Correspondence Homomorphisms

Pavol Hell
and
Tomás Feder

LAGOS 2017

Classical Colouring

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Graph G with n vertices

G is k-colourable $\iff$ $G \,\square\, K_k$ has n independent vertices

Plesnevič + Vizing 1965

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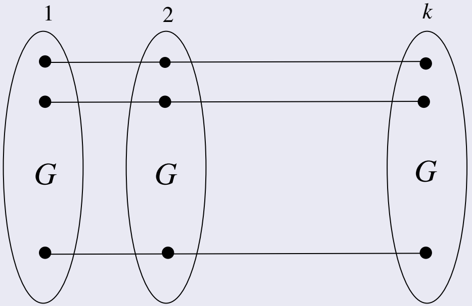
Cartesian product



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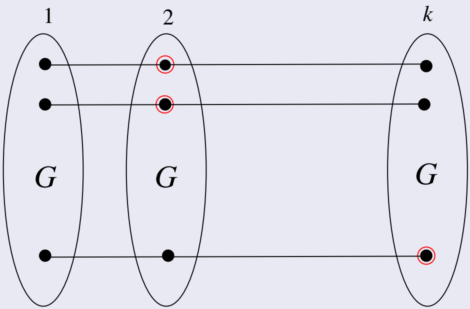

Graph G with n vertices

G is k-colourable $\iff$ $G \square K_k$ has an independent transversal

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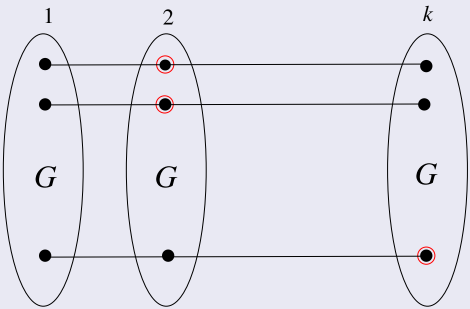
Cartesian product



Classical Colouring

C_4 is 2-colourable

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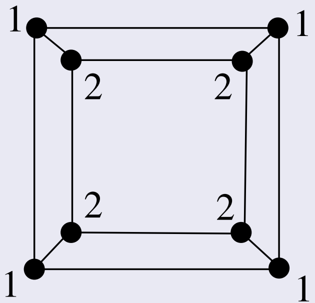

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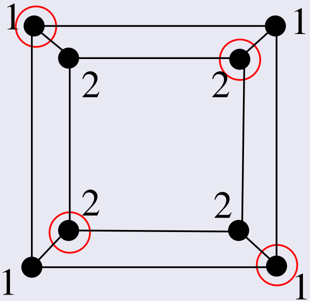

List Colouring

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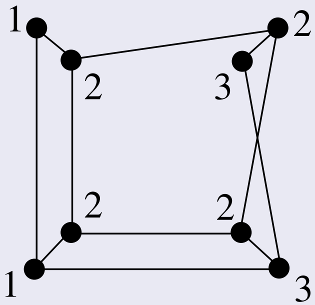
C_4 is 2-choosable



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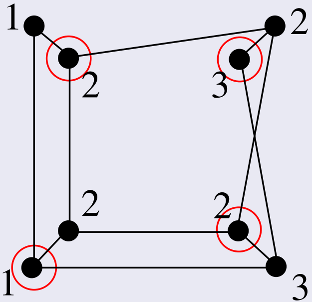
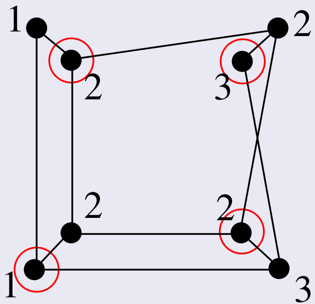


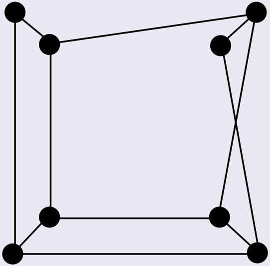
List Colouring

C_4 is 2-list-colourable



Correspondence Colouring

Forget the labels



Correspondence Colouring

Products $G \square^* K_k$

Put a partial matching between every two adjacent copies of K_k

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Correspondence Colouring

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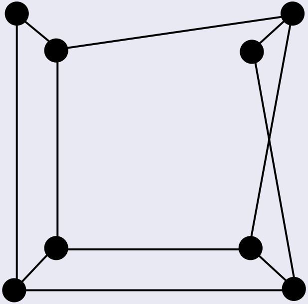
Correspondence Colouring

Example $C_4 \square^* K_2$



Correspondence Colouring

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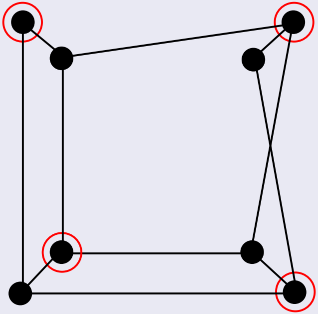

Correspondence Colouring

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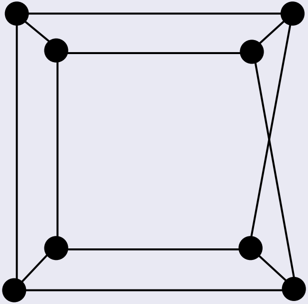
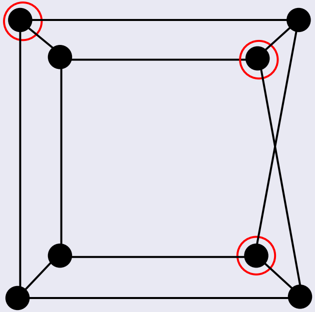


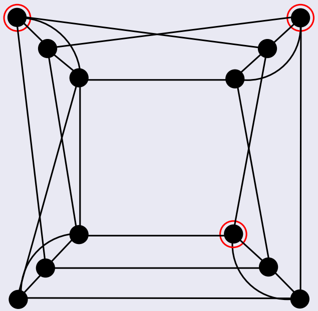
Correspondence Colouring

C_4 is not 2-correspondence-colourable



Correspondence Colouring

C_4 is 3-correspondence-colourable



Correspondence Colouring

Correspondence chromatic number of G

Smallest k such that G is k-correspondence-colourable

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$$\chi(G) \leq \chi_{list}(G) \leq \chi_{corr}(G)$$

Correspondence Colouring

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Smallest k such that G is k-correspondence-colourable

Correspondence chromatic number of G

$$\chi(G) \leq \chi_{list}(G) \leq \chi_{corr}(G)$$

Example results

- $\chi_{corr}(G) \leq k + 1$ for k-degenerate graphs
- $\chi_{corr}(G) \leq 3$ for triangle-free planar graphs
- $\chi_{corr}(G) \leq 5$ for planar graphs

Dvořák + Postle 2016

Fleischner + Steibitz 1992

Any cycle+triangles graph is 3-colourable

(Asked by Erdős 1990)

Useful Generalizations of Colouring

Fleischner + Steibitz 1992

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Every planar graph without 4-, 5-, 6-, 7-, and 8-cycles is 3-choosable.

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Dvořák Postle 2016

Every planar graph without 4-, 5-, 6-, 7-, and 8-cycles is 3-choosable.

(Asked by Borodin 2013)
(Actually prove 3-correspondence colourability for certain cases that include list colouring)

Change of Notation

Always consider perfect matchings

Get partial matching by restricting via lists

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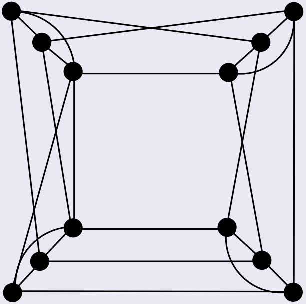
Example



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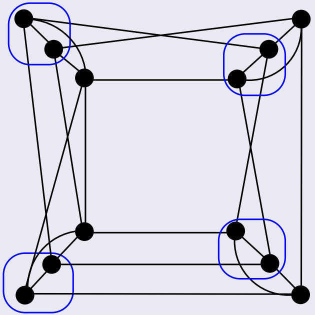
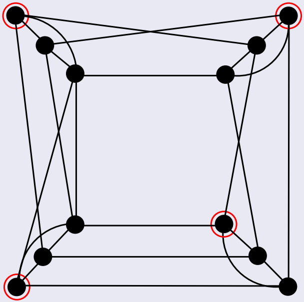


Change of Notation

G is correspondence k-colourable

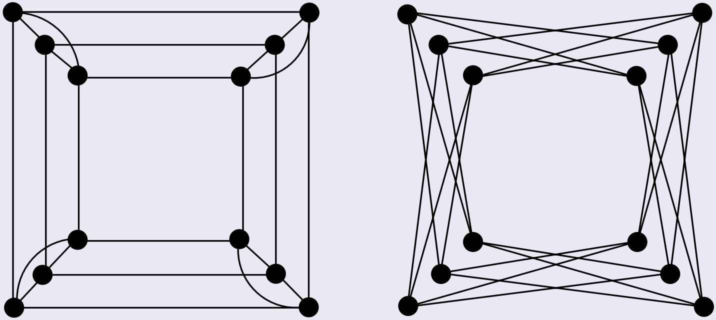
Each product $G \square^* K_k$ has an independent transversal

C_4 is 3-correspondence-colourable



Change of Notation

$G \square K_k$ versus $G \times K_k$



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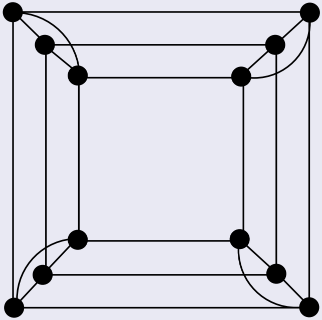

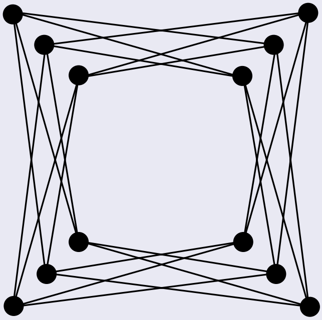

$K_2 \times K_k$ is the complement of a matching

Change of Notation

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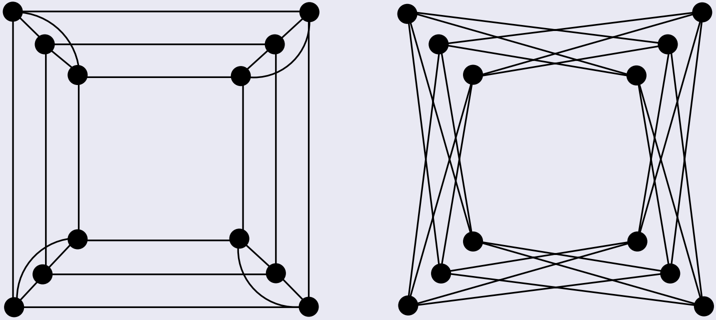


$K_2 \times K_k$ is the complement of a matching

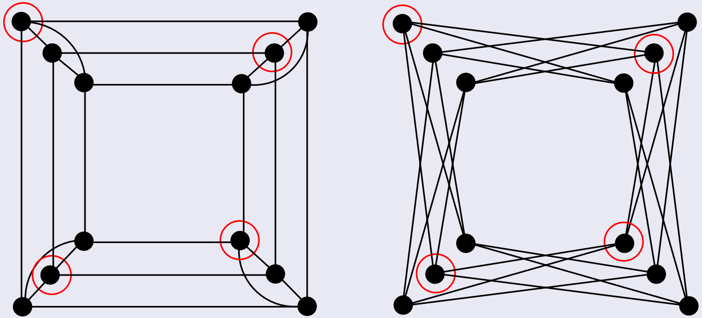
Switch to complements of matchings

Change of Notation

$G \,\square\, K_k$ versus $G \times K_k$

Independent transversal in $G \,\square\, K_k$ = G-transversal in $G \times K_k$

Example

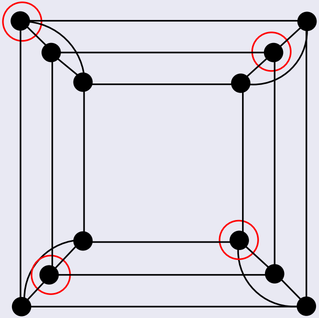


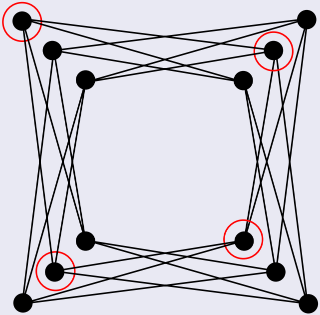
Change of Notation

$G \square K_k$ versus $G \times K_k$

G is k-colourable $\iff$ $G \times K_k$ has a G-transversal

Example



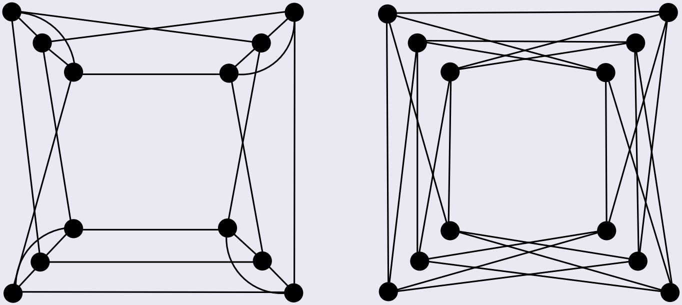


Change of Notation

Correspondence k-colouring

Consider products $G \times^* K_k$ with complements of matchings

Example



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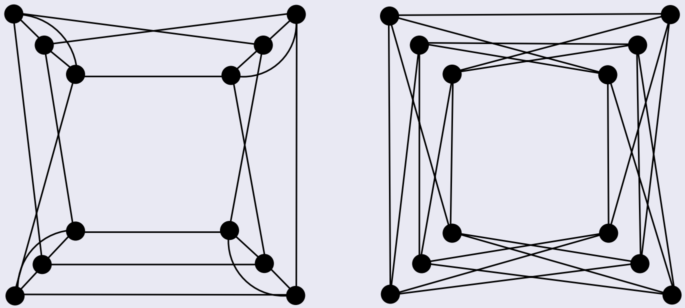

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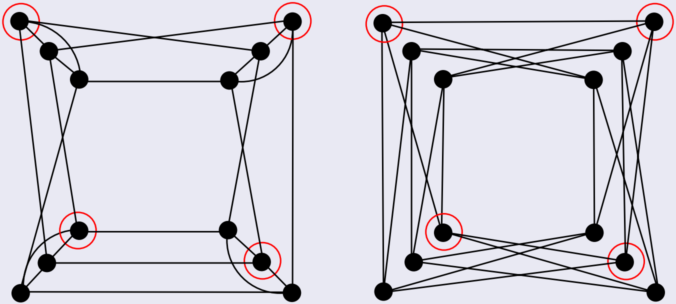

Homomorphism

H-colouring of G

A G-transversal of $G \times H$

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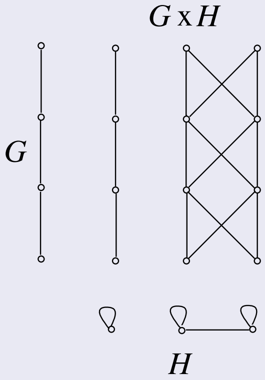

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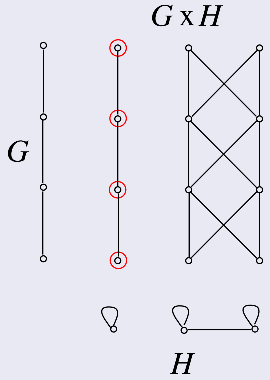

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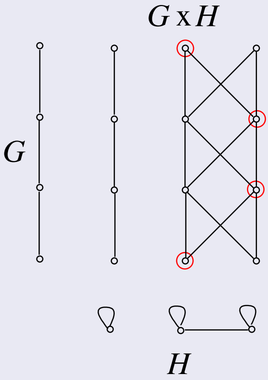

Correspondence Homomorphism

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Products $G \times^* H$

Put an isomorphic copy of $K_2 \times H$ between every two adjacent copies of H

Correspondence Homomorphism

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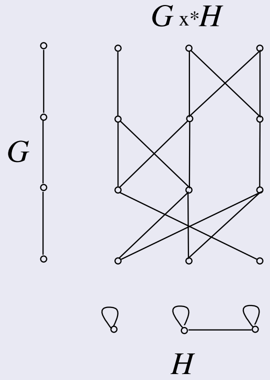

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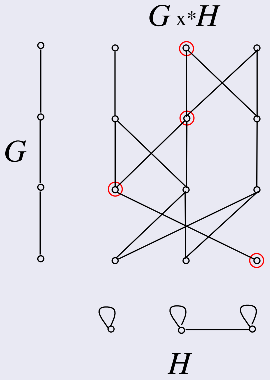

The Correspondence Homomorphism Problem

The correspondence H-colouring problem

Given some $G \times^* H$, is there a G-transversal?

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The H-colouring problem

Given G, is there an H-colouring of G?

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(I.e., given G, does $G \times H$ have a G-transversal?)

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The H-colouring problem

Given $G \times H$, is there a G-transversal? (I.e., is there a homomorphism $G \to H$?)

The k-colouring problem

Given $G \times K_k$, is there a G-transversal? (I.e., is there a k-colouring of G?)

The k-colouring problem

Polynomial-time solvable when $k \leq 2$, NP-complete otherwise

The Correspondence Homomorphism Problem

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The H-colouring problem

Given $G \times H$, is there a G-transversal? (I.e., is there a homomorphism $G \to H$?)

The H-colouring problem

Polynomial-time solvable when H has a loop or is is bipartite, NP-complete otherwise

H+Nesetril 1990

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Given some $G \times^* H$, is there a G-transversal?

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Is there a dichotomy classification?

The Correspondence Homomorphism Problem

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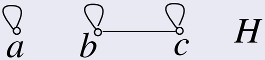
Given some $G \times^* H$, is there a G-transversal?

NP-complete when H is reflexive $K_1 + K_2$



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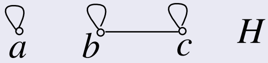
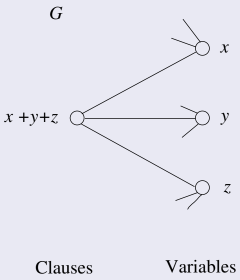


Reduce from 1-in-3-SAT without negated variables



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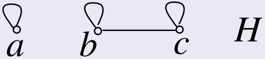

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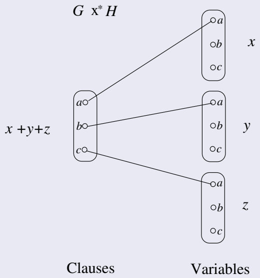

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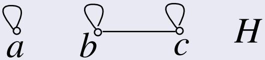

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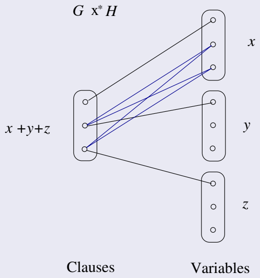

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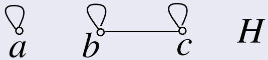

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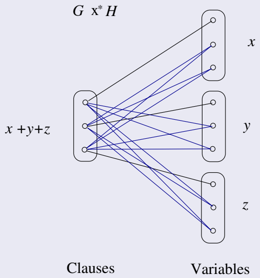

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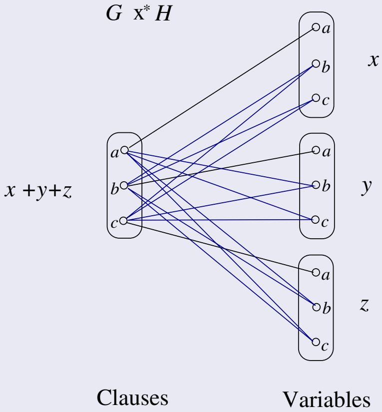

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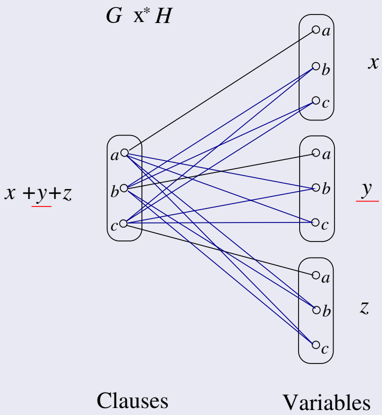

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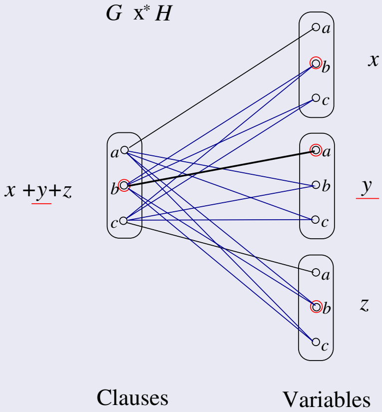

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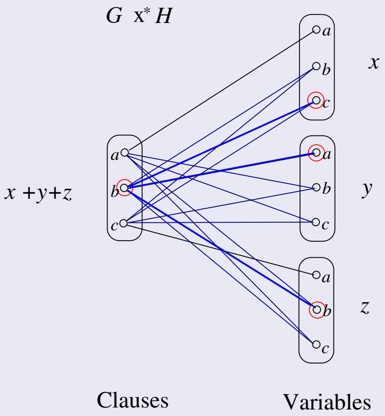

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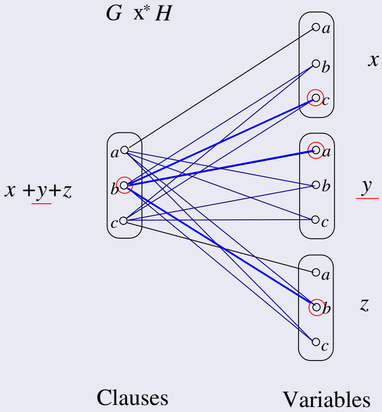
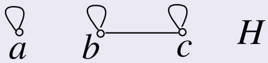


The Correspondence Homomorphism Problem

NP-complete when H is reflexive $K_1 + K_2$



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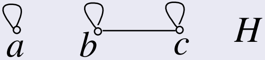
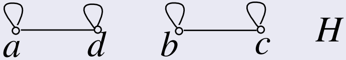


Polynomial-time solvable when H is reflexive $K_2 + K_2$



Dichotomy classification

Dichotomy classification

The correspondence H-colouring problem is NP-complete except in the following polynomial-time solvable cases:

1. H is a reflexive $K_2 + K_2$
2. H is a reflexive K_k or a reflexive kK_1
3. H is an irreflexive $K_{2,2}$
4. H is an irreflexive $pK_2 + qK_1$
5. H is an irreflexive pK_2 plus a reflexive qK_1
6. H is a star with only one loop at the center

The Correspondence List Homomorphism Problem

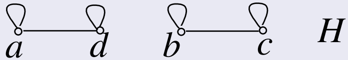
Dichotomy classification

The correspondence **list** H-colouring problem is NP-complete except in the following polynomial-time solvable cases:

1. H is a reflexive K_k or a reflexive kK_1
2. H is an irreflexive $pK_2 + qK_1$
3. H is an irreflexive pK_2 plus a reflexive qK_1
4. H is a star with only one loop at the center

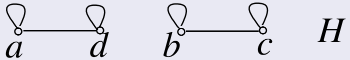
The Correspondence Homomorphism Problem

Polynomial-time solvable when H is reflexive $K_2 + K_2$

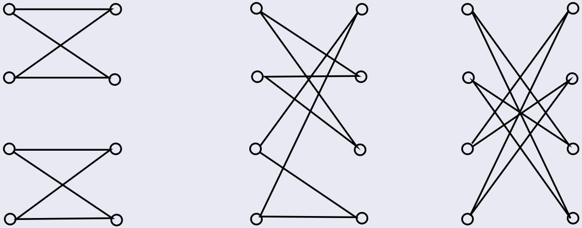


The Correspondence Homomorphism Problem

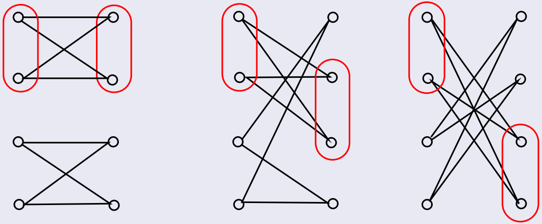
Polynomial-time solvable when H is reflexive $K_2 + K_2$



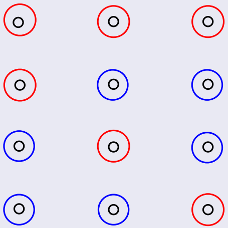
Possible $K_2 \times^* (K_2 + K_2)$



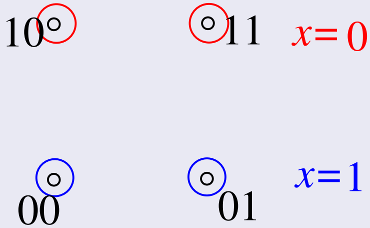
Possible $K_2 \times^* (K_2 + K_2)$



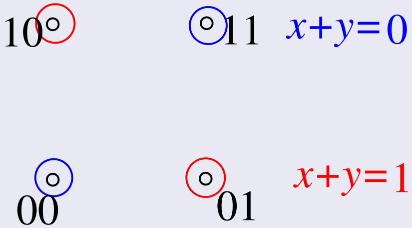
Possible partitions of $K_2 + K_2$



Possible partitions of $K_2 + K_2$

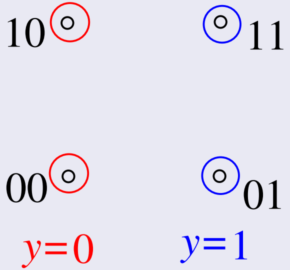


Possible partitions of $K_2 + K_2$



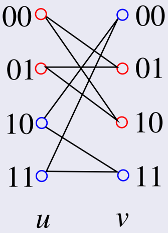
The Correspondence Homomorphism Problem

Possible partitions of $K_2 + K_2$



The Correspondence Homomorphism Problem

Example $K_2 \times^* (K_2 + K_2)$



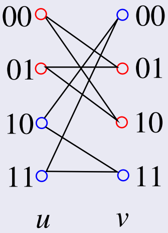
$x_u = 0$ $\quad$ $x_v + y_v = 0$

$x_u = 1$ $\quad$ $x_v + y_v = 1$

$x_u + x_v + y_v = 1$

The Correspondence Homomorphism Problem

Example $K_2 \times^* (K_2 + K_2)$



$x_u = 0$ $\qquad$ $x_v + y_v = 0$

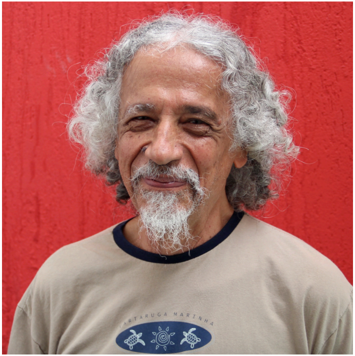
$x_u = 1$ $\qquad$ $x_v + y_v = 1$

$x_u + x_v + y_v = 1$

Solved by Gaussian elimination

G-transversal $\iff$ satisfies all such equations

Happy Birthday!



Happy Birthday!