Biclique graph of bipartite permutation graphs

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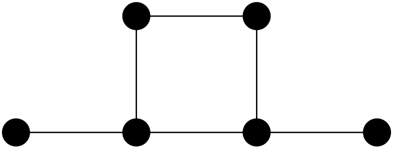

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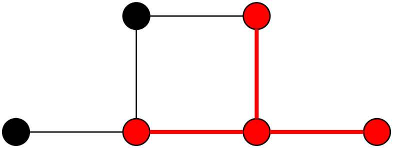

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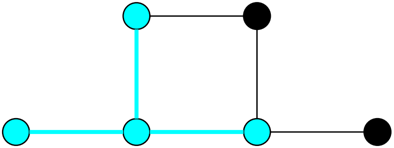

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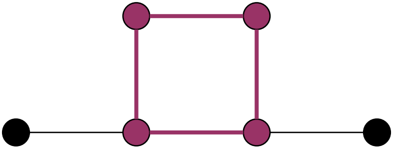

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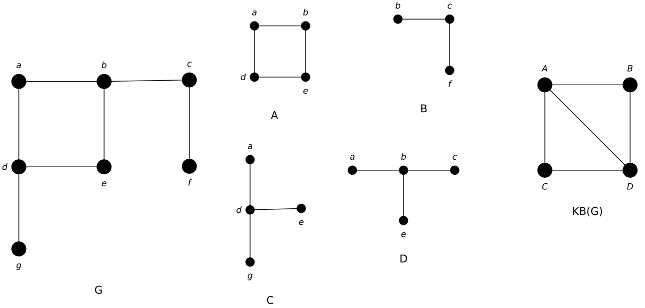

Biclique Graph

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- The characterization does not lead to a polynomial time recognition algorithm.
- It remains open the time complexity of the problem of recognizing biclique graphs.

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We study the problem of deciding if a graph G is the biclique graph of a bipartite permutation graph ($\mathcal{BPG}$).

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Results:

- $KB(\mathcal{BPG}) \subset K_{1,4}$-free interval graph.
- $KB(\mathcal{BPG}) = (L(\mathcal{BPG}))^2$ (the square of the line graph).
- a characterization of the biclique graph of a particular subclass of $\mathcal{BPG}$, which lead to a polynomial time recognition algorithm.

Bipartite permutation graphs

Definition

A graph G is a permutation graph if there are two permutations, π_1 and π_2 of $V(G)$ such that there is an edge $\{u, v\}$ if and only if u and v are in one order in π_1 and in the reversed order in π_2.

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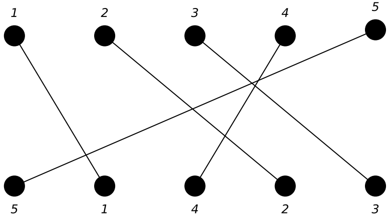

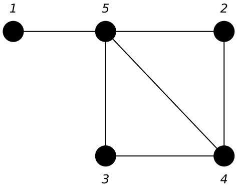

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$\mathcal{BPG}$ = proper bi-interval graphs (Hell and Huang, 2004).

Ordering properties

Definition (adjacency property)

The order $<_G$ has the **adjacency property** if the vertices in $N_G(v)$ are consecutive in $<_G$ for each $v \in V(G)$.

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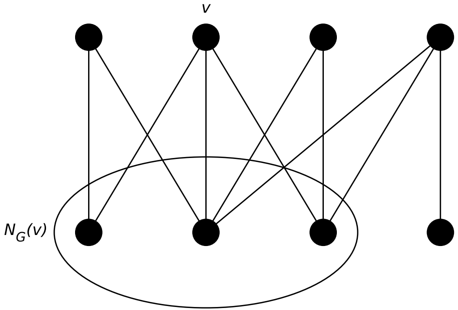

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The order $<_G$ has the **enclosure property** when for all $u, v \in V(G)$, if $N_G(u) \subseteq N_G(v)$ then the vertices in $N_G(v) \setminus N_G(u)$ are consecutive.

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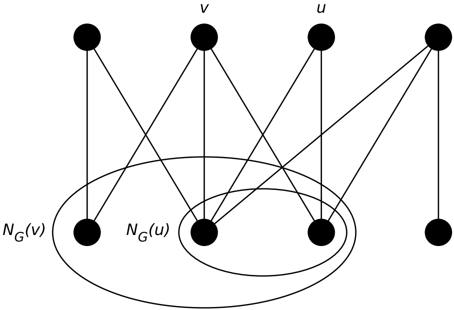

Ordering properties

Definition (strong ordering property)

The order $<_G$ of the vertices of a bipartite graph $G = (A \cup B, E)$ has the **strong ordering property** if $u, v \in A$, $u <_G v$, $w, x \in B$ where $w <_G x$, and $\{u, x\}, \{v, w\} \in E$, then $\{u, w\}, \{v, x\} \in E$.

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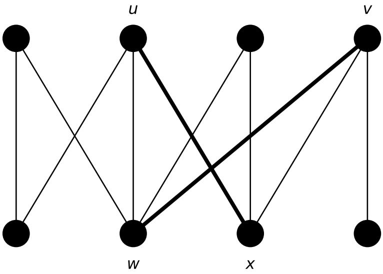

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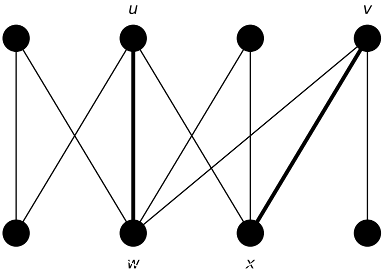

Ordering properties

A $\mathcal{BPG}$ G has an ordering $<_G$ with these properties (Spinrad at al., 1987).

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Definition

- $f_G(u) = \min_{<_G}\{v|\{u,v\} \in E\}$ - first neighbour of u in G, and
- $l_G(u) = \max_{<_G}\{v|\{u,v\} \in E\}$ - last neighbour of u in G.

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Bicliques of a bipartite permutation graph

Lemma (first and last)

S is a biclique of a $\mathcal{BPG}$ G if and only if

- $l_G(f_A(S)) = l_B(S)$, $f_G(l_A(S)) = f_B(S)$,
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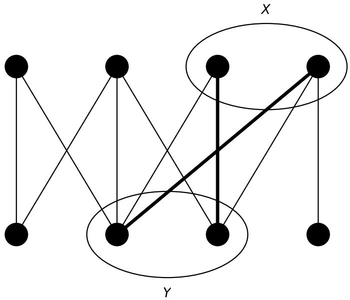

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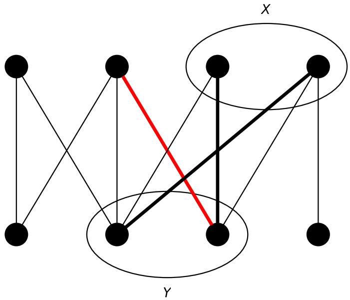

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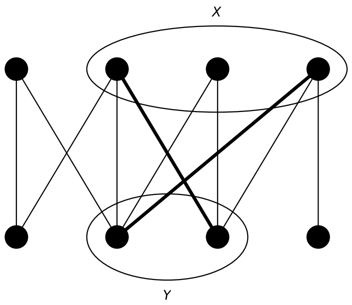

Bicliques of a bipartite permutation graph

Lemma (order)

S and S' are bicliques of a $\mathcal{BPG}$ G.

- *$f_A(S) <_G f_A(S')$ iff $l_B(S) <_G l_B(S')$.*
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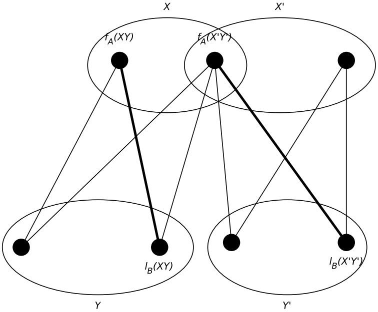

Non intersecting bicliques

Corollary (non intersecting)

S and S' be two non intersecting bicliques of a $\mathcal{BPG}$ G. If $S \cap A$ is completely before $S' \cap A$ then $S \cap B$ is completely before $S' \cap B$.

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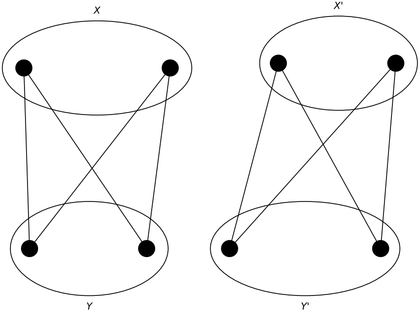

Asteroidal triple

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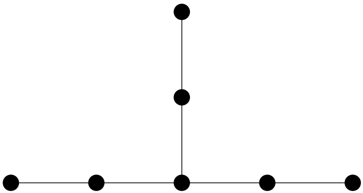

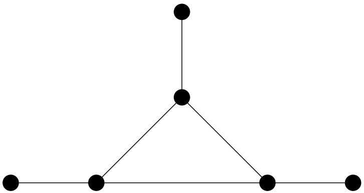

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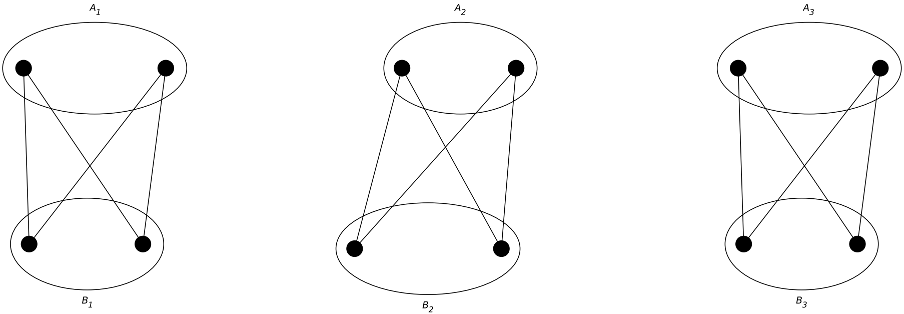

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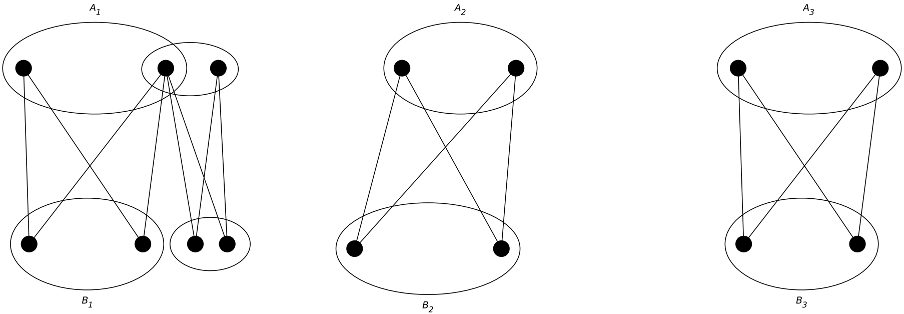

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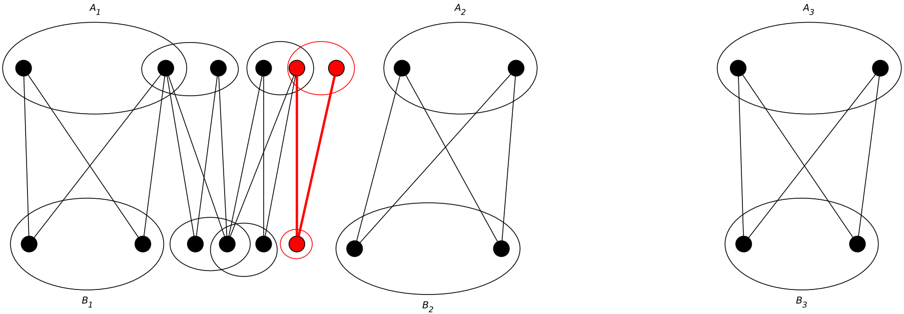

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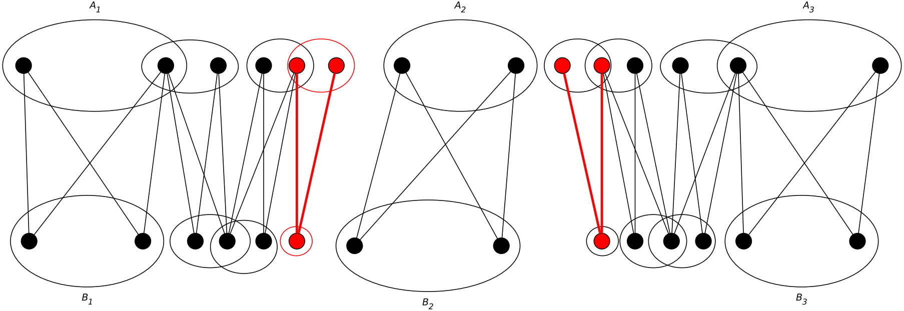

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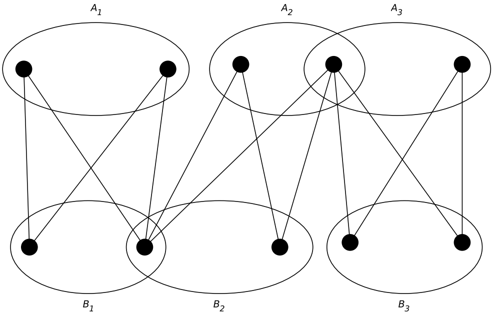

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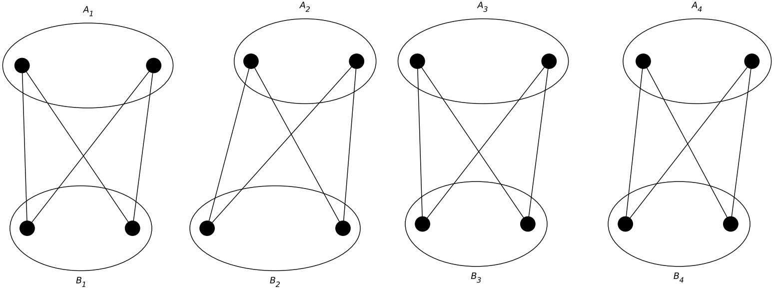

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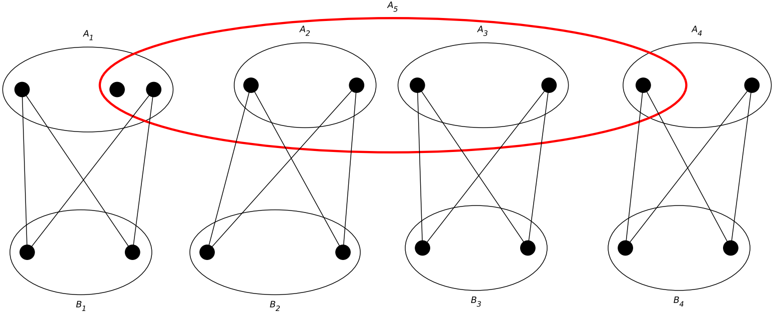

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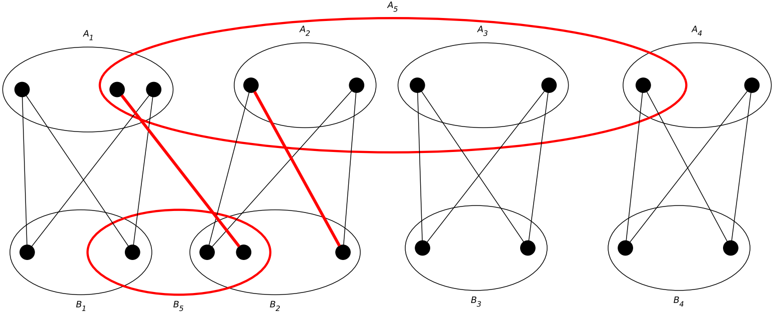

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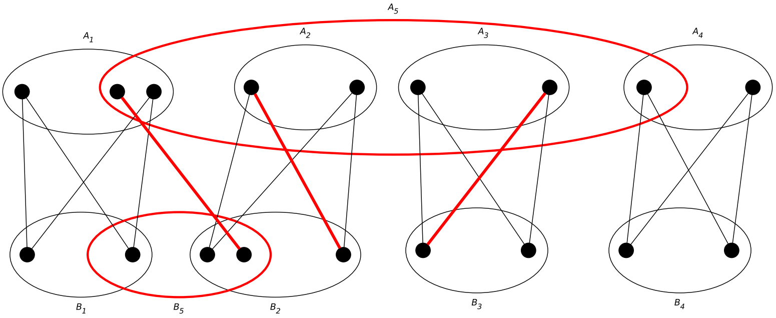

$K_{1,4}$-free interval graph

Theorem ($K_{1,4}$-free interval)

The biclique graph of $\mathcal{BPG}$ is a $K_{1,4}$-free interval graph.

Simplification of a $\mathcal{BPG}$

Definition

The set of extremal edges of G is the set $T(G)$ defines as:

$$\{\{u, v\} \in E \mid (u = f_G(v) \text{ and } \ v = l_G(u)) \text{ or } \ (u = l_G(v) \text{ and } \ v = f_G(u))\}.$$

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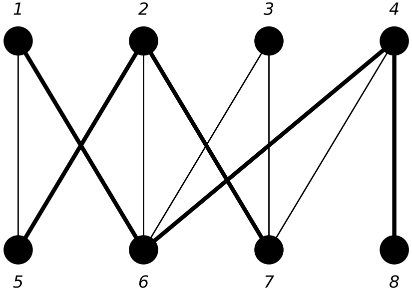

Simplification of G

Definition

The *simplification of* G is the graph $S(G) = (T(G), E')$ such that:

$$E' = \{\{a, b\} \mid a \text{ and } b \text{ of } G \text{ are ``crossed'' or has a vertex in common}\}$$

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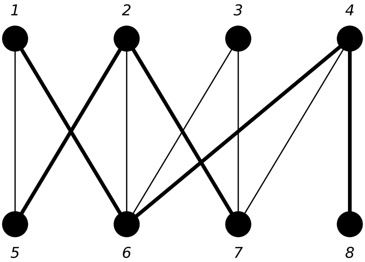

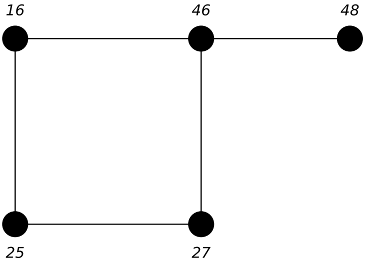

Simplification of G

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- bijection: bicliques of $G \leftrightarrow$ edge "crossings" of $T(G)$.

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- $S(G)$ is a $\mathcal{BPG}$ (permutations are easily obtained).
- for every $\mathcal{BPG}$ H there is a $\mathcal{BPG}$ G such that $H \simeq S(G)$.

Line graph and the square of a graph

Definition

$L(H)$ is the intersection graph of its edges.

Line graph and the square of a graph

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Definition

G^2 is the graph G plus edges between vertices of distance 2.

Square of the line graph of the simplification graph of G

Theorem

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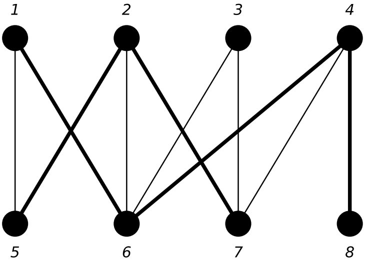

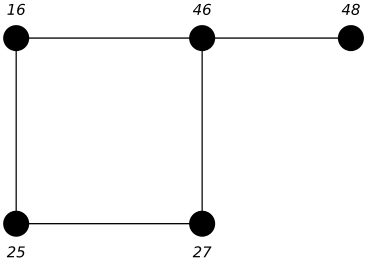

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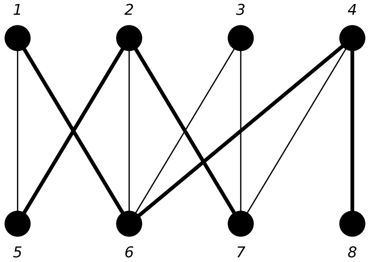

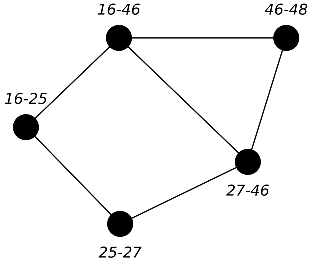

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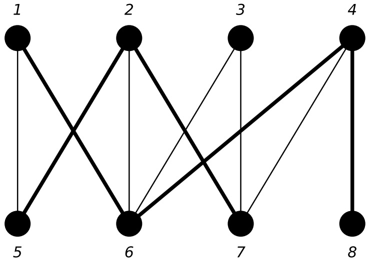

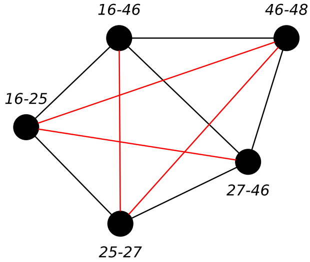

Characterization of $KB(\mathcal{BPG})$

Theorem

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- $\forall H \in \mathcal{BPG}$, there is a $G \in \mathcal{BPG}$ such that $H \simeq S(G)$
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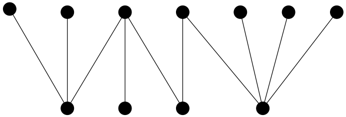

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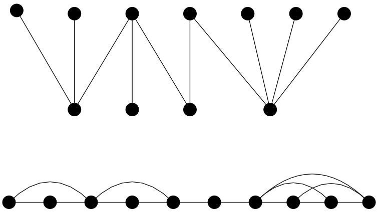

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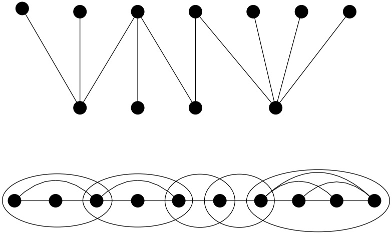

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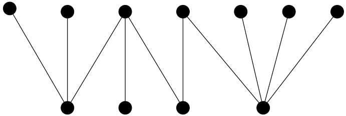

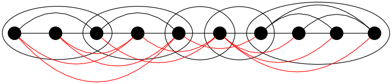

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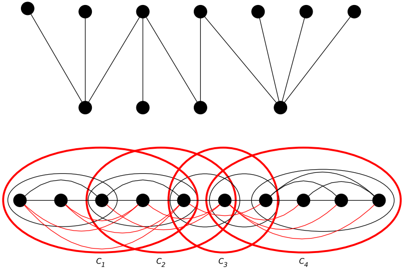

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Definition

A *linear proper interval graph* is a proper interval graph with the cliques $(C_1, C_2, \dots, C_k)$ in the above order and such that $|C_i \cap C_{i+1}| > 1$, for $1 \leq i \leq k-1$, and $|C_i \cap C_{i+2}| = 1$, for $1 \leq i \leq k-2$.

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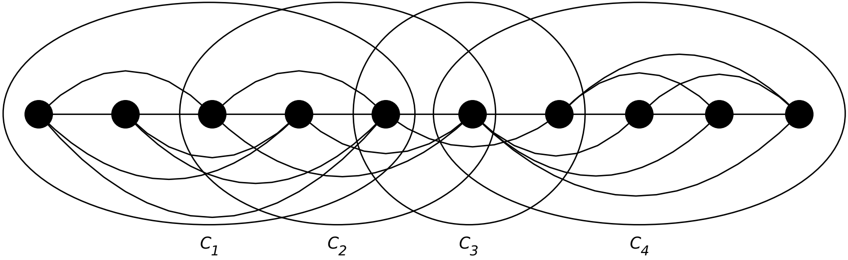

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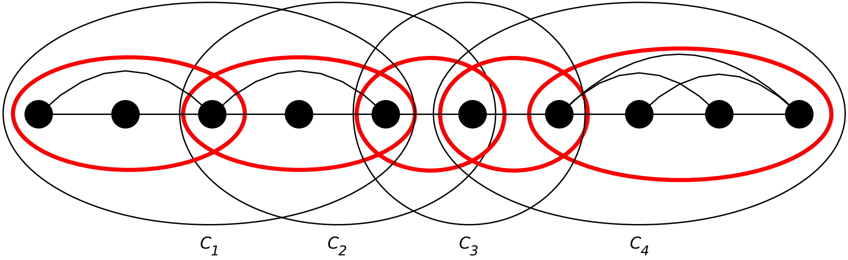

Biclique graph of bipartite permutation graphs

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