Scaffolding skeletons using spherical Voronoi diagrams

A. J. Fuentes Suárez[1,2]   E. Hubert[1]

[1]AROMATH team
Inria Sophia Antipolis Mediterraneé

[2]Université Côte d'Azur

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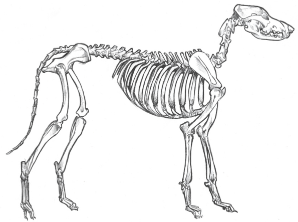
Outline

Skeleton based modeling

In nature



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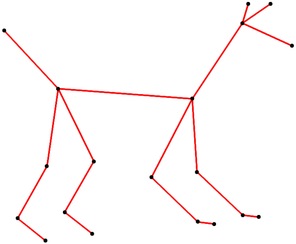
Mathematically



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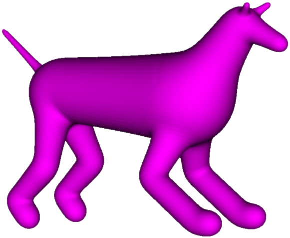
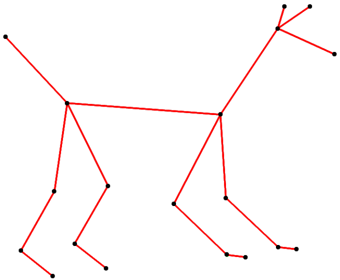


Skeleton

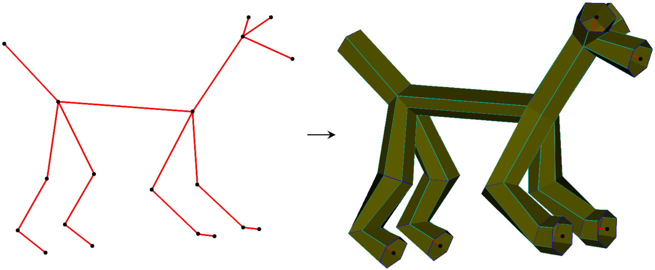
Definition

A *skeleton* is a finite set $\mathcal{S}$ of spatial line segments satisfying the following property: any two line segments intersect at most at one of their endpoints (then called joints).



Scaffold

Scaffold for a skeleton



A quad mesh around the skeleton, such that the volume it encloses can be contracted towards the skeleton, and the distance from the mesh to the skeleton is bounded by some constant.

Goal and previous work

Objective

To construct a quad mesh around a segment-based skeleton that follows the structure of the skeleton.

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We want:

- General method for skeleton with cycles.
- Optimal solution in terms of the number of quads.

General idea

Steps to construct a scaffold:

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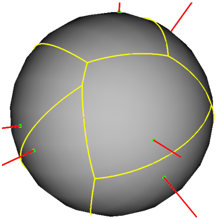
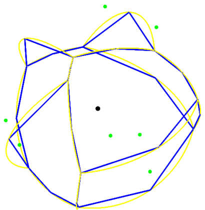
1. Partition the unit sphere centered at the joints into regions around the segments incident to the joint.



General idea

Steps to construct a scaffold:

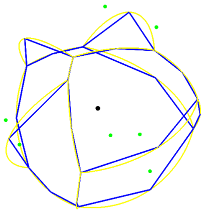
1. Partition the unit sphere centered at the joints into regions around the segments incident to the joint.
2. Discretize the regions as a set of points in the boundary to form cells.



General idea

Steps to construct a scaffold:

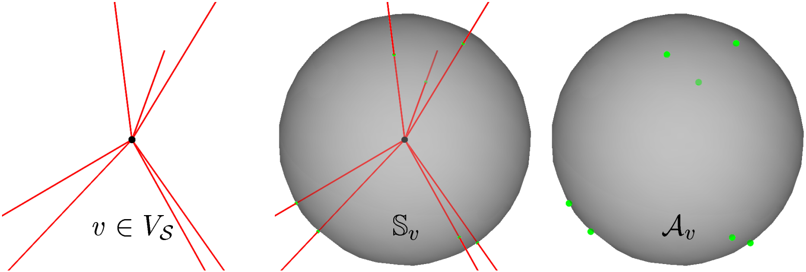
1. Partition the unit sphere centered at the joints into regions around the segments incident to the joint.
2. Discretize the regions as a set of points in the boundary to form cells.
3. Link the cells corresponding to the same segment to form quads.



Constrain: The linked cells must have the same number of points.

Formalization

- $\mathcal{S}$ defines naturally a graph $G_{\mathcal{S}} = (V_{\mathcal{S}}, E_{\mathcal{S}})$ (embedded in $\mathbb{R}^3$).
- $\mathbb{S}_v$ the unit sphere centered at $v \in V_{\mathcal{S}}$.
- $\mathcal{A}_v = \{e \cap \mathbb{S}_v \mid e \in E_{\mathcal{S}}, e \multimap v\}$ ($e \multimap a \equiv e$ incident to a).



Formalization (cont.)

- The regions $\{R_e^v\}_{e \multimap v}$ partition the sphere $\mathbb{S}_v$, such that $(\mathbb{S}_v \cap e) \in R_e^v$.
- The cell C_e^v is an ordered set of points describing the boundary of the region R_e^v.

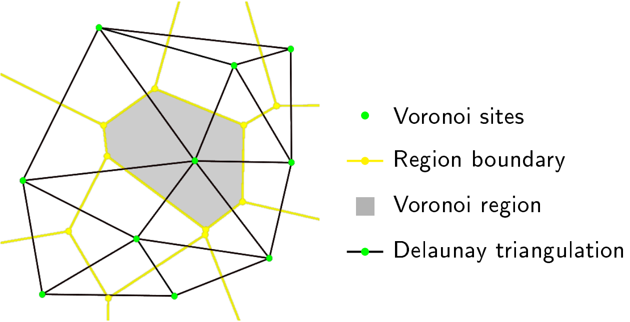
Definition (Scaffold)

A *scaffold* $\mathcal{K}_\mathcal{S}$ is a pair $(P_\mathcal{S}, \Phi_\mathcal{S})$, satisfying

1. $P_\mathcal{S} = \{\mathcal{C}_v \mid v \in V_\mathcal{S}\}$ where each $\mathcal{C}_v = \{C_e^v \mid e \in E_\mathcal{S}, e \multimap v\}$ is a family of *cells* representing a partition of $\mathbb{S}_v$.
2. $\Phi_\mathcal{S} = \{\phi_e \mid e \in E_\mathcal{S}\}$ is a family of bijections ϕ_e between C_e^a and C_e^b for $e = ab$.

Quads are defined as $(p_i, \phi_e(p_i), \phi_e(p_{i+1}), p_{i+1})$ for $C_e^v = (p_1, p_2, \cdots, p_n)$.

Voronoi diagram and Delaunay triangulation



Voronoi diagram and Delaunay triangulation [from Wikipedia]

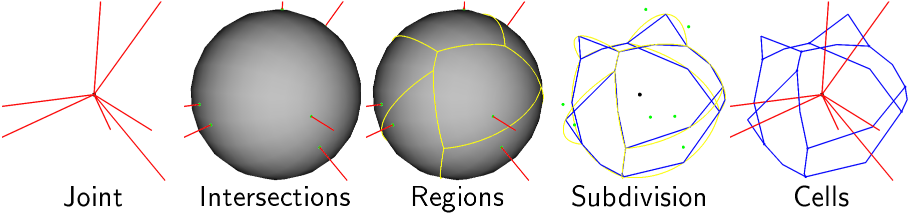
Notation

$\mathcal{A}$ set of points on a 2-dimensional sphere.
$\mathrm{Vor}(\mathcal{A})$ 2-dimensional *Voronoi diagram* of $\mathcal{A}$ on the sphere.
$\mathrm{Del}(\mathcal{A})$ *Delaunay triangulation* of $\mathrm{Vor}(\mathcal{A})$ (dual of $\mathrm{Vor}(\mathcal{A})$).

Partition and cells in practice

In our method:

- The regions $\{R_e^v\}_{e \multimap v}$ are defined as $\mathrm{Vor}(\mathcal{A}_v)$.
- Boundary between regions are arcs of great circles
- C_e^v is defined by taking points in each arc on the boundary.
- Points in an arc form a polyline, the number of segments in the polyline is called the *number of subdivision* of the arc.



Partition and cells (cont.)

Notation

E_v the set of edges of $\mathrm{Del}(\mathcal{A}_v)$ (for $v \in V_{\mathcal{S}}$).
x_f^v the number of subdivisions of the corresponding arc (for $f \in E_v$).

Observation

The number of elements in a cell C_e^v ($v \in V_{\mathcal{S}}$, $e \in E_{\mathcal{S}}$ and $e \multimap v$) is given by

$$|C_e^v| = \sum_{\substack{f \in E_v \\ f \multimap (\mathbb{S}_v \cap e)}} x_f^v.$$

Integer programming model

Minimize:

$$\sum_{v \in V_{\mathcal{S}}} \sum_{h \in E_v} x_h^v$$

Subject to:

$$|C_e^a| = |C_e^b| \quad \forall e = ab \in E_{\mathcal{S}}$$
$$x_h^v \in \mathbb{Z}, x_h^v \geq 2, \quad \forall h \in E_v, v \in V_{\mathcal{S}}$$

Recall that

$$|C_e^v| = \sum_{\substack{f \in E_v \\ f \multimap (\mathbb{S}_v \cap e)}} x_f^v$$

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Is this feasible?

Theorem

$$\sum_{\substack{f \in E_a \\ f \multimap (\mathbb{S}_a \cap e)}} x_f^a = \sum_{\substack{g \in E_b \\ g \multimap (\mathbb{S}_b \cap e)}} x_g^b, \ \forall e = ab \in E_{\mathcal{S}}$$

Theorem

There is a solution for the system with all entries in the set of positive integers.

Proof

Idea: "all cells with the same number of points"

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For $v \in V_{\mathcal{S}}$, the *local* linear system

$$|C_e^v| = \sum_{\substack{f \in E_v \\ f \multimap (\mathbb{S}_v \cap e)}} x_f^v = \lambda_v \quad \forall e \multimap v, e \in E_{\mathcal{S}} \tag{1}$$

has a solution in the positive integers with λ_v also a positive integer.
Assuming there are local solutions $(\tilde{x}_f^v, \tilde{\lambda}_v)$ for each $v \in V_{\mathcal{S}}$ then there is a global solution $\hat{x}_f$ given by $\hat{x}_f = \hat{x}_f^v = (\lambda/\tilde{\lambda}_v) \cdot \tilde{x}_f^v$ where $\lambda = \prod_{u \in V_{\mathcal{S}}} \tilde{\lambda}_u$.

Proof (cont.)

There is a positive real solution for the local linear system:

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- $\mathrm{Del}(\mathcal{A}_v)$ is combinatorially equivalent to the convex hull of $\mathcal{A}_v$.

  3. Brown KQ. *Geometric Transforms for Fast Geometric Algorithms.* Phd thesis, Carnegie-Mellon University. **1979**.

  $\mathrm{Del}(\mathcal{A}_v)$ dual of $\mathrm{Vor}(\mathcal{A}_v)$ $\implies$ E_v set of edges of convex hull of $\mathcal{A}_v$.

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  $\mathrm{Del}(\mathcal{A}_v)$ dual of $\mathrm{Vor}(\mathcal{A}_v) \quad \Longrightarrow \quad E_v$ set of edges of convex hull of $\mathcal{A}_v$.

- Numerical characterization for inscribable polyhedral graphs due to Rivin.

  4. Rivin I. *A Characterization of Ideal Polyhedra in Hyperbolic 3-Space*. Annals of Mathematics. **1996**;143:51-70.

Proof (cont.)

5. Dillencourt MB, Smith WD. *Graph-theoretical conditions for inscribability and Delaunay realizability*. Discrete Mathematics. **1996**;161(1):63-77.

Proposition (I. Rivin 1996, extracted from [5])

If a graph is of inscribable type then weights w *can be assigned to its edges such that:*

1. *For each edge* e, $0 < w(e) < 1/2$.
2. *For each vertex* v*, the total weight of all edges incident to* v *is equal to 1.*

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$x_f^v = w(f)$ defines a positive real solution for the local system (with $\lambda_v = 1$).

Proof (end)

Claim

A homogeneous linear system with integer coefficients has a positive integer solution whenever it has a positive real solution.

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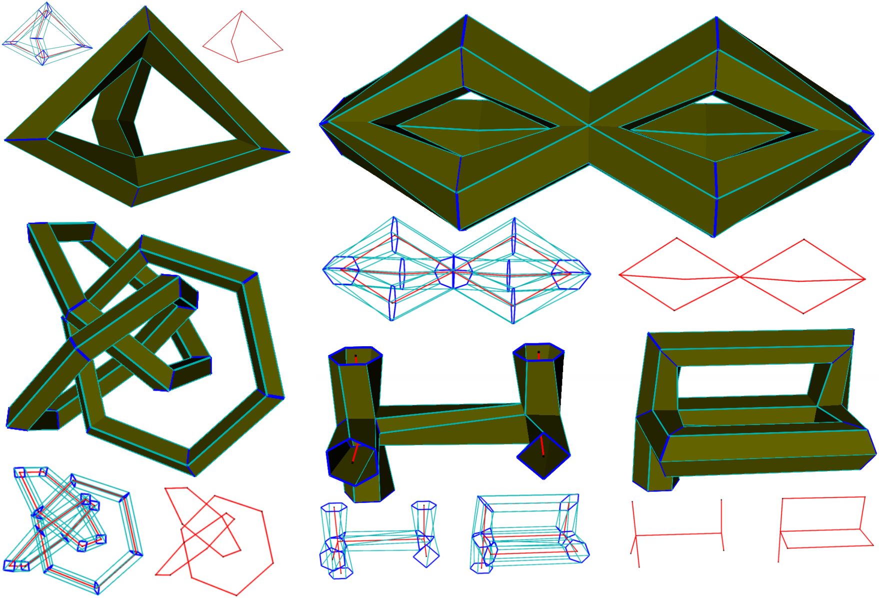
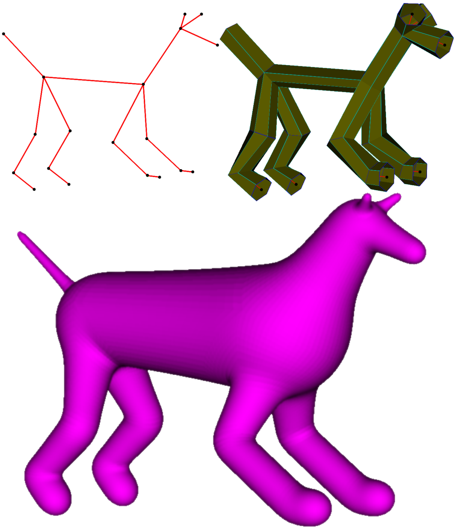
Proof sketch (of claim).

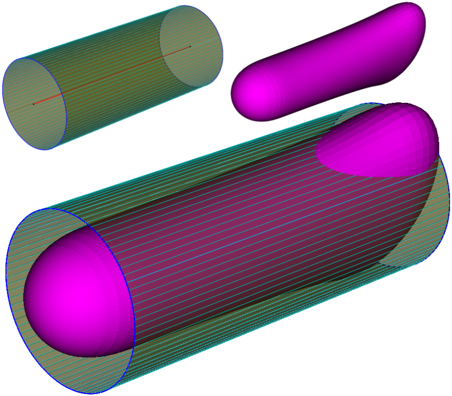
A positive real solution implies the space of solutions is non-empty.
Take a rational basis for the space.
The rational solutions are dense in this space.
Approximate the real solution by a positive rational solution.
Multiply by a common multiple of denominators of the entries. □

Examples



Application





Thank you