Complexity-separating graph classes for vertex, edge and total coloring

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Overview

Classification into P or NP-complete of challenging problems in graph theory

Full dichotomy: class of problems where each problem is classified into P or NP-complete

Coloring problems: vertex, edge, total

NP-completeness ongoing guide

Identification of an interesting problem, of an interesting graph class

Categorization of the problem according to its complexity status

Problems and complexity-separating graph classes

Graph classes and complexity-separating problems

Johnson's NP-completeness column 1985

Spinrad's book 2003

Ongoing Guide – graph restrictions and their effect

Graph Class	Member		IndSet		Clique		CliPar		ChrNum	
Trees/Forests	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[T]
Almost Trees (k)	P		P	[24]	P	[T]	P?		P?	
Partial k-Trees	P	[2]	P	[1]	P	[T]	P?		P	[1]
Bandwidth-k	P	[68]	P	[64]	P	[T]	P?		P	[64]
Degree-k	P	[T]	N	[GJ]	P	[T]	N	[GJ]	N	[GJ]
Planar	P	[GJ]	N	[GJ]	P	[T]	N	[10]	N	[GJ]
Series Parallel	P	[79]	P	[75]	P	[T]	P?		P	[74]
Outerplanar	P		P	[6]	P	[T]	P	[6]	P	[67]
Halin	P		P	[6]	P	[T]	P	[6]	P	[74]
k-Outerplanar	P		P	[6]	P	[T]	P	[6]	P	[6]
Grid	P		P	[GJ]	P	[T]	P	[GJ]	P	[T]
$K_{3,3}$-Free	P	[4]	N	[GJ]	P	[T]	N	[10]	N	[GJ]
Thickness-k	N	[60]	N	[GJ]	P	[T]	N	[10]	N	[GJ]
Genus-k	P	[34]	N	[GJ]	P	[T]	N	[10]	N	[GJ]
Perfect	O!		P	[42]	P	[42]	P	[42]	P	[42]
Chordal	P	[76]	P	[40]	P	[40]	P	[40]	P	[40]
Split	P	[40]	P	[40]	P	[40]	P	[40]	P	[40]
Strongly Chordal	P	[31]	P	[40]	P	[40]	P	[40]	P	[40]
Comparability	P	[40]	P	[40]	P	[40]	P	[40]	P	[40]
Bipartite	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[T]
Permutation	P	[40]	P	[40]	P	[40]	P	[40]	P	[40]
Cographs	P	[T]	P	[40]	P	[40]	P	[40]	P	[40]
Undirected Path	P	[39]	P	[40]	P	[40]	P	[40]	P	[40]
Directed Path	P	[38]	P	[40]	P	[40]	P	[40]	P	[40]
Interval	P	[17]	P	[44]	P	[44]	P	[44]	P	[44]
Circular Arc	P	[78]	P	[44]	P	[50]	P	[44]	N	[36]
Circle	P	[71]	P	[GJ]	P	[50]	O?		N	[36]
Proper Circ. Arc	P	[77]	P	[44]	P	[50]	P	[44]	P	[66]
Edge (or Line)	P	[47]	P	[GJ]	P	[T]	N	[GJ]	N	[49]
Claw-Free	P	[T]	P	[63]	O?		N	[GJ]	N	[49]

Graph Class	ChrInd		HamCir		DomSet		MaxCut		StTree		GraIso	
Trees/Forests	P	[GJ]	P	[T]	P	[GJ]	P	[GJ]	P	[T]	P	[GJ]
Almost Trees (k)	P?		P?		P	[45]	P?		P?		P?	
Partial k-Trees	O?		P	[3]	P	[3]	P?		P?		O?	
Bandwidth-k	P?		P?		P	[64]	P	[64]	P?		P	[58]
Degree-k	N	[49]	N	[GJ]	N	[GJ]	N	[GJ]	N	[GJ]	P	[58]
Planar	O		N	[GJ]	N	[GJ]	P	[GJ]	N	[35]	P	[GJ]
Series Parallel	P	[74]	P	[74]	P	[54]	P	[GJ]	P	[82]	P	[GJ]
Outerplanar	P	[67]	P	[T]	P	[6]	P	[GJ]	P	[81]	P	[GJ]
Halin	P	[74]	P	[T]	P	[6]	P	[GJ]	P?		P	[GJ]
k-Outerplanar	O?		P	[6]	P	[6]	P	[GJ]	P?		P	[GJ]
Grid	P	[GJ]	N	[51]	N	[55]	P	[T]	N	[35]	P	[GJ]
$K_{3,3}$-Free	O?		N	[GJ]	N	[GJ]	P	[5]	N	[GJ]	O?	
Thickness-k	N	[49]	N	[GJ]	N	[GJ]	N	[7]	N	[GJ]	O?	
Genus-k	O?		N	[GJ]	N	[GJ]	O?		N	[GJ]	P	[61]
Perfect	O?		N	[1]	N	[14]	O?		N	[GJ]	I	[GJ]
Chordal	O?		N	[22]	N	[14]	O?		N	[83]	I	[GJ]
Split	O?		N	[22]	N	[19]	O?		N	[83]	I	[15]
Strongly Chordal	O?		O?		P	[32]	O?		P	[83]	O?	
Comparability	O?		N	[1]	N	[28]	O?		N	[GJ]	I	[GJ]
Bipartite	P	[GJ]	N	[1]	N	[28]	P	[T]	N	[GJ]	I	[GJ]
Permutation	O?		O		P	[33]	O?		P	[23]	P	[21]
Cographs	O?		P	[25]	P	[33]	O?		P	[23]	P	[25]
Undirected Path	O?		O?		N	[16]	O?		O?		I	[GJ]
Directed Path	O?		O?		P	[16]	O?		P	[83]	O?	
Interval	O?		P	[53]	P	[16]	O?		P	[83]	P	[57]
Circular Arc	O?		O?		P	[13]	O?		P	[83]	O?	
Circle	O?		P	[12]	O?		O?		P	[70]	O?	
Proper Circ. Arc	O?		P	[12]	P	[13]	O?		P	[83]	O?	
Edge (or Line)	O?		N	[11]	N	[GJ]	O?		N	[70]	I	[15]
Claw-Free	O?		N	[11]	N	[GJ]	O?		N	[70]	I	[15]

Complexity-separating graph classes

	VERTEXCOL	EDGECOL
perfect	P	N
chordal	P	O
split	P	O
strongly chordal	P	O
comparability	P	N
bipartite	P	P
permutation	P	O
cographs	P	O
indifference	P	O
split-indifference	P	P

N: NP-complete P: polynomial O: open

Johnson's NP-completeness column 1985

I. Holyer – *SIAM J. Comput.* 1981

Complexity-separating problems

	VERTEXCOL	EDGECOL
perfect	P	N
chordal	P	O
split	P	O
strongly chordal	P	O
comparability	P	N
bipartite	P	P
permutation	P	O
cographs	P	O
indifference	P	O
split-indifference	P	P

N: NP-complete P: polynomial O: open

L. Cai, J. Ellis – *Discrete Appl. Math.* 1991

Spinrad's book 2003

Complexity-separating problems

	VERTEXCOL	EDGECOL
perfect	P	N
chordal	P	O
split	P	O
strongly chordal	P	O
comparability	P	N
bipartite	P	P
permutation	P	O
cographs	P	O
indifference	P	O
split-indifference	P	P

N: NP-complete P: polynomial O: open

C. Ortiz Z., N. Maculan, J. Szwarcfiter – *Discrete Appl. Math.* 1998

C. Simone, C. Mello – *Theoret. Comput. Sci.* 2006

Full dichotomies

Classes of problems for which every problem is classified into P or NP-complete

Problems: EDGE COLORING, TOTAL COLORING

Graph classes: unichord-free, split-indifference, chordless

Unichord-free graphs

χ-bounded graph class: $\chi \leq f(\omega)$

Perfect graph: $\chi = \omega$

Line graph: $\chi \leq \omega + 1$, the Vizing bound

A. Gyárfás – *Zastos. Mat.* 1987

Unichord-free graphs

χ-bounded graph class: $\chi \leq f(\omega)$

Perfect graph: $\chi = \omega$

Line graph: $\chi \leq \omega + 1$, the Vizing bound

Which choices of forbidden induced subgraphs give χ-bounded class?

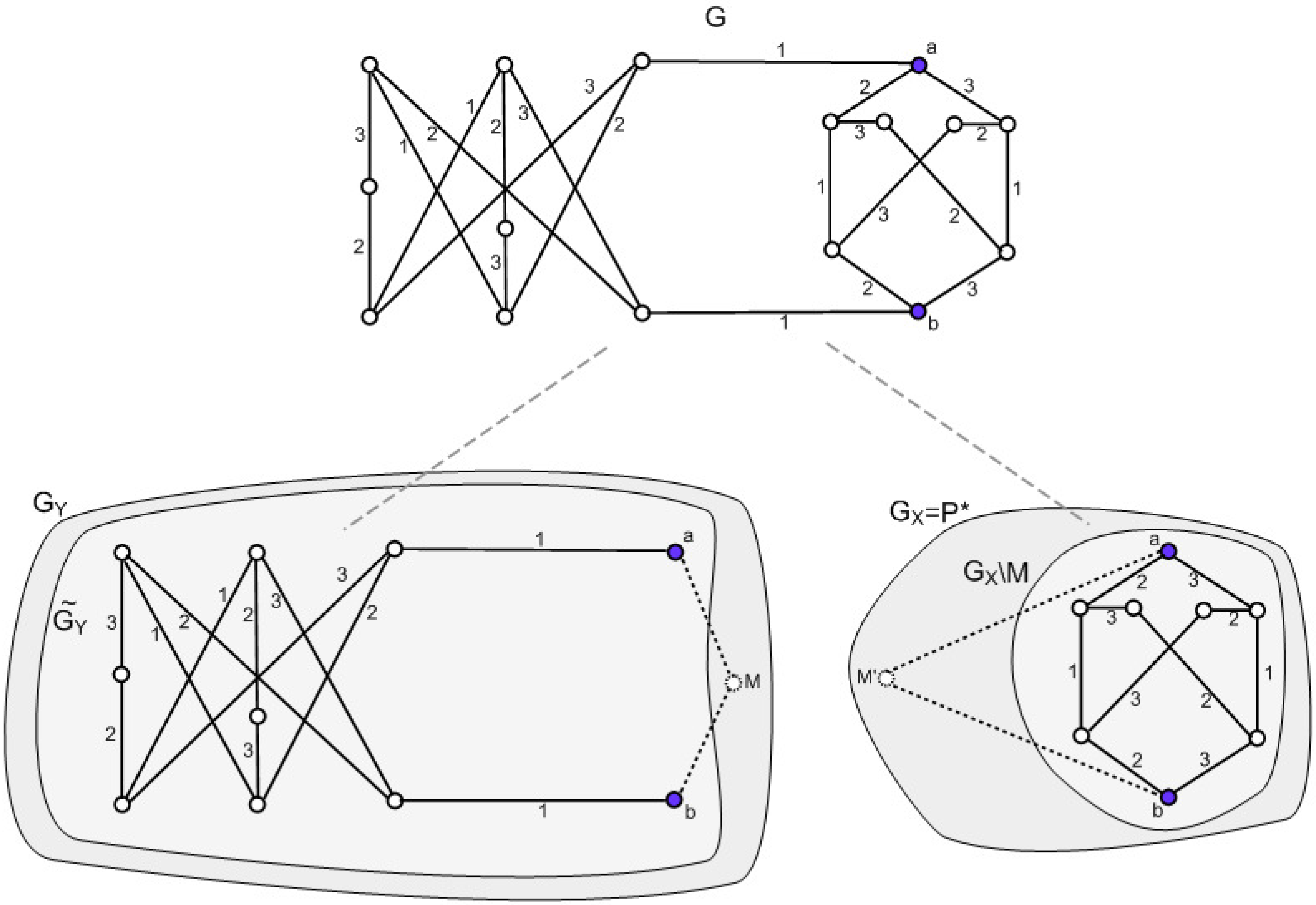
Unichord-free graphs: $\chi \leq \omega + 1$

Structure theorem:

every graph in the class can be built from basic graphs

N. Trotignon, K. Vušković – *J. Graph Theory* 2009

Combining edge-colorings with respect to 2-cutset



Decomposition with respect to a proper 2-cutset $\{a, b\}$
G is Class 1: Δ colors suffice, but $G_X = P^*$ is Class 2: $\Delta + 1$ colors needed

Edge-coloring unichord-free graphs

Class $\mathcal{C}$ = unichord-free graphs

	$\Delta = 3$	$\Delta \geq 4$	regular
graphs of $\mathcal{C}$	N	N	N
4-hole-free graphs of $\mathcal{C}$	N	P	P
6-hole-free graphs of $\mathcal{C}$	N	N	N
$\{$4-hole, 6-hole$\}$-free graphs of $\mathcal{C}$	P	P	P

“Chromatic index of graphs with no cycle with a unique chord”

Theoret. Comput. Sci. 2010 (with Raphael Machado, Kristina Vušković)

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$\{$4-hole, 6-hole$\}$-free graphs of $\mathcal{C}$	P	P	P

EDGECOL is N for k-partite r-regular, for each $k \geq 3$, $r \geq 3$

	$k \leq 2$	$k \geq 3$
k-partite graphs	P	N

“Chromatic index of graphs with no cycle with a unique chord”

Theoret. Comput. Sci. 2010 (with Raphael Machado, Kristina Vušković)

Class 2 = overfull implies EDGECOL is P

Overfull graph: $|E| > \Delta \left\lfloor \frac{|V|}{2} \right\rfloor$

Complete multipartite: Class 2 = overfull

Graphs with a universal vertex: Class 2 = overfull

Split-indifference graphs: Class 2 = subgraph overfull

$\{$4-hole,unichord$\}$-free graphs, with $\Delta \neq 3$: Class 2 = subgraph overfull

D. Hoffman, C. Rodger – *J. Graph Theory* 1992

M. Plantholt – *J. Graph Theory* 1981

C. Ortiz Z., N. Maculan, J. Szwarcfiter – *Discrete Appl. Math.* 1998

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M. Plantholt – *J. Graph Theory* 1981

C. Ortiz Z., N. Maculan, J. Szwarcfiter – *Discrete Appl. Math.* 1998

Conjecture for edge-coloring chordal graphs:

Class 2 = subgraph overfull

“On edge-colouring indifference graphs”

Theoret. Comput. Sci. 1997 (with João Meidanis, Célia Mello)

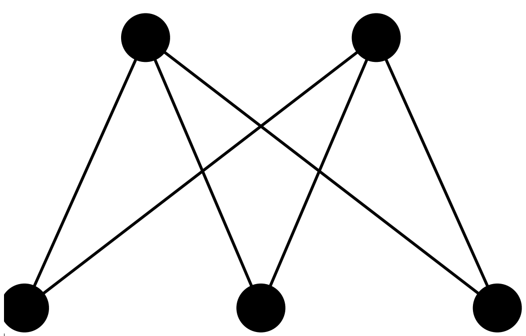
Total coloring conjecture

Vizing's edge coloring theorem: every graph is $(\Delta + 1)$-edge colorable

Total coloring conjecture: every graph is $(\Delta + 2)$-total colorable
Type 1 = $(\Delta + 1)$-total colorable, Type 2 = $(\Delta + 2)$-total colorable

M. Molloy, B. Reed – *Combinatorica* 1998

Natural to consider classes of graphs for which TCC is established



TCC for bipartite: 2-color vertices, Δ-color edges

Total coloring is hard

NP-hard for k-regular bipartite

Reduction from edge-coloring

Consider classes of graphs for which edge-coloring is polynomial

Edge-coloring is polynomial for split-indifference graphs

C. McDiarmid, A. Sánchez-Arroyo – *Discrete Math.* 1994

C. Ortiz Z., N. Maculan, J. Szwarcfiter – *Discrete Appl. Math.* 1998

Type 2 = Hilton condition implies TOTALCOL is P

	Δ even	Δ odd
complete	Type 1	Type 2 (Hilton condition)
univ. vertex	Type 1	Hilton condition
split	Type 1	open
indifference	Type 1	open
split-indifference	Type 1	Hilton condition
3 max cliques	Type 1	open

A. Hilton – *Discrete Math.* 1989

What is the largest class of graphs for which:

G Type 2 iff Hilton condition holds for closed neighborhood of Δ vertex

Necessary condition:

Δ even implies Type 1

“The total chromatic number of split-indifference graphs”

Discrete Math. 2012 (with Christiane Campos, Raphael Machado, Célia Mello)

Total chromatic number of unichord-free graphs

	VERTEXCOL	EDGECOL	TOTALCOL
unichord-free	P	N	N
{4-hole,unichord}-free, $\Delta \geq 4$	P	P	P
{4-hole,unichord}-free, $\Delta = 3$	P	N	P

Surprising full-dichotomy wrt EDGECOL:

$\Delta \geq 4$ is polynomial whereas $\Delta = 3$ is NP-complete

Surprising complexity-separating graph class:

EDGECOL is NP-complete whereas TOTALCOL is polynomial

"Complexity of colouring problems restricted to unichord-free and {square,unichord}-free graphs", *Discrete Appl. Math.* 2014 (with Raphael Machado and Nicolas Trotignon)

Edge coloring chordless graphs

G is chordless iff $L(G)$ is wheel-free

Chordless, with $\Delta = 3$ is Class 1 implies
$\{$wheel,$ISK_4\}$-free is 3 vertex colorable

B. Lévêque, F. Maffray, N. Trotignon – J. Comb. Theory, Ser. B 2012

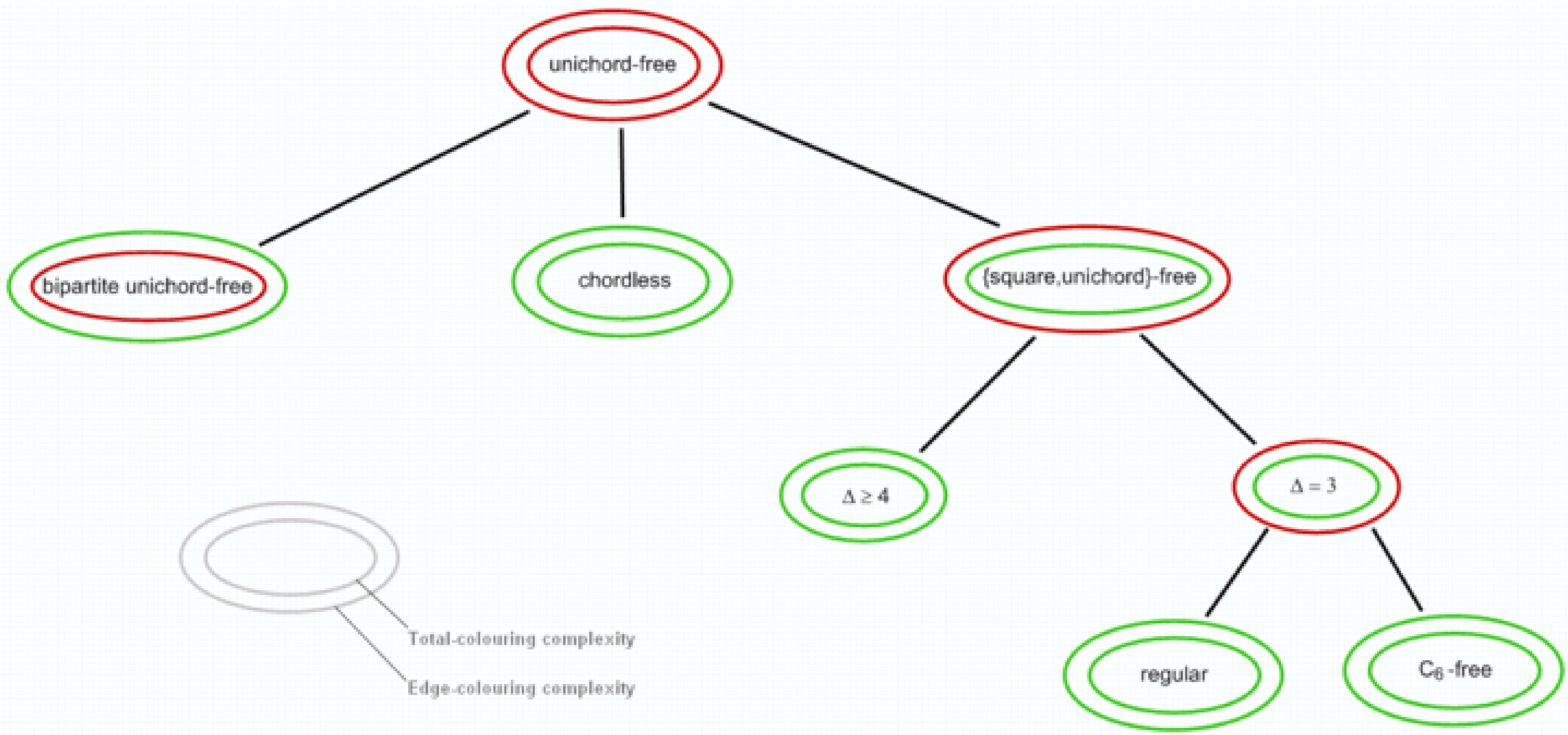
Chordless is a subclass of unichord-free
EDGECOL is NP-complete for unichord-free graphs

Every chordless, with $\Delta > 3$ is Class 1

"Edge-colouring and total-colouring chordless graphs"

Discrete Math. 2013 (with Raphael Machado and Nicolas Trotignon)

Edge and total coloring complexity-separating classes

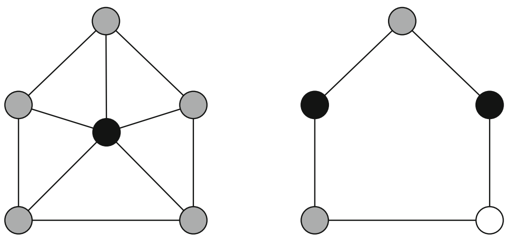


When restricted to $\{$square,unichord$\}$-free graphs,
edge coloring is NP-complete whereas total coloring is polynomial

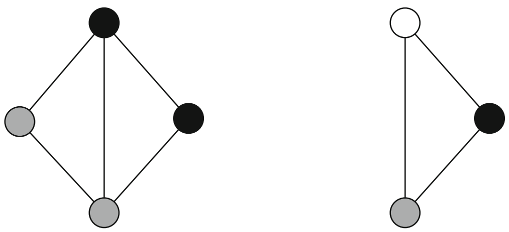
Clique-colouring unichord-free graphs

A clique-colouring of G is an assignment of colours to the vertices of G such that no inclusion-wise maximal clique of size at least 2 is monochromatic

Colouring of hypergraphs arising from graphs: clique, biclique



subgraphs may even have a larger clique-chromatic number



subgraphs may even have a larger biclique-chromatic number

“Efficient algorithms for clique-colouring and biclique-colouring unichord-free graphs”
Algorithmica 2017 (with Hélio Macedo and Raphael Machado)

Complexity restricted to unichord-free and special subclasses

Colouring problem \ class	General	Unichord-free	{□, unichord}-free	{△, unichord}-free
Vertex-col.	$\mathcal{NPC}$ [14]	$\mathcal{P}$ [26]	$\mathcal{P}$ [26]	$\mathcal{P}$ [26]
Edge-col.	$\mathcal{NPC}$ [13]	$\mathcal{NPC}$ [18]	$\mathcal{NPC}$ [18]	$\mathcal{NPC}$ [18]
Total-col.	$\mathcal{NPC}$ [21]	$\mathcal{NPC}$ [17]	$\mathcal{P}$ [16,17]	$\mathcal{NPC}$ [17]
Clique-col.	$\Sigma_2^p\mathcal{C}$ [20]	$\boldsymbol{\mathcal{P}}$	$\boldsymbol{\mathcal{P}}$	$\mathcal{P}$ $(\kappa = \chi)$
Biclique-col.	$\Sigma_2^p\mathcal{C}$ [10]	$\boldsymbol{\mathcal{P}}$	$\boldsymbol{\mathcal{P}}$	$\boldsymbol{\mathcal{P}}$ $(\boldsymbol{\kappa_B = 2})$

[10] M. Groshaus, F. Soulignac, P. Terlisky – *J. Graph Algorithms Appl.* 2014

[20] D. Marx – *Theoret. Comput. Sci.* 2011

“Efficient algorithms for clique-colouring and biclique-colouring unichord-free graphs”

Algorithmica 2017 (with Hélio Macedo and Raphael Machado)

Are all perfect graphs 3-clique-colourable?

Every diamond-free perfect graph is 3-clique-colourable

G. Bacsó, S. Gravier, A. Gyárfás, M. Preissmann, A. Sebő – *SIAM J. Discrete Math.* 2004

M. Chudnovsky, I. Lo – *J. Graph Theory* 2017

Every unichord-free graph is 3-clique-colourable
A unichord-free graph is 2-clique-colourable if and only if it is perfect

"Efficient algorithms for clique-colouring and biclique-colouring unichord-free graphs"

Algorithmica 2017 (with Hélio Macedo and Raphael Machado)

Every graph is easy or hard: dichotomy theorems for graph problems

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Budapest, Hungary

ICGT 2014
Grenoble, France
July 3, 2014

Dichotomy theorems

- Dichotomy theorems give good research programs: easy to formulate, but can be hard to complete.
- The search for dichotomy theorems may uncover algorithmic results that no one has thought of.
- Proving dichotomy theorems may require good command of both algorithmic and hardness proof techniques.