LAGOS 2017

Delta-Wye Transformations and the Efficient Reduction of Almost-Planar Graphs

Isidoro Gitler and Gustavo Sandoval–Angeles
ABACUS-Department of Mathematics
CINVESTAV

September 2017

Almost-planar graphs

A non–planar graph G is called almost-planar if for every edge e of G, at least one of $G \setminus e$ and G/e is planar.

We can also say that a graph G is almost–planar if and only if G is not $\{K_5, K_{3,3}\}$–free but for every edge e of G, at least one of $G \setminus e$ and G/e is $\{K_5, K_{3,3}\}$–free (also known as a $\{K_5, K_{3,3}\}$–fragile graph).

Series-parallel extension

A graph G is a series-parallel extension of a graph H if there is a sequence of graphs H_1, H_2, ..., H_n such that $H_1 = H$, $H_n = G$ and, for all i in $\{2, 3, \ldots, n\}$, H_{i-1} is obtained from H_i by the deletion of a parallel edge or by the contraction of an edge incident to a degree two vertex.

The following results are found in: [Gubser, B. S. A characterization of almost-planar graphs, Combinatorics, Probability and Computing, 5, Num. 3, pp. 227-245, 1996.]

Lemma 1 (Gubser 1996)

If G is an almost-planar graph, then G is a series-parallel extension of a simple, 3–connected, almost-planar graph.

Sets of edges whose deletion and contraction give a planar graph

Let G be an almost-planar graph. We define the sets $\mathcal{D}(G)$ and $\mathcal{C}(G)$ formed by edges f and g in G such that $G \setminus f$ and G/g are planar graphs, respectively.

Lemma 2 (Gubser 1996)

Let G be an almost planar graph, and let $e \in E(G)$ then:

- If $e \in \mathcal{C}(G)$, we can always add edges parallel to e and obtain an almost-planar graph.
- If $e \in \mathcal{D}(G)$, we can always subdivide e and obtain an almost-planar graph.

Lemma 3 (Gubser 1996)

Let G be a simple 3–connected almost-planar graph. Then, either G is isomorphic to K_5, or G has a spanning subgraph that is a subdivision of $K_{3,3}$. Moreover, every non–planar subgraph of G is spanning.

Lemma 4 (Gubser 1996)

If G is an almost-planar graph and H is a non–planar minor of G, then H is almost-planar.

We define the main families of almost-planar graphs.

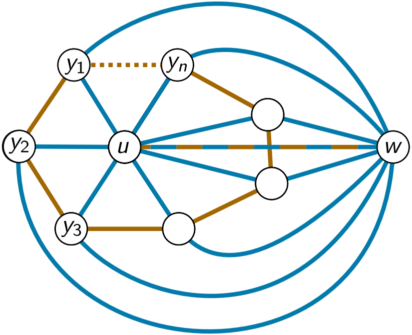
Double wheels $\boldsymbol{DW(n)} \in \boldsymbol{\mathcal{DW}}$ obtained from a cycle of length n and two adjacent vertices not in the cycle, both incident to all vertices in the cycle.

Möbius ladders $\boldsymbol{M(n)} \in \boldsymbol{\mathcal{M}}$ obtained from a cycle of length $2n$ by joining opposite pairs of vertices on the cycle.

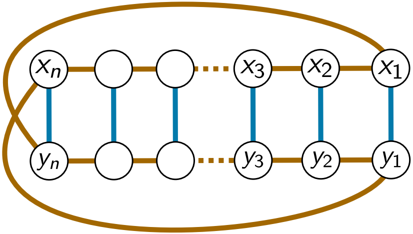
and the family

Sums of three wheels $\boldsymbol{W(l, m, n)} \in \boldsymbol{\mathcal{W}}$ which is the set of all graphs constructed by identifying three triangles from three wheels. In other words, each graph $G \in \mathcal{W}$ admits a partition (V_0, V_1, V_2, V_3) of its vertex set such that $G[V_0]$ is a triangle, $G[V_0 \bigcup V_i]$ is a wheel $(i = 1, 2, 3)$, and G has no edges other than those in these three wheels. Notice that graphs in $\mathcal{W}$ can be naturally divided into three groups, $\boldsymbol{\mathcal{W}_1}$, $\boldsymbol{\mathcal{W}_2}$ and $\boldsymbol{\mathcal{W}_3}$ depending on how the three hubs are distributed on the common triangle: all 3 hubs in one vertex of the triangle, 2 hubs in one vertex of the triangle and 1 hub in each vertex of the triangle, respectively.

Double wheel $\boldsymbol{DW(n)} \in \mathcal{DW}$.
$\mathcal{C}(DW(n)) = \{uw, vy_i | v = u, w\}$
$\mathcal{D}(DW(n)) = \{uw, y_1 y_n, y_i y_{i+1}\}$



Möbius ladder $\boldsymbol{M(n)} \in \mathcal{M}$.
$\mathcal{C}(M(n)) = \{x_i y_i\}$
$\mathcal{D}(M(n)) = \{x_i x_{i+1}, y_i y_{i+1}, x_n y_1, y_n x_1\}$



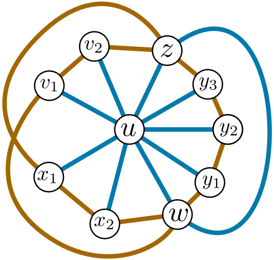
The edges in orange are in $\mathcal{D}$ and the edges in blue are in $\mathcal{C}$.

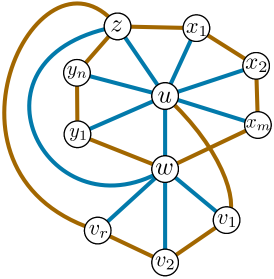
The family sum of three wheels $\mathcal{W}$.

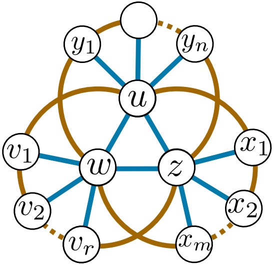
Let G_1 be a graph of the subfamily $\boldsymbol{\mathcal{W}_1}$. $\mathcal{C}(G_1) = \{uw, wz, zu, uv_i, uy_j, ux_k\}$. $\mathcal{D}(G_1)$ is the complement of $\mathcal{C}(G_1)$.

Let G_2 be a graph of the subfamily $\boldsymbol{\mathcal{W}_2}$. $\mathcal{C}(G_2) = \{uw, wz, zu, wv_i, wy_j, zx_k\}$. $\mathcal{D}(G_2)$ is the complement of $\mathcal{C}(G_2)$.

Let G_3 be a graph of the subfamily $\boldsymbol{\mathcal{W}_3}$. $\mathcal{C}(G_3) = \{uw, wz, zu\; zv_i, uy_j, wx_k\}$. $\mathcal{D}(G_3)$ is the complement of $\mathcal{C}(G_3)$.







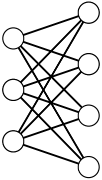
The following results are found in: [Ding, G., Fallon, J. and Marshall, E. On almost-planar graphs, arXiv preprint arXiv:1603.02310, 2016].

Theorem 1 (Ding, G., Fallon, J. and Marshall, E., 2016)

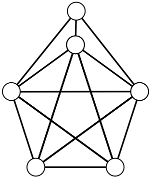
Let G be a simple, 3–connected, non–planar graph. Then the following are equivalent.

- *G is almost-planar*
- *G is a minor of a double wheel ($\mathcal{DW}$), a Möbius ladder ($\mathcal{M}$), or a sum of 3 wheels ($\mathcal{W}$)*
- *G is $\{K_{4,3}, K_5^{\oplus}, K_{3,3}^{H}, K_5^{H}, K_{3,3}^{\oplus}\}$-free*

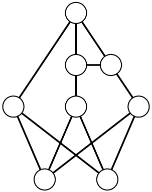
$K_{3,4}$



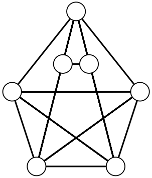
$K_5^{\oplus}$



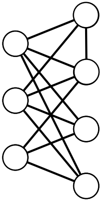
$K_{3,3}^{H}$



K_5^{H}



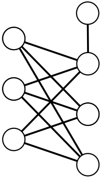
$K_{3,3}^{\oplus}$



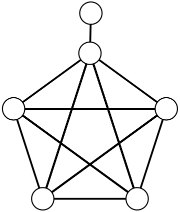
Theorem 2 (Ding, G., Fallon, J. and Marshall, E., 2016)

A connected graph G is almost-planar if and only if G is $\{K_5^+, K_{3,3}^+, K_5^H, K_{3,3}^H\}$-free.

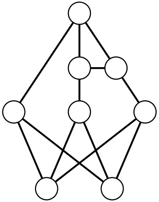
$K_{3,3}^+$



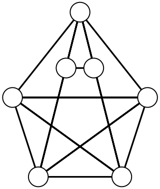
K_5^+



$K_{3,3}^H$



K_5^H



A characterization of almost-planar graphs

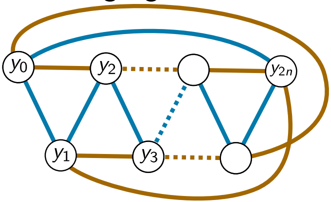
For our purpose we specify some important subfamilies of almost–planar graphs that can be obtained as minors of the ones defined above.

Zigzag ladders, denoted by $\mathcal{C}^2$, are the graphs obtained from a cycle of length n (n odd) by joining all pairs of vertices of distance two on the cycle.

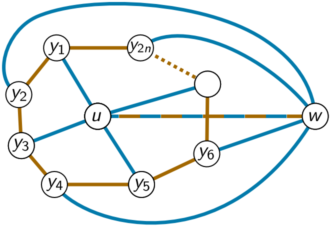
Alternating double wheels of length $2n$ ($n \geq 2$), denoted by $\mathcal{AW}$, are the graphs obtained from a cycle $v_1 v_2 \dots v_{2n} v_1$ by adding two new adjacent vertices u_1, u_2 such that u_i is adjacent to v_{2j+i} for all $i = 1, 2$ and $j = 0, 1, \dots, n-1$.

We also consider the families of almost–planar graphs $\mathcal{C}^{2*}$ and $\mathcal{DW}^*$ obtained by the 3-sum of wheels with graphs in $\mathcal{C}^2$ and $\mathcal{DW}$ on specified triangles in each of the original graphs. Together with the family $3\mathcal{W}^*$ obtained by the deletion of some edges, of a specified triangle of each graph in $3\mathcal{W}$.

Zigzag ladders



Alternating double wheels



The edges in orange are in $\mathcal{D}$ and the edges in blue are in $\mathcal{C}$.

Lemma 5

The following statements hold:

- Möbius ladders $\mathcal{M}$ are in $\mathcal{C}^{2*}$, (except $K_{3,3}$).
- Alternating double wheels $\mathcal{AW}$ are in $\mathcal{DW}^*$, (except $K_{3,3}$).

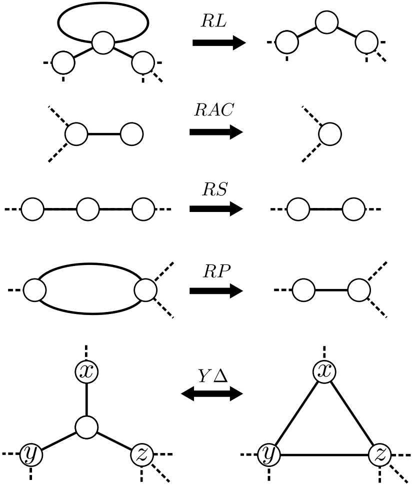
Extending results of Ding et al. (2016) we obtain:

Theorem 3

Let G be a simple, 3–connected, non–planar graph. Then G is almost–planar if and only if G is a graph in $\mathcal{DW}^*$, $\mathcal{C}^{2*}$ or $3\mathcal{W}^*$. Furthermore, if $\mathcal{D}(G) \cap \mathcal{C}(G) \leq 1$, then G belongs to exactly one of the following families: $\mathcal{DW}^*$, $\mathcal{C}^{2*}$, or $3\mathcal{W}^*$.

This theorem gives an explicit list of all simple 3–connected almost-planar graphs.

Delta–wye reduction of almost-planar graph



It is known (I.G. 2015) that any almost-planar graph is delta-wye reducible to a graph containing a single vertex, which extends Epifanov's result for planar graphs. This result follows by combining three known results. Specifically, D. Archdeacon, C. Colbourn, I.G. and S. Provan (2000) showed that any graph with crossing number one is delta-wye reducible to a single vertex; Gubser showed that every almost–planar graph is a minor of some almost–planar graph that has crossing number one; and Truemper showed that any minor of a delta-wye reducible graph is also delta-wye reducible.

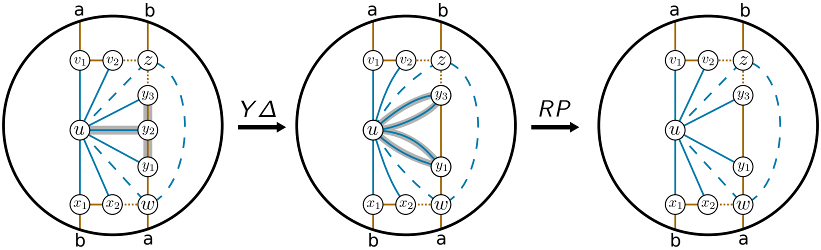
The question we are interested is to reduce almost–planar graphs to $K_{3,3}$ in such a way that all graphs in the reduction sequence are almost–planar.

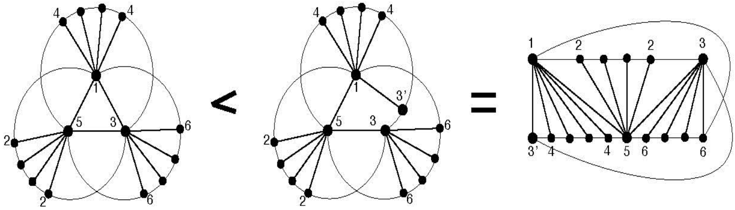
Theorem 4 (Wagner, D. K. Delta–wye reduction of almost-planar graphs, Discrete Applied Mathematics, 180, pp. 158-167, 2015.)

Let G be an almost-planar graph. Then, G is delta-wye reducible to $K_{3,3}$. Moreover, there exists a reduction sequence in which every graph is almost-planar.

We give a simpler proof of this result that is also algorithmic. Let G be a connected almost-planar graph. In order to $\Delta \leftrightarrow Y$ reduce G to $K_{3,3}$ we first do all series and parallel reductions to obtain a 3–connected almost–planar graph (Corollary 1). Now we can assume that G is a simple and 3–connected almost–planar graph. By Lemma 3 it is enough to show how to reduce the graphs in the families $\mathcal{M}$, $\mathcal{DW}^*$, $\mathcal{AW}$, $\mathcal{C}^{2*}$ and $\mathcal{W}$.

It is a simple task to delta-wye reduce to $K_{3,3}$ graphs in $\mathcal{W}$ so that each graph in the reduction sequence is almost–planar.



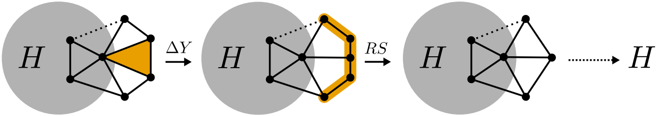

4
4
1
5
3
6
2
2
6
<
4
4
1
3'
5
3
6
2
2
6
=
1
2
2
3
3'
4
4
5
6
6

Lemma 6

Let G and H be a pair of almost–planar graphs such that G is obtained by a $3 - sum$ of H and a wheel $W(n)$. Then, G is delta–wye reducible to H, and the graphs in the reduction sequence are all almost–planar.

Sketch of Proof. Let v be a degree 3 vertex in $V(W(n))$, we obtain the reduction sequence G_1, G_2, G_3 by applying a $Y \to \Delta$ at the vertex v, followed by two parallel reductions on the edges e and f of the delta that are spokes of W.

Now, G_3 is almost–planar since it is a non–planar minor of G. Also, H is a minor of $G_3 \setminus e$ and $G_3 \setminus f$ then e and f are in $\mathcal{C}(G_3)$ and by lemma 2 G_1 and G_2 are almost–planar graphs. Note that, G_3 can be obtained by a 3–sum of H and $W(n-1)$, by induction we can reduce G to H such that the graphs in the reduction sequence are all almost–planar. □



Corollary 1

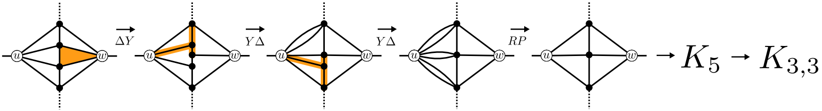
If G is a graph in $\mathcal{C}^{2*}$ o $\mathcal{DW}^*$ then, G can be reduced to a graph in $C^2(2n+1)$ or $DW(n)$, by a reduction sequence in which all graphs are almost–planar.

So by this Corollary and Theorem 3, we only need to prove that we can $\Delta \leftrightarrow Y$ reduce the graphs in the families $\mathcal{C}^2$ and $\mathcal{DW}$.

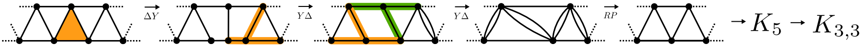
Lemma 7

Graphs in $\mathcal{DW}$, $\mathcal{C}^2$ and $3\mathcal{W}$ are delta–wye reducible to $K_{3,3}$, such that all graphs in the reduction sequence are almost–planar.

A part of Double Wheel.



A part of Zigzag Ladder.



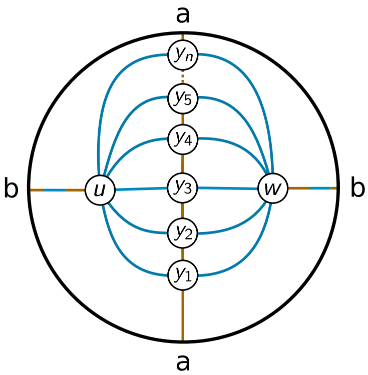
Outline of the algorithm.

1. Let G be a graph with n vertices. To decide if a graph is almost–planar we generate the sets $\mathcal{D}(G)$ and $\mathcal{C}(G)$, by applying a planarity test to the graphs $G \setminus e$ and G/e for each edge e of G ($\mathcal{O}(n^2)$).
2. We apply to G all allowable series and parallel reductions ($\mathcal{O}(n)$).
3. If $\mathcal{D}(G) \cap \mathcal{C}(G)$ has more than one edge, then G belongs to one of the specific simple cases, being easily reducible to $K_{3,3}$ ($\mathcal{O}(n^2)$).

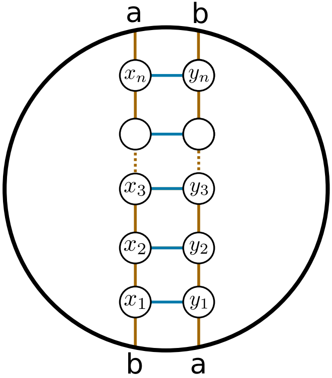
4. By Theorem 3 we have one of the following cases:
    4.1. When G is in $3\mathcal{W}^*$: We reduce each wheel that forms the graph to a K_4 and, consequently the graph G reduces to $K_{3,3}$, except for some edges between one chromatic class that are eliminated by simple reductions ($\mathcal{O}(n)$).
    4.2. When G is in $\mathcal{DW}^*$ or $\mathcal{C}^{2*}$: We reduce from G the wheels introduced by a 3-sum as shown in Corollary 1. In this way we reduce G to a *double wheel* or *zigzag ladder*, respectively ($\mathcal{O}(n)$).
5. Now, we are left with the reduction of double wheels and zigzag ladders to K_5. This is done inductively (Lemma 7). Finally, we reduce K_5 to $K_{3,3}$ ($\mathcal{O}(n)$).

Almost-Planar Graphs on the Projective Plane

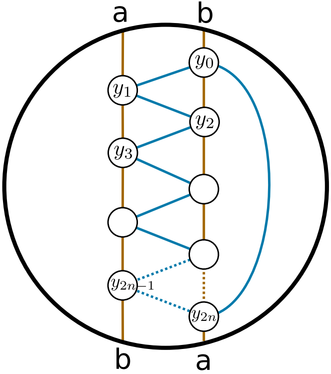
Double wheel $\boldsymbol{DW(n)} \in \mathcal{DW}$.



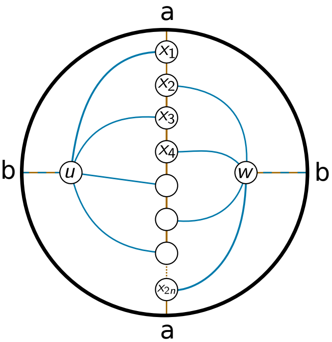
Möbius ladder $\boldsymbol{M(n)} \in \mathcal{M}$.



Zigzag ladders $C^2(2n+1) \in \mathcal{C}^2$

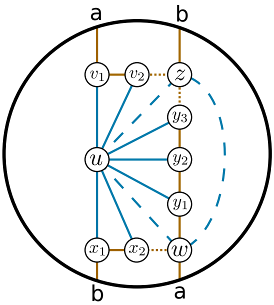


Alternating wheel $AW(2n) \in \mathcal{AW}$

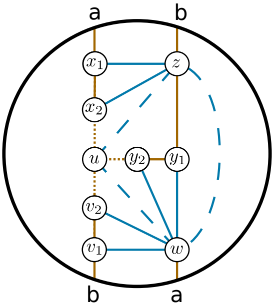


The family $\mathcal{W}$

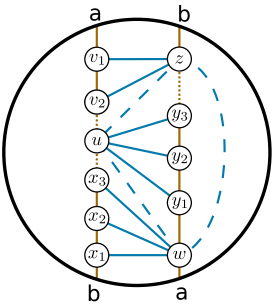
Let G_1 be a graph of the subfamily $\mathcal{W}_1$.



Let G_2 be a graph of the subfamily $\mathcal{W}_2$.



Let G_3 be a graph of the subfamily $\mathcal{W}_3$.



In the paper [Mohar, B., Robertson, N. and Vitray, R. P, (Planar graphs on the projective plane, Discrete Mathematics, 149, Num. 1, pp. 141-157, 1996] the authors show that embeddings of planar graphs in the Projective Plane have very specific structure.

They exhibit this structure and characterize graphs on the Projective Plane whose dual graphs are planar.

Furthermore, Whitney's Theorem about 2-switching equivalence of planar embeddings is generalized: Any two embeddings of a planar graph in the Projective Plane can be obtained from each other by means of simple local reembeddings, very similar to Whitney's switchings.

Corollary 2

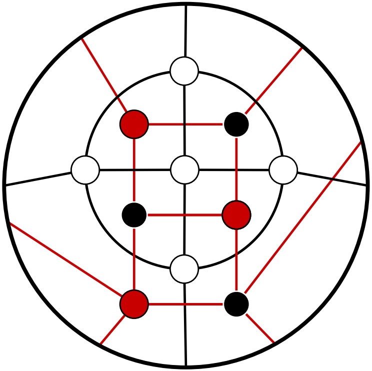
Any of the graphs in $\mathcal{M}$, $\mathcal{C}^{2*}$ or $\mathcal{W}$ has a Projective Planar Embedding such that its geometric dual is a planar graph.

Corollary 3

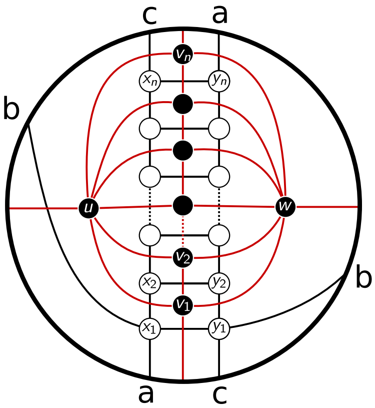
If G is a graph in $\mathcal{AW}$ or $\mathcal{DW}^*$, then G does not have an embedding on the Projective Plane such that its geometric dual is a planar graph.

It is natural to ask if almost-planar graphs admit embeddings in the Projective Plane such that the geometric dual graph is also an almost-planar graph. The next results answer this question.

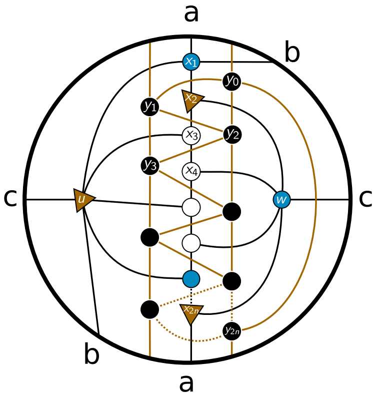
Let $K'_{3,3}$ be $K_{3,3}$ with a parallel edge e, by Lemma 2 $K'_{3,3}$ is an almost-planar graph. In the Projective Plane K_5 and $K'_{3,3}$ are duals.



Let $M'(n)$ be a Möbius ladder $M(n)$ with a parallel edge x_1y_1. By Lemma 2 the graph $M'(n)$ is almost-planar and it has an embedding on the Projective Plane such that its dual is a double wheel (a member of $\mathcal{DW}$).



Let $AW'(2n)$ be a alternating wheel $AW(2n)$ with a parallel edge ux_1. By Lemma 2 the graph $AW'(2n)$ is almost-planar and it has an embedding on the Projective Plane such that its dual is a zigzag ladder (a member of $C^2(2n+1)$).



Lemma 8

Any graph in $\mathcal{M}$, $\mathcal{DW}^*$, $\mathcal{AW}$, $\mathcal{C}^{2*}$ or $\mathcal{W}$ has an embedding in the Projective Plane such that its geometric dual is not planar nor almost-planar.

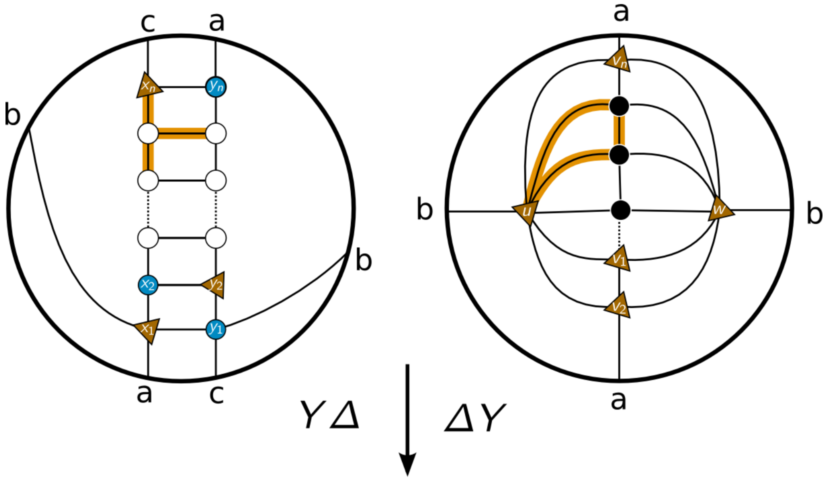
Theorem 5

For each graph G in $\mathcal{C}^{2*}$ (with an added parallel edge) there exists an embedding in the projective plane such that its dual graph G^* is in $\mathcal{DW}^*$. Furthermore, G is delta–wye reducible to $K_{3,3}$ (with an added parallel edge) while G^* reduces to K_5. Both sequences consist of almost–planar graphs.

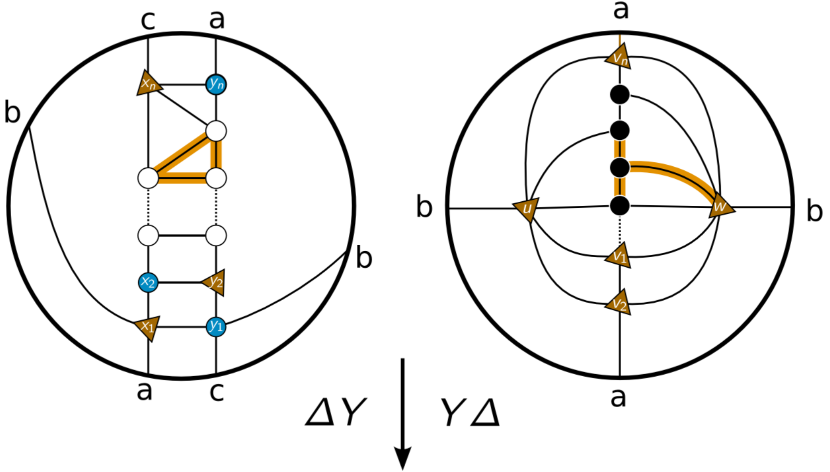
Conjecture

No graph in $\mathcal{W}$ has an embedding in the Projective Plane such that its dual is an almost-planar graph.

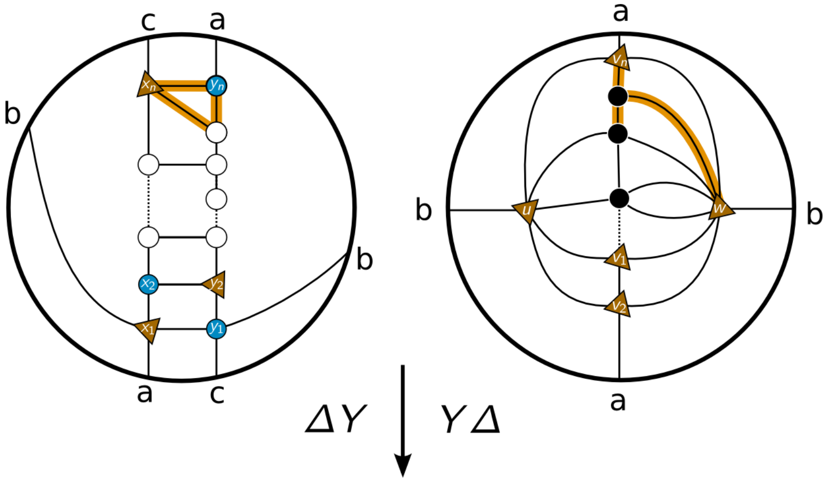
Reduction of $M'(n)$ to $M'(n-1)$ and double wheel $DW(n)$ to $DW(n-1)$.



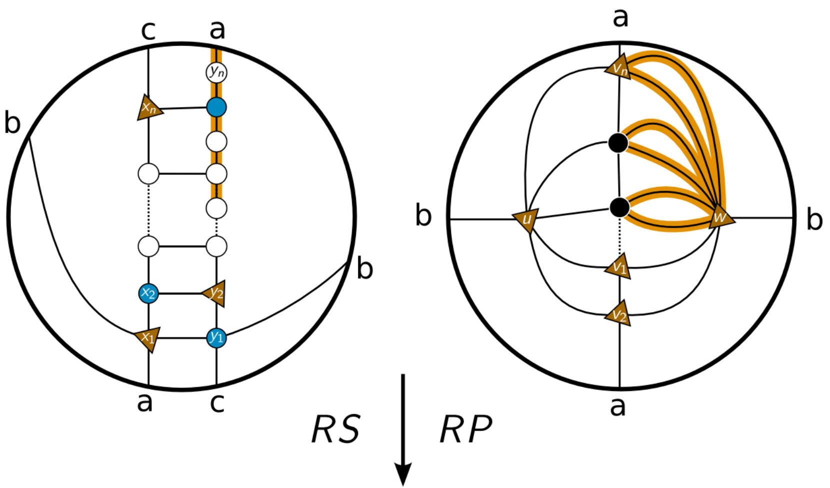
Reduction of $M'(n)$ to $M'(n-1)$ and double wheel $DW(n)$ to $DW(n-1)$.

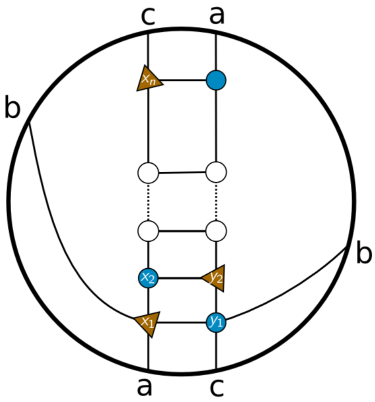


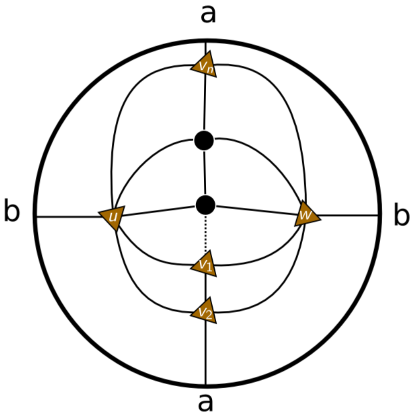
Reduction of $M'(n)$ to $M'(n-1)$ and double wheel $DW(n)$ to $DW(n-1)$.



Reduction of $M'(n)$ to $M'(n-1)$ and double wheel $DW(n)$ to $DW(n-1)$.

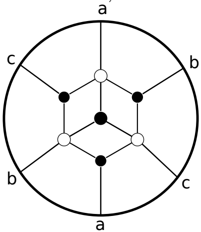




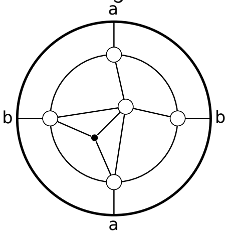


So we can reduce $M'(n)$ to $K'_{3,3}$ and by duality $DW(n)$ to K_5.
Analogously, we reduce $AW'(2n)$ to $K'_{3,3}$ and $C^2(2n+1)$ to K_5.

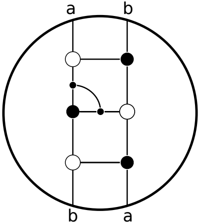
$K_{3,4}$



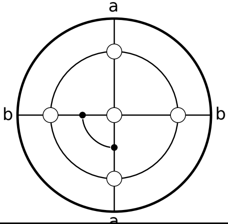
$K_5^{\oplus}$



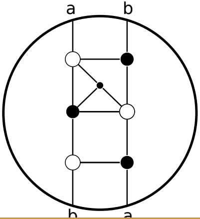
$K_{3,3}^H$



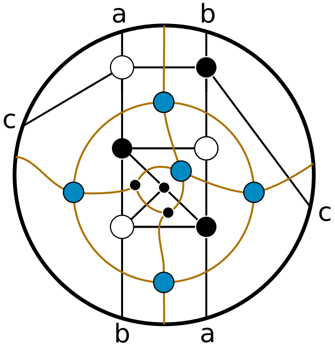
K_5^H



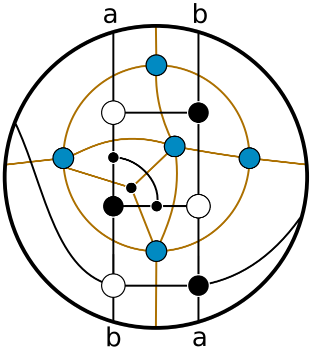
$K_{3,3}^{\oplus}$



The 3–connected forbidden minors are dual by pairs on the Projective Plane (for a specific embedding); except $K_{4,3}$.



$K_{3,3}^{\oplus}$ with a parallel edge and K_5^H are duals.



$K_{3,3}^{H}$ with a parallel edge and $K_5^{\oplus}$ are duals.

Delta–wye reducticibility of terminal almost–planar graphs

$\widetilde{K_6}$

Let a, b, c, d, e y f be the vertices of K_6. We define the graph $\boldsymbol{\widetilde{K_6}}$ as $K_6 \backslash \{ab, bc, ca\}$ with terminal vertices a, b, c

Lemma 9

The graph $\widetilde{K_6}$ is not 3–terminal reducible and is minimal with this property.

Theorem 6

If G is a connected almost–planar graph with 3 terminals then G is 3–terminal reducible to $\widetilde{K_6}$ or to a K_3 with all its vertices as terminals.

A corollary of these results is that given any two almost–planar graphs, then one can get from one of these graphs to the other, and stay within the class of almost–planar graphs, by delta-wye exchanges, series and parallel reductions, and series and parallel extensions (where extensions are the inverse of reductions).

I. An interesting problem is to characterize all embeddings of almost–planar graphs in the Projective Plane and the Torus analogous to the characterization of Mohar, Robertson and Vitray (1996) (among other works) for the case of planar graphs.

II. Similarly we would like to characterize which embeddings of projective planar and toroidal graphs have an almost–planar graph as a geometric dual.

III. We would like to understand delta–wye reductions and transformations for terminal graphs under duality (dual terminals are now terminal faces).

IV. Characterize Toroidal reducible graphs.

V. Characterize terminal reducible Projective Planar and Toroidal graphs.

VI. Characterize terminal almost–planar graphs where the reduction sequence consists of almost–planar graphs.

Thank You!