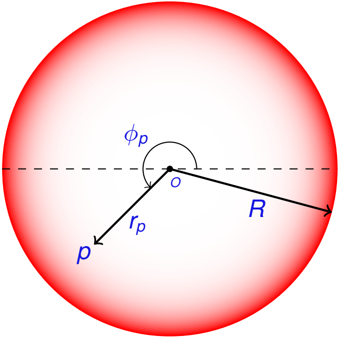
Space expands at exponential rate!

Continuous analogue of regular trees.

Good for making cool pictures!

[Rendered with M. Christersson hyperbolic tiling applet]

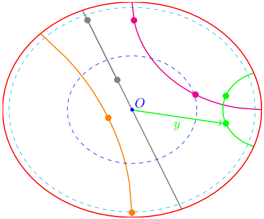
Native representation of $\mathbb{H}^2$



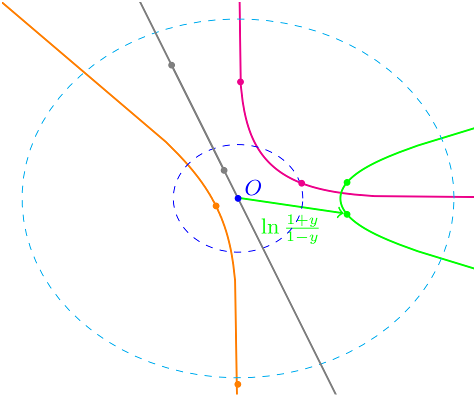
$B_O(R)$: Ball of radius R centered at origin O with perimeter $2\pi \sinh R = \Theta(e^R)$.

- $\mathbb{H}^2$ is represented as $\mathbb{R}^2$.
- A point p is represented in polar coordinates.
- r_p is the hyperbolic distance between p and O

Poincaré vs Native representation of $\mathbb{H}^2$



Poincare model



Native representation.

Hiperbolicland can be dangerous!

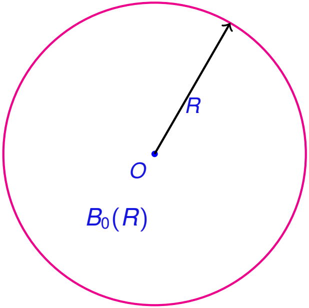
"Just because you keep getting lost on the way to work is no proof that the Universe is hiperbolic!"

Formal definition of RHG model: $G_{\alpha,\nu}(n)$

(Gugelmann, Panagiotou, Peter [ICALP'12])

Model parameters:
$\alpha, \nu \in \mathbb{R}_+, n \in \mathbb{N}_+$.

Set $R := 2 \ln \frac{n}{\nu}$.



Choose an n-node graph $G = (V, E)$ as follows:

- Each $v \in V$ uniformly and independently in $B_O(R)$.
- $uv \in E$ iff $u \in B_v(R)$.

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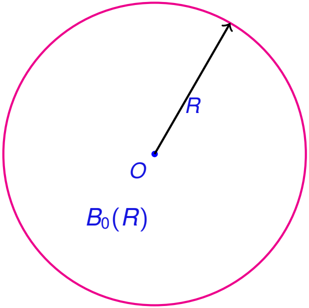


Choose an n-node graph $G = (V, E)$ as follows:

- Each $v \in V$ so $\phi_v \sim \mathrm{Unif}[0, 2\pi)$ independent of r_v with density:

$$f(r) := \frac{\alpha}{C_{\alpha,R}} \sinh(\alpha r) \approx \alpha e^{-\alpha(R-r)} \qquad \text{if } 0 \leq r < R \text{ and } 0 \text{ otherwise.}$$

  (Here, $C_{\alpha,R}$ is a normalizing constant).

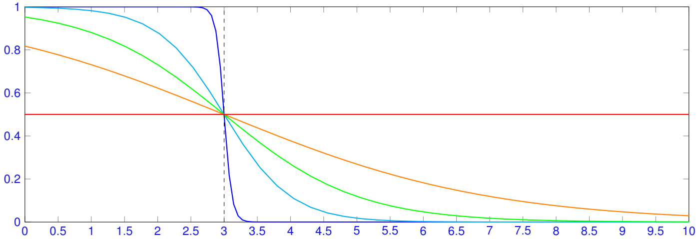
- $uv \in E$ iff $u \in B_v(R)$.

Soft version

Incorporates a temperature T and a *probability of connecting* u and v:

$$p(d) := \frac{1}{1 + e^{\frac{1}{2T}(d-R)}}$$

where $d := d_{\mathbb{H}^2}(u, v)$ is the (hyperbolic) distance between $u, v \in \mathbb{H}^2$.

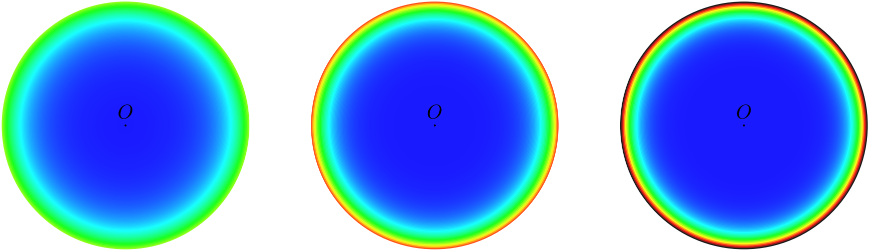


$$+\infty \approx T > T > T > T > T \approx 0$$

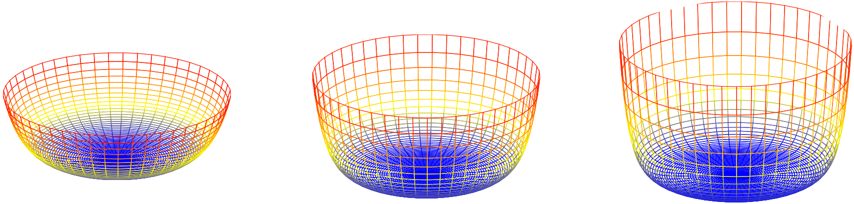
$$R = 3.0.$$

Pdf of (r_v, ϕ_v) and its heat plot

(Colder colors correspond to smaller density)



$\alpha = \frac{1}{2}$ $\alpha = \frac{3}{4}$ $\alpha = 1$



Calculating distances

Hyperbolic distance from v to origin O, ... easy! Just r_v.

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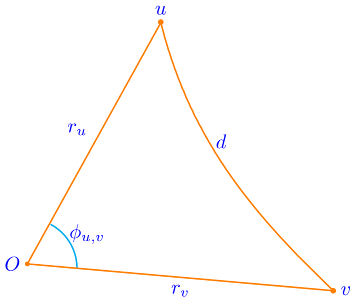
Hyperbolic distance from v to origin O, ... easy! Just r_v.

In general, use hyperbolic law of cosines

$$\cosh(d) = \cosh(r_u)\cosh(r_v) - \sinh(r_u)\sinh(r_v)\cos(\phi_{u,v}).$$



Calculating distances

Hyperbolic distance from v to origin O, ... easy! Just r_v.

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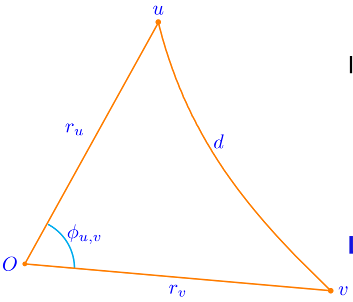


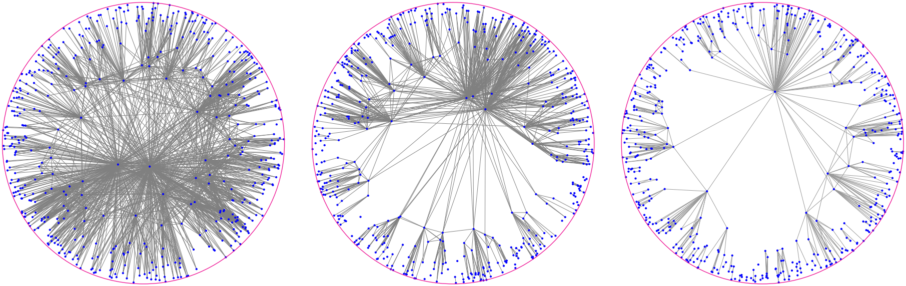
If $d = R$ and $r_u + r_v > R$, then[GPP'12]

$$\theta_R(r_u, r_v) := 2e^{\frac{1}{2}(R-r_u-r_v)}(1 + \Theta(e^{R-r_u-r_v})) = \Theta(e^{\frac{1}{2}(R-r_u-r_v)}).$$

Lemma: $\phi_{u,v} \leq \theta_R(r_u, r_v) \iff d_{\mathbb{H}^2}(u, v) \leq R$.

Examples of RHGs

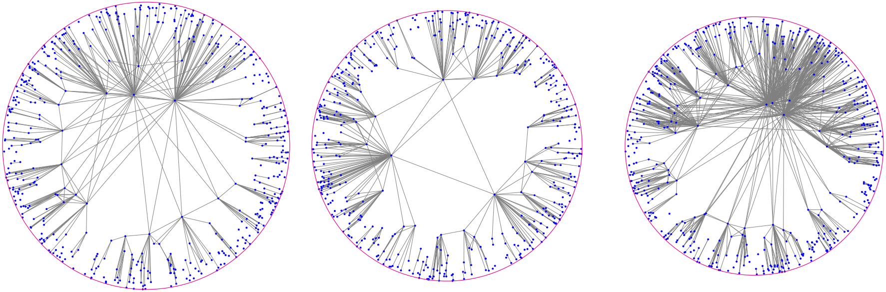
($\nu = 1$ fixed, $n = 500$)



$\alpha = 0.60$ $\alpha = 0.75$ $\alpha = 0.90$

Examples of RHGs

($\alpha = \frac{3}{4}$ fixed, $n = 500$)



$\nu = 0.50$ $\nu = 0.75$ $\nu = 1.00$

Nice, but *who cares?*

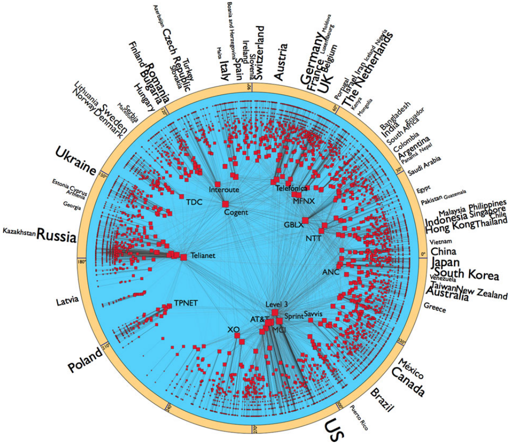
First model that “naturally” exhibits:

- Scale freeness, AND
- Non-negligible clustering.

But, what really drew attention ...

Mapping of Internet's Autonomous Systems (ASs)

(2009 data collected by infrastructure developed by CAIDA)



[From Boguña, Papadopoulus, Krioukov (Nat. Comm. '10)]

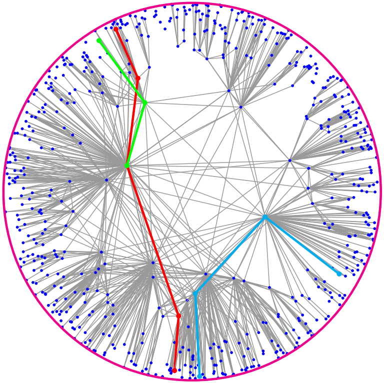
Data set:

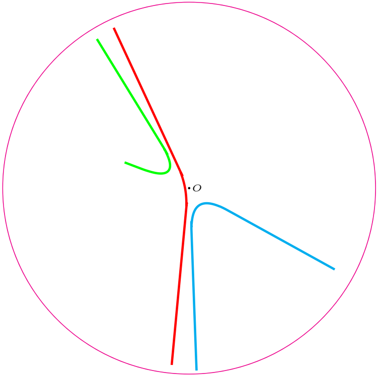
- $23,752$ ASs
- $58,416$ links
- Average degree 4.92

"Maximum Likelihood" fit:

- $\alpha = 0.55$
- $R = 27$
- Temperature $T = 0.69$

Greedy Forwarding





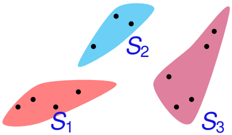
Papadopoulos et al. [INFOCOM 2010], in a experimental study (but without “real” data) report excellent stretch (average ~ 1, max ~ 1.4) and success ratio (0.99920 for $\alpha \sim \frac{1}{2}$ to 0.92 for $\alpha \sim 1$, with α, ν as in the Internet).

Part II: Analysis of model

Poissonized model of RHGs: $\mathcal{G}_{\alpha,\nu}(n)$

It is more natural to consider a Poissonized version of $G_{\alpha,\nu}(n)$.

I.e., a process where given



$\cdots$ it holds that

- $\mathbb{E}|V \cap S|$ is proportional to $n\mu(S)$ where $\mu(S) := \iint_S f(r,\phi)drd\phi$.
- $|V \cap S_1|$, $|V \cap S_2|$ ... are independent.

Equivalently, $\forall S \subseteq \mathbb{H}^2$, $|S \cap V| \sim \mathrm{Poisson}(n\mu(S))$, i.e, $\forall k \in \mathbb{N}$.

$$\mathbb{P}(|S \cap V| = k) = e^{-n\mu(S)} \frac{1}{k!}(n\mu(S))^k.$$

Key fact

If $\mu(S)$

is "large" → $|S \cap V| \overset{\text{w.e.p.}}{\cong} n\mu(S)$.

is "small" → $|S \cap V|$ is small, w.e.p.

Key fact

If $\mu(S)$

- is $\omega(\frac{\ln n}{n})$ $\rightarrow$ $|S \cap V| \overset{\text{w.e.p.}}{\cong} n\mu(S)$.
- otherwise $\rightarrow$ $|S \cap V| \overset{\text{w.e.p.}}{=} (\ln n)^{1+o(1)}$.

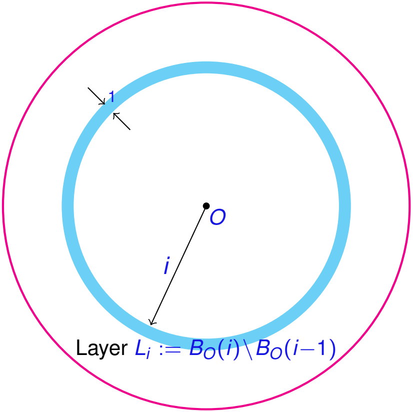
Moreover, it is possible to depoisonize

Henceforth $\frac{1}{2} < \alpha < 1$.

Do Not Forget!

Vertices per layer

(measure centered balls)



Calculations yield[GPP'12]

$$\mu(L_i) \cong \frac{\mu(B_O(i))}{1-e^{-\alpha}}.$$

$$\mu(B_O(i)) \cong e^{-\alpha(R-i)}.$$

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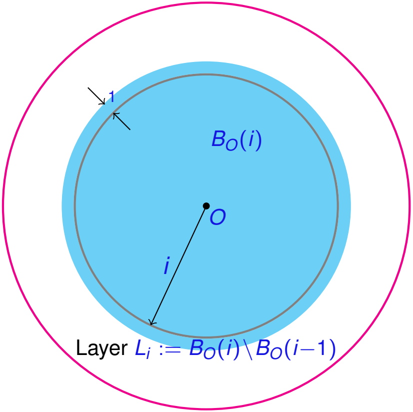

Layer $L_i := B_O(i) \setminus B_O(i-1)$

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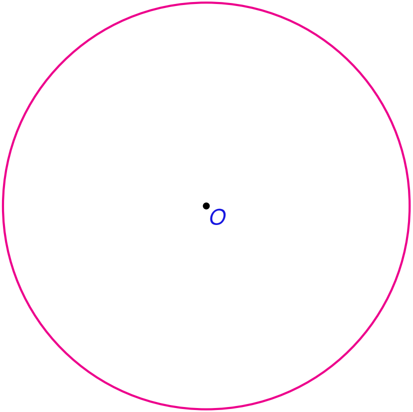

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Define, $V_{\leq i} := V \cap B_O(i)$.

Let $i_0 := (1 - \frac{1}{2\alpha})R$. so $\mu(B_O(i_0)) = \frac{1}{n}$.

Vertices per layer

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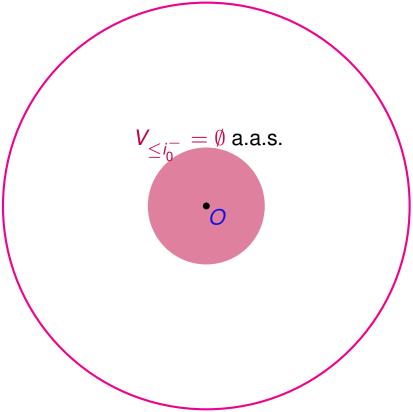

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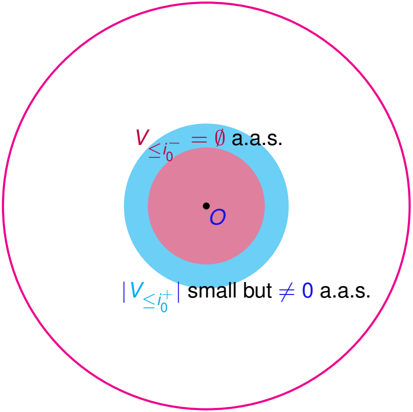
If $i_0^- = i_0 - \frac{\ln R}{\alpha} - \omega(1)$, then $\mathbb{E}|V_{\leq i_0^-}| = n\mu(B_O(i_0^-)) = o(1)$.

Vertices per layer

(measure centered balls)



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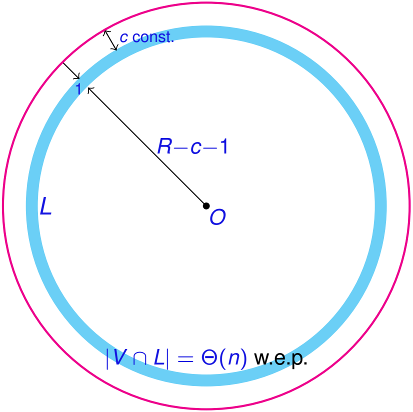
If $i_0^+ = i_0 + \frac{\ln R}{\alpha} - \omega(1)$, then $\mathbb{P}(|V_{\leq i_0^+}| > \ln n) \leq \frac{1}{\ln n}\mathbb{E}|V_{\leq i_0^+}| = o(1)$.

Vertices per layer

(measure centered balls)



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Vertex degrees

(measure of non-centered balls)

Calculations yield

$$\mu(B_P(R)) = C_\alpha e^{-\frac{r_P}{2}}(1 + o(e^{-(\alpha-\frac{1}{2})r_P}).$$

Vertex degrees

(measure of non-centered balls)

Calculations yield

$$\mu(B_P(R)) = C_\alpha e^{-\frac{r_P}{2}}(1 + o(e^{-(\alpha-\frac{1}{2})r_P})).$$

Thus,

$$\deg(P) = \begin{cases} O(\ln n) \text{ (no concentration)}, \\ \qquad \text{if } r_P = R - 2\ln R + O(1), \\ \\ \Theta(ne^{-\frac{r_P}{2}}) \text{ w.e.p.}, \\ \qquad \text{otherwise}. \end{cases}$$

Consequences

- A.a.s, a max degree vertex is in $V_{i_0^+}$ and has degree $n^{1-\frac{1}{2\alpha}+o(1)}$ w.e.p.
- If $k = C_\alpha n e^{-\frac{j}{2}}$, $j \geq i_0^+$, then w.e.p. the number of degree $\leq k$ nodes is

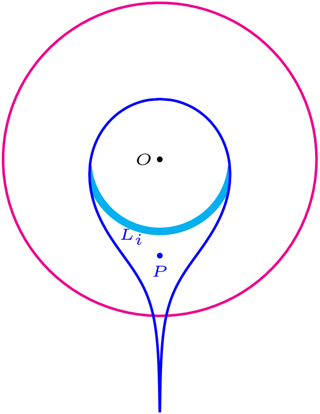
$$\cong n e^{-\alpha(R-j)} = n\left(\frac{\nu C_\alpha}{k}\right)^{-2\alpha}.$$

  I.e., power law degree distribution with exponent $2\alpha + 1$.
- The average degree is $\pi\nu C_\alpha^2(1+o(1))$, i.e., constant!.
- If $v \notin V_{\leq R-c}$, c constant,

$$\mathbb{P}(\deg(v) = 0) \cong C_\alpha e^{-c/2}$$

  and w.e.p. there are $\Theta(n)$ such vertices.
- $V_{\leq R/2}$ induces a clique K (w.e.p. $|V_{\leq R/2}| = \Theta(n^{1-\alpha})$)

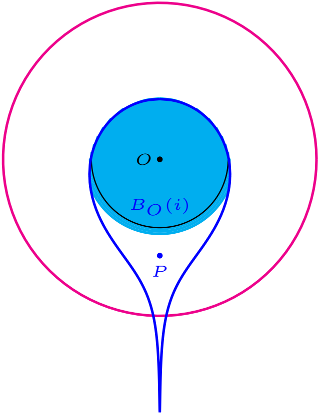
Location of neighbors of a vertex



Calculations yield

$$\mu(B_P(R) \cap L_i) = \Theta(e^{-(\alpha-\frac{1}{2})(R-i)} e^{-\frac{1}{2}(R-r_P)})$$

Location of neighbors of a vertex



Calculations yield

$$\mu(B_P(R) \cap L_i) = \Theta(e^{-(\alpha-\frac{1}{2})(R-i)} e^{-\frac{1}{2}(R-r_P)})$$
$$= (1 - e^{-(\alpha-\frac{1}{2})})(1 + o(1))\mu(B_P(R) \cap B_O(i-1)).$$

As a function of i grows like $e^{-\alpha i}$.

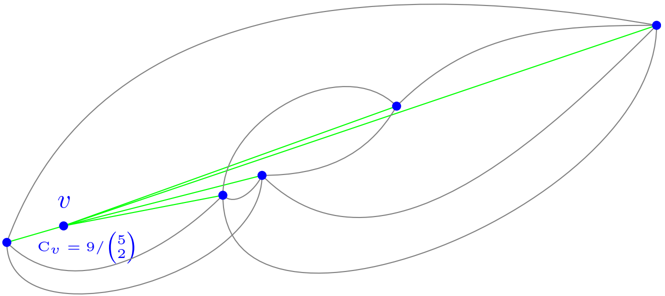
So, P has:

- more neighbors towards $\partial B_O(R)$
- const. fraction of neighbors "near" $\partial B_O(R)$

Visualization of claims

Non-negligible local clustering coefficient

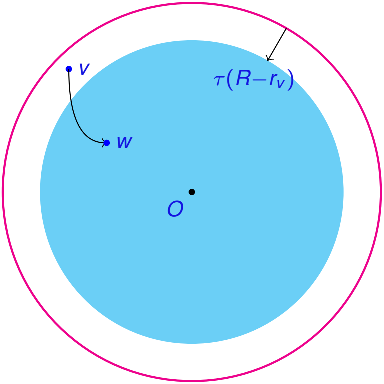
[GPP'12]



If $C_v := \mathbb{P}_{s,t}(st \in E | s, t \in \mathcal{N}_v)$, then $\mathbb{E}_v C_v = \Omega(1)$.

Giant component

[BFM, EJC'15; FM, AAP'17]



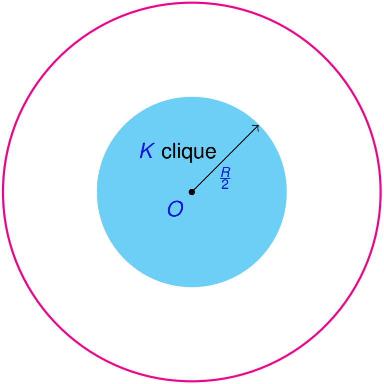
Let $v \in V$ be s.t. $R - r_v = \Omega(\ln R)$.

There is a $\tau > 1$ so that w.e.p. $\exists w \sim v$ s.t.

$$R - r_w > \tau(R - r_v).$$

Giant component

[BFM, EJC'15; FM, AAP'17]



$|K| \overset{\text{wep}}{=} \Theta(n^{1-\alpha})$

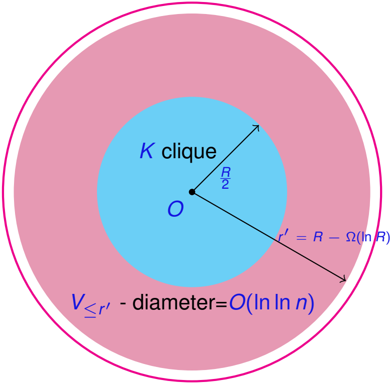
Let $v \in V$ be s.t. $R - r_v = \Omega(\ln R)$.

There is a $\tau > 1$ so that w.e.p. $\exists w \sim v$ s.t.

$$R - r_w > \tau(R - r_v).$$

Giant component

[BFM, EJC'15; FM, AAP'17]



$|K| \overset{\text{wep}}{=} \Theta(n^{1-\alpha})$

$|V_{\leq r'}| \overset{\text{wep}}{=} \Theta(\frac{n}{\text{polylog(n)}})$

Let $v \in V$ be s.t. $R - r_v = \Omega(\ln R)$.

There is a $\tau > 1$ so that w.e.p. $\exists w \sim v$ s.t.

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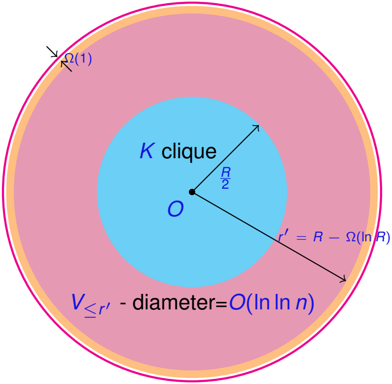

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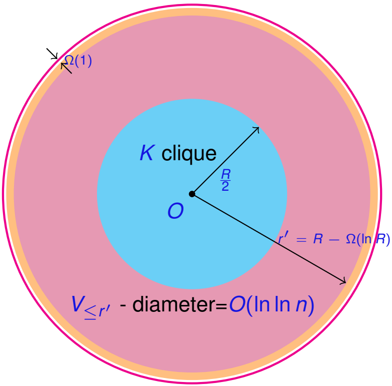

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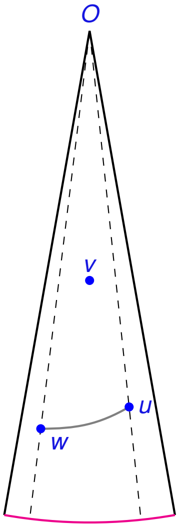
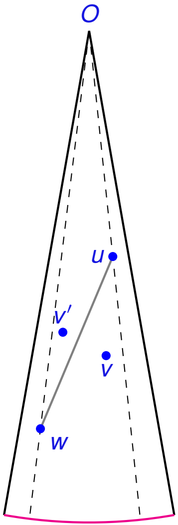
$|\text{2nd component}| \overset{\text{wep}}{=} \Theta(\text{polylog}(n))$

[KM, ANALCO'15]

Forbidden configurations

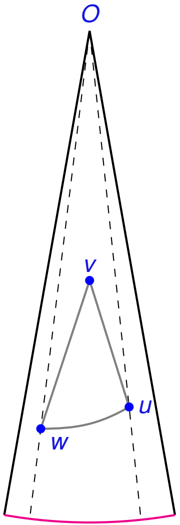
[FK, ICALP'15]

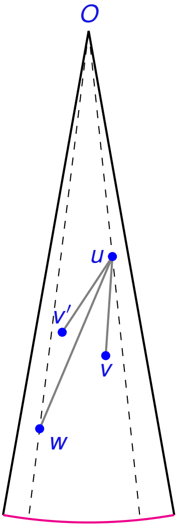




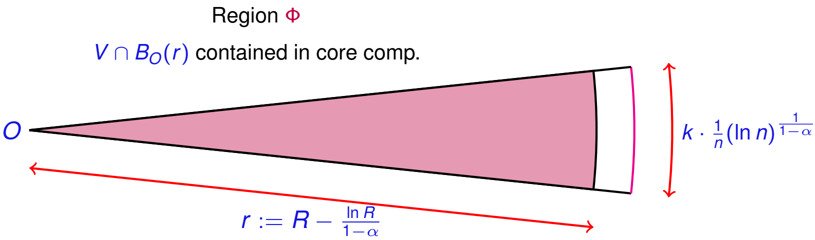
Forbidden configurations

[FK, ICALP'15]





Some observations



If $k = c \cdot (\ln n)^{\frac{1}{1-\alpha}}$ and c large enough, then

$$\mathbb{P}(\Phi \cap B_O(R) \cap V = \emptyset) = e^{-\Theta(1)k(\ln n)^{-\frac{\alpha}{1-\alpha}}} = O(n^{-3}).$$

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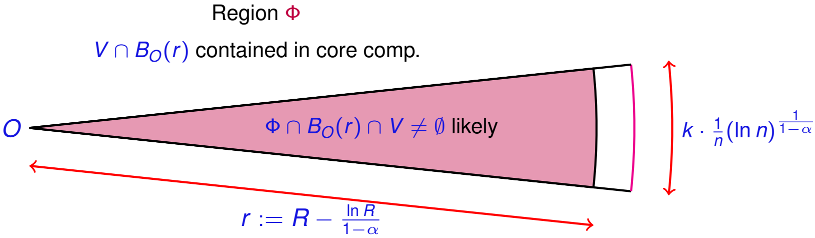

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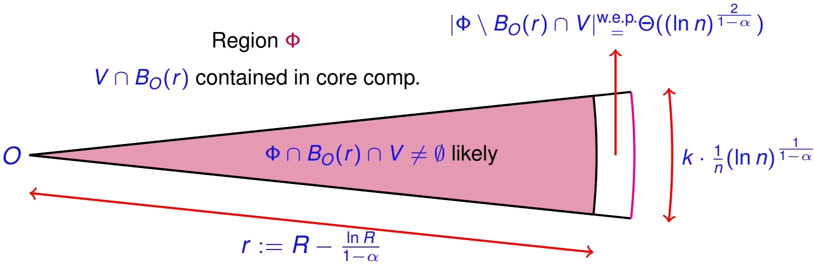

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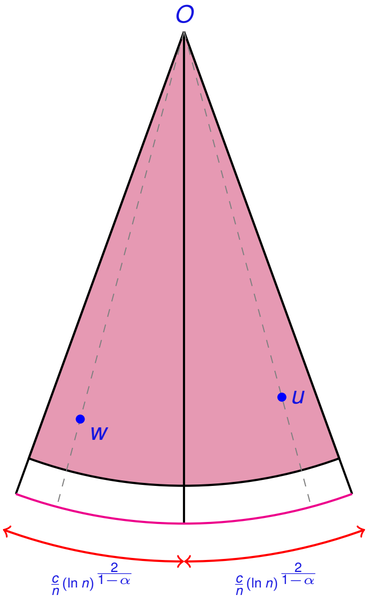
An $O((\ln n)^{\frac{2}{1-\alpha}})$ bound on the diameter and $2nd$ component

[KM, ANALCO'15; FK, ICALP'15; MS, arXiv17]

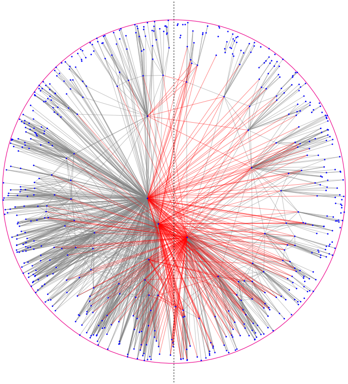


Conductance and spectral gap

The graph conductance of the core component H of $G_{\alpha,\nu}(n)$ is:

$$\varphi(H) := \min_{\substack{S \subseteq V(H) \\ 0 < \mathrm{vol}(S) \leq |E(H)|}} \frac{E_H(S, V(H) \setminus S)}{\mathrm{vol}(S)}.$$

The spectral gap of H is $\lambda_1(H)$ – the 2nd smallest eigenvalue of the normalized Laplacian of H



By Cheeger's inequality:

$$\frac{1}{2}\varphi^2(H) \leq \lambda_1(H) \leq 2\varphi(H).$$

Upper bound is almost tight[KM, AAP'17] and

$$\overset{\text{wep}}{\approx} \Theta\left(\frac{1}{n^{2\alpha-1}}\right)$$

Fairly small!

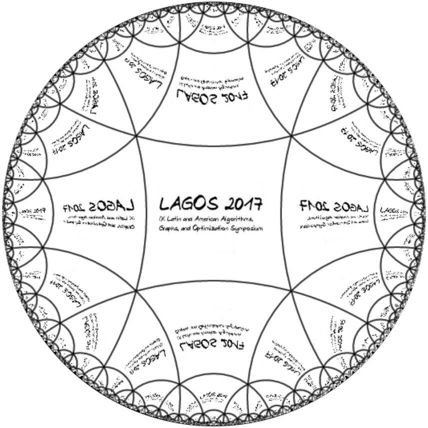
Other ...

- Bipartite[KPK, Phys. Rev. E'17] and higher dimensional analogues, as well as generalizations[BKL, ESA'17] have also been considered.
- Average distance[BKL, arXiv'16]
- Separators and treewidth[BFK, ESA'16]: Balanced separator hierarchies with separators of size $O(n^{1-\alpha})$ and $O(n^{1-\alpha})$ treewidth, a.a.s.
- Minimum and maximum bisection [KM, AAP'17].
- Fast generation[BKL, ESA'17; vLSMP, ISAAC'15] and embedding[BFKL, ESA'16].
- Connectivity threshold[BFM, RS&A'16]
- Bootstrap percolation[CF, SP&A'16; KL, ICALP'16; etc.] in RHGs and GIRGs.
- Greedy routing[BKLMM, arXiv'17].

What next?

(some of my favorite questions)

- Is there a compelling model that explains the emergence of “RHG like” networks and how they evolve?
- How do epidemics/information spread through RHGs?
- When $n \to \infty$, are the graph metric of RHGs and $\mathbb{H}^2$ related? If so, how?



[Rendered with M. Christersson hyperbolic tiling applet]

The End!