Minimum density of identifying codes of king grids

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Joint work with

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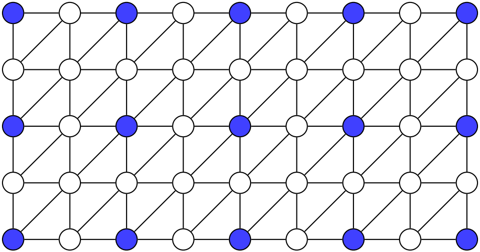
Definitions

neighbourhood of v : $N(v) = \{u \mid uv \in E(G)\}$.
closed neighbourhood of v : $N[v] = N(v) \cup \{v\}$.

$C \subseteq V(G)$
identifier of v : $I(v) = N[v] \cap C$.

C is an identifying code if

- $I(v) \neq \emptyset$ for all $v \in V(G)$;
- $I(v) \neq I(u)$ for any $v \neq u$.



Existence theorem

u and v are twins if $N[u] = N[v]$.

For all C, two twins have the same identifier.

Theorem: G admits an identifying code iff G has no twins.

Proof: If two twins, no identifying code.
If no twins, then $V(G)$ is an identifying code. □

Problem 1: Let G be a finite graph with no twins.
What is the **minimum size** of an identifying code ?

Problem 2: Let G be an infinite graph with no twins.
What is the **minimum density** $d^*(G)$ of an identifying code ?

Formal definition of density

v_0 vertex in G

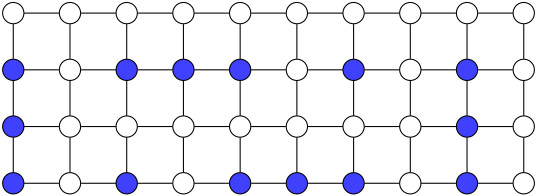
$B_r(v_0)$ ball of radius r in G : $B_r(v_0) = \{x \mid d(v_0, x) \leq r\}$.

density of C in G :

$$d(C, G) = \limsup_{r \to +\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|}$$

Identifying codes in the infinite square grid

Cohen et al. (1999) $d^*(\mathbb{Z}^2) \leq 7/20$.



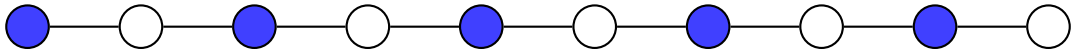
shifted by vectors $(10x, x+4y)$ $x, y \in \mathbb{Z}$.

Benhaim and Litsyn (2005) $d^*(\mathbb{Z}^2) \geq 7/20$.

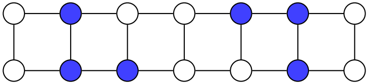
Identifying codes of infinite square strips

$\mathcal{S}_k = \mathbb{Z} \Box [1, k]$ = square grid on k rows.

Daniel et al. (2004) $d^*(\mathcal{S}_1) = 1/2$.

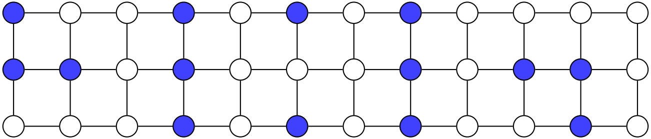


Daniel et al. (2004) $d^*(\mathcal{S}_2) = 3/7$.



Identifying codes in infinite square strips

Bouzníf, H. and Preissman (2016) $d^*(\mathcal{S}_3) = 7/18 = 0.388\ldots.$



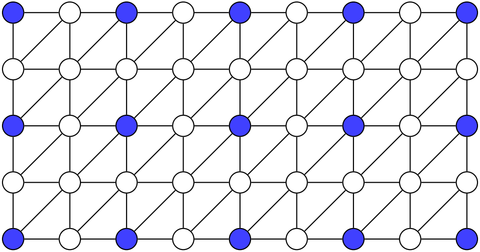
Jiang (2016+) $d^*(\mathcal{S}_4) = 11/28 = 0.392\ldots$
and $d^*(\mathcal{S}_5) = 19/50 = 0.38.$

Bouzníf, H. and Preissman (2016)

$$\frac{7}{20} + \frac{1}{20k} \leq d^*(\mathcal{S}_k) \leq \frac{7}{20} + \frac{3}{10k}.$$

Identifying codes in the triangular grid

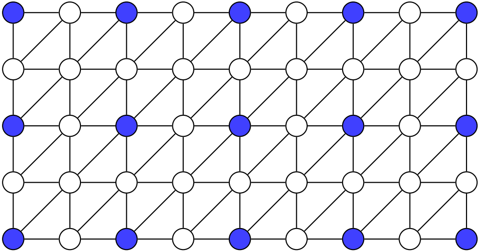
Karpovsky et al. (1998) $d^*(\mathcal{G}_T) = 1/4$.



Identifying codes in infinite triangular strips

Dantas, H., and Sampaio (2017)

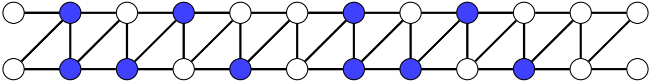
- $d^*(T_k) = \frac{1}{4} + \frac{1}{4k}$,   if k is **odd**.



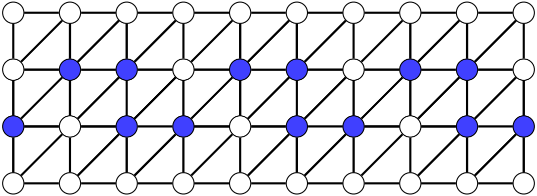
- $\frac{1}{4} + \frac{1}{4k} \leq d^*(T_k) \leq \frac{1}{4} + \frac{1}{2k}$,   if k is **even**.

Identifying codes in infinite triangular strips

Dantas, H., and Sampaio (2017) $d^*(T_2) = 1/2 = \frac{1}{4} + \frac{1}{2\times 2}$.

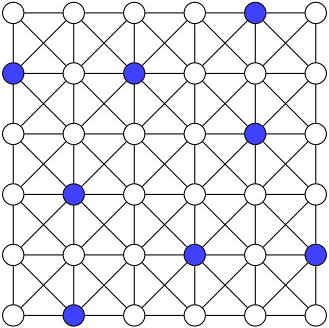


$$\frac{5}{16} = \frac{1}{4} + \frac{1}{4\times 4} < d^*(T_4) = 1/3 < \frac{1}{4} + \frac{1}{2\times 4} = \frac{3}{8}.$$



Identifying codes in the infinite king grid

Charon et al. (2002) $d^*(\mathcal{G}_K) = 2/9$.



Our results

Theorem 1 : for every king grid G, $d^*(G) \geq 2/9$.

Theorem 2 : for every finite king grid G, $d^*(G) > 2/9$.

$\mathcal{K}_k$: king strip of height k.
$d^*(\mathcal{K}_1) = 1/2$; $\mathcal{K}_2$ has no id. code; $d^*(\mathcal{K}_3) = 1/3$;
$d^*(\mathcal{K}_4) = 5/16$; $d^*(\mathcal{K}_5) = 4/15$; $d^*(\mathcal{K}_6) = 5/18$.

Theorem 3 :

$$2/9 + \frac{8}{81k} \leq d^*(\mathcal{K}_k) \leq \begin{cases} \frac{2}{9} + \frac{6}{18k}, & \text{if } k \equiv 0 \mod 3, \\ \frac{2}{9} + \frac{8}{18k}, & \text{if } k \equiv 1 \mod 3, \\ \frac{2}{9} + \frac{7}{18k}, & \text{if } k \equiv 2 \mod 3. \end{cases}$$

Using the Discharging Method

General Idea

- Assume there is an identifying code C of G.
- **Initial charge** : $w(v) = 1$ if $v \in C$; $w(v) = 0$ if $v \in V(G) \setminus C$.
- Apply some **local discharging rules**. (constant weight)
- **Final charge** : $w^*(v) \geq \alpha$ for all $v \in V(G)$.

$$\Longrightarrow d^*(G) \geq \alpha.$$

Theorem 1 : $d^*(G) \geq 2/9$ for every king grid G.

Notations

C code $\qquad U = V(G) \setminus C$.

$X_i = \{v \in X \mid |I(v)| = i\}$,

$X_{\geq i} = \{v \in X \mid |I(v)| \geq i\}$, $\qquad X_{\leq i} = \{v \in X \mid |I(v)| \leq i\}$,

full vertex v : $|N[v]| = 9$ $\qquad$ **side** vertex v : $|N[v]| \leq 6$.

Initial charge : $w(v) = 1$ if $v \in C$; $w(v) = 0$ if $v \in V(G) \setminus C$.

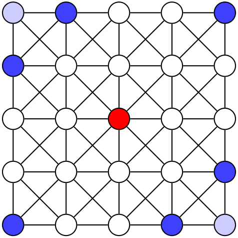
Goal : $w^*(v) \geq 2/9$ for all $v \in V(G)$.

Theorem 1 : $d^*(G) \geq 2/9$ for every king grid G.

(R1) Every C-vertex sends $\frac{2}{9i}$ to each of its neighbours in U_i.

Every U-vertex get charge $2/9$.

BUT some C-vertices might become **defective** (have charge less than $2/9$).



Theorem 1 : $d^*(G) \geq 2/9$ for every king grid G.

(R1) Every C-vertex sends $\frac{2}{9i}$ to each of its neighbours in U_i.

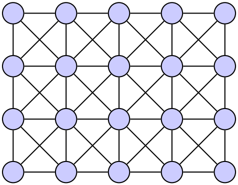
(R2) Every defective vertex receives $\frac{1}{54}$ from each of its partners.

At the end, **the charge of every vertex is at least $2/9$**.

Theorem 2 : $d^*(G) > 2/9$ for every finite king grid G.

If v is a side C-vertex, then $w^*(v) \geq \frac{2}{9} + \frac{11}{54}$.
If v is a full $C_{\geq 3}$-vertex, then $w^*(v) \geq 2/9 + \frac{1}{27}$.

In each "corner", there is a side C-vertex of a full $C_{\geq 3}$-vertex.



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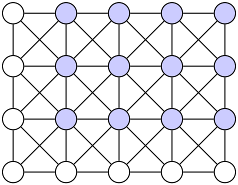

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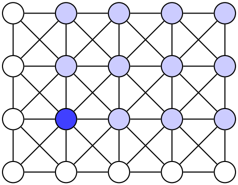

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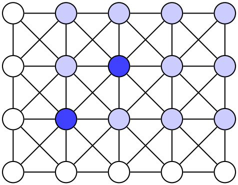
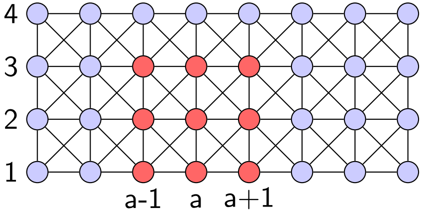


Theorem 3 : $d^*(\mathcal{K}_k) \geq 2/9 + \frac{8}{81k}$.

excess : $\text{exc}(v) \geq w^*(v) - \frac{2}{9}$

(R3) Every side C-vertex sends $\frac{2}{27}$ to its two side neighbours.

$$\begin{aligned} B[a] &= \{(a-1,1),(a-1,2),(a-1,3),(a,1),(a,2),(a,3),\} \\ &\cup \{(a+1,1),(a+1,2),(a+1,3)\}. \end{aligned}$$



$\text{exc}(B[a]) \geq \frac{4}{27}$ for every integer a.