On graphs with a single large Laplacian eigenvalue

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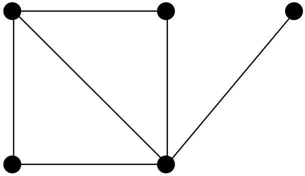
Laplacian Matrix

Given a simple graph G on n vertices, let $A = A(G)$ be its adjacency matrix, where rows and columns are indexed by $V(G)$ and $a_{uv} = 1$ if u is adjacent to v and $a_{uv} = 0$ otherwise. If $D = D(G)$ is the diagonal matrix with vertex degrees on the diagonal, the matrix $L(G) = D - A$ is called the Laplacian Matrix.

Example $G = (P_3 + K_1) \vee K_1$



$$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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- The largest Laplacian eigenvalue μ_1 is greater than or equal to $\overline{d}(G) = \frac{2m}{n}$, where m is the number of edges of G.
- The parameter $\sigma(G)$ counts the number of Laplacian eigenvalues of G greater than or equal to $\overline{d}(G)$ (introduced by Das, Mojallal and Trevisan, 2016).
- $1 \leq \sigma(G) \leq n$, for every graph G. If G has at least one edge, then $\sigma(G) \leq n - 1$.

Motivation

One of the motivations to study $\sigma(G)$ is the Laplacian energy (introduced by Gutman and Zhou, 2006)

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It is easy to see that

$$LE(G) = 2\mathcal{S}_\sigma - \frac{4m\sigma}{n},$$

where $S_\sigma = \sum_{i=1}^{\sigma} \mu_i(G)$.

Basic properties of σ

Theorem (Das, Mojallal and Trevisan, 2016)

Let G be a graph on n vertices. Then

$$n - 1 \leq \sigma(G) + \sigma(\overline{G}) \leq 2n - 1.$$

Moreover the right equality holds if and only if $G \cong K_n$.

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Characterize all graphs G for which $\sigma(G) + \sigma(\overline{G}) = n - 1$.

Example: Every graph $G \cong K_t + (n-t)K_1$ satisfies $\sigma(G) + \sigma(\overline{G}) = n - 1$ ($2 \leq t \leq n - 1$).

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Proposition (Das, Mojallal and Gutman, 2015)

If G is a graph on n vertices, then the following conditions hold:

1. $\sigma(G) = n$ iff $G \cong nK_1$.
2. $\sigma(G) = n - 1$ iff $G \cong K_{\underbrace{t, \ldots, t}_{k}}$ with $k > 1$ and $n = kt$.

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Example: $\sigma(K_{1,n-1}) = 1$, because the eigenvalues of $K_{1,n-1}$ are n, 1 with multiplicity $n-2$ and 0 and $\overline{d}(K_{1,n-1}) = 2\left(1 - \frac{1}{n}\right)$.

Our conjecture and the Laplacian spectrum of $G_1 + G_2$ and $G_1 \vee G_2$

Conjecture 1 (Allem, Cafure, Dratman, **G.**, Safe, Trevisan, 2017+)

Let G be a graph. Then $\sigma(G) = 1$ if and only if G is isomorphic to K_1, $K_2 + sK_1$ for some $s \geq 0$, or $K_{1,r} + sK_1$ for some $r \geq 2$ and $0 \leq s < r - 1$.

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Theorem (Folklore)

Let G_1 and G_2 be two graphs with Laplacian spectrums $\mathcal{S}(G_1) = \{\mu_1, \ldots, \mu_{n_1-1}, 0\}$ and $\mathcal{S}(G_2) = \{\lambda_1, \ldots, \lambda_{n_2-1}, 0\}$, respectively. Then

1. $\mathcal{S}(G_1 + G_2) = \mathcal{S}(G_1) \cup \mathcal{S}(G_2)$,
2. the Laplacian eigenvalues of $G_1 \vee G_2$ are $n_1 + n_2$; $n_2 + \mu_i$, for $1 \leq i \leq n_1 - 1$; $n_1 + \lambda_i$, for $1 \leq i \leq n_2 - 1$ and 0.

Previous results

Lemma (Allem, Cafure, Dratman, G., Safe and Trevisan, 2017+)

If $G = G_1 \vee \cdots \vee G_k$, with $k \geq 1$, is a graph on n vertices, then n is a Laplacian eigenvalue of G with multiplicity at least $k - 1$.

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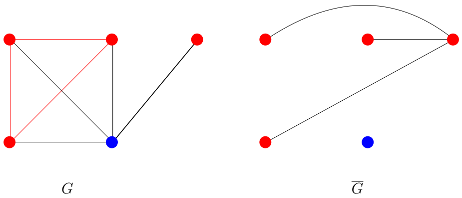
An anticomponent of a graph G is the subgraph of G induced by the vertex set of a connected component of $\overline{G}$.



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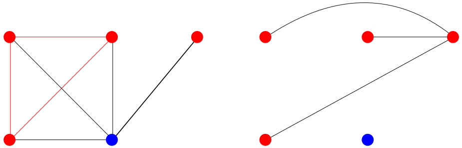

G $\qquad\qquad$ $\overline{G}$

Corollary (Allem, Cafure, Dratman, **G.**, Safe and Trevisan, 2017+)

If G has k anticomponents, then $k \leq \sigma(G) + 1$.

Case where $k = \sigma(G) + 1$ holds

The graph $G = 4K_2 \vee \overbrace{K_1 \vee \cdots \vee K_1}^{s}$ has average degree $s + 7 - \frac{48}{s+8}$ and $s + 1$ anticomponents. Its eigenvalues are $s + 8$, $s + 2$, s, and 0 with multiplicities s, 4, 3, and 1, respectively. Therefore, $\sigma(G) = s$.

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Theorem (Allem, Cafure, Dratman, G., Safe and Trevisan, 2017+)

Let G be a graph having $k = \sigma(G) + 1$ anticomponents. Then $\ell(G) \leq \sigma(G)$. Moreover, if $\sigma(G) = \ell(G)$, then the remaining anticomponent of G is empty but nontrivial.

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Corollary (Allem, Cafure, Dratman, G., Safe and Trevisan, 2017+)

If G is a graph with $\sigma(G) = 1$ and $\overline{G}$ is disconnected, then G is a complete bipartite graph.

Reduction to co-connected graphs

Theorem (Allem, Cafure, Dratman, **G.**, Safe and Trevisan, 2017+)

Let G be a graph on n vertices such that $\overline{G}$ is disconnected. Then $\sigma(G) = 1$ if and only if $G = K_{1,n-1}$.

Proof sketch: $\Leftarrow$ If $G \cong K_{1,n-1}$, it can be easily shown that $\sigma(G) = 1$.

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A weaker conjecture

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Conjecture 2 (Allem, Cafure, Dratman, G., Safe, Trevisan, 2017+)

Let G be a graph with connected complement. Then, $\sigma(G) = 1$ if and only if G is isomorphic to K_1, $K_2 + sK_1$ for some $s > 0$, or $K_{1,r} + sK_1$ for some $r \geq 2$ and $0 < s < r - 1$.

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Theorem (Allem, Cafure, Dratman, **G.**, Safe, Trevisan, 2017+)

Let $\mathcal{G}$ be a graph class closed by taking components. If Conjecture 3 holds for $\mathcal{G}$, then Conjecture 1 also holds for $\mathcal{G}$.

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Notice that the class of all graphs is closed by taking components. Therefore, the validity of Conjecture 1 can be reduced to the validity of Conjecture 3.

Validity of conjecture 1 for forests

Theorem (Li and Pan, 2000)

Let G be a graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ and Laplacian spectrum $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. Then $\mu_2 \geq d_2$.

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Proof sketch: Let T be a connected and co-connected forest.

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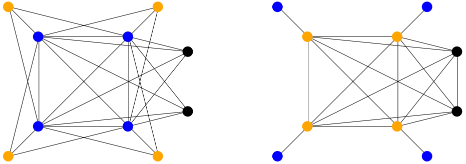
Spiders

A spider is a graph whose vertex set can be partitioned into three sets S, C, and R, where $S = \{s_1, \ldots, s_k\}$ $(k \geq 2)$ is a stable set; $C = \{c_1, \ldots, c_k\}$ is a clique; s_i is adjacent to c_j if and only if $i = j$ (a *thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick spider*); R is allowed to be empty and all the vertices in R are adjacent to all the vertices in C and nonadjacent to all the vertices in S. The sets S, C and R are called legs, body and head of the spider, respectively.



Extended P_4-laden graphs

Theorem (Giakoumakis, 1996)

Each connected and co-connected extended P_4-laden graph G satisfies one of the following assertions:

1. G is isomorphic to K_1, P_5, $\overline{P_5}$, or C_5;
2. G is a spider or arises from a spider by adding a twin to a vertex of the body or the legs; or
3. G is a split graph.

Conjecture 3 for extended P_4-laden graphs

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Proof sketch:

1. Let G be a spider or a graph obtained from a spider by adding a twin to a vertex of the body or the legs. Let k be the number of vertices in the body and n_H the number of vertices in the head. We prove that $d_2(G) \geq \overline{d}(G)$ whenever $n_H \neq 0$ or $k > 1$. Hence $\sigma(G) \geq 2$.

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4. Therefore, the only connected and co-connected extended P_4-laden graph with $\sigma = 1$ is K_1.$\square$

Thank you for your attention!