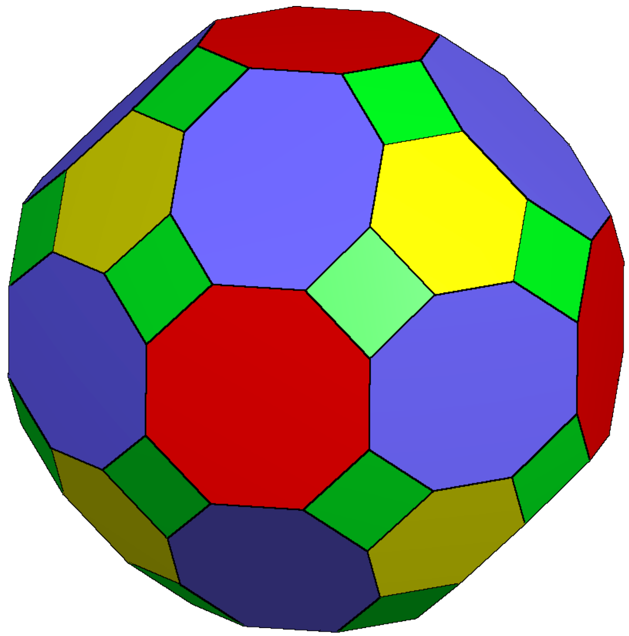
Computational determination of the largest lattice polytope diameter



Antoine Deza, Paris Sud

based on joint works with: **Nathan Chadder**, McMaster

George Manoussakis, Paris Sud

Lionel Pournin, Paris XIII

Shmuel Onn, Technion

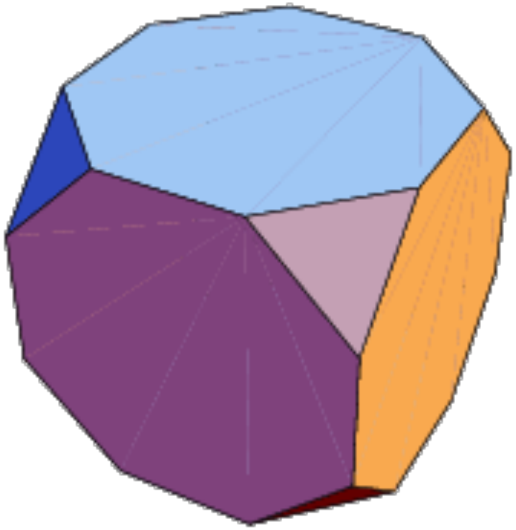
Lattice polytopes with large diameter

lattice (***d***,***k***)-polytope : convex hull of points drawn from $\{0,1,\ldots,k\}^d$

diameter δ(***P***) of polytope ***P*** : smallest number such that **any two vertices** of ***P*** can be connected by a **path with at most** δ(***P***) **edges**

δ(***d***,***k***): largest diameter over all **lattice** (***d***,***k***)-polytopes

ex. δ(3,3) = 6 and is achieved by a ***truncated cube***



Lattice polytopes with large diameter

lattice (d,k)-polytope : convex hull of points drawn from $\{0,1,\ldots,k\}^d$

diameter $\delta(P)$ of polytope P : smallest number such that **any two vertices** of P can be connected by a **path with at most $\delta(P)$ edges**

$\delta(d,k)$: largest diameter over all **lattice** (d,k)-polytopes

- $\delta(P)$: lower bound for the worst case number of iterations required by *pivoting methods* (simplex) to optimize a linear function over P
- *Hirsch conjecture* : $\delta(P) \leq n - d$ (n number of inequalities) was *disproved* [Santos 2012]

Lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

upper bounds :

$\delta(d,1) \leq d$		[Naddef 1989]
$\delta(2,k) = O(k^{2/3})$		[Balog-Bárány 1991]
$\delta(2,k) = 6(k/2\pi)^{2/3} + O(k^{1/3} \log k)$		[Thiele 1991] [Acketa-Žunić 1995]
$\delta(d,k) \leq kd$		[Kleinschmid-Onn 1992]
$\delta(d,k) \leq kd - \lceil d/2 \rceil$	for $k \geq 2$	[Del Pia-Michini 2016]
$\delta(d,k) \leq kd - \lceil 2d/3 \rceil - (k - 3)$	for $k \geq 3$	[Deza-Pournin 2017]

Lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

lower bounds :

$\delta(d,1) \geq d$ [Naddef 1989]

$\delta(d,2) \geq \lfloor 3d/2 \rfloor$ [Del Pia-Michini 2016]

$\delta(d,k) = \Omega(k^{2/3} d)$ [Del Pia-Michini 2016]

$\delta(d,k) \geq \lfloor (k+1)d/2 \rfloor$ for $k < 2d$ [Deza-Manoussakis-Onn 2017]

Lattice polytopes with large diameter

$\delta(d,k)$		*k*								
		1	2	3	4	5	6	7	8	9
d	2	2								
	3	3								
	4	4								
	5	5								

$\delta(d,1) = d$

[Naddef 1989]

Lattice polytopes with large diameter

δ(*d*,*k*)		*k*								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3								
	4	4								
	5	5								

δ(*d*,1) = *d* [Naddef 1989]

δ(2,*k*) : close form [Thiele 1991] [Acketa-Žunić 1995]

Lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4							
	4	4	6							
	5	5	7							

$\delta(d,1) = d$ [Naddef 1989]

$\delta(2,k)$: close form [Thiele 1991] [Acketa-Žunić 1995]

$\delta(d,2) = \lfloor 3d/2 \rfloor$ [Del Pia-Michini 2016]

Lattice polytopes with large diameter

δ(*d*,*k*)		*k*								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7	9				
	4	4	6	8						
	5	5	7							

δ(*d*,1) = *d* [Naddef 1989]

δ(2,*k*) : close form [Thiele 1991] [Acketa-Žunić 1995]

δ(*d*,2) = $\lfloor 3d/2 \rfloor$ [Del Pia-Michini 2016]

δ(4,3) = 8 [Deza-Pournin 2017]

δ(3,4) = 7, δ(3,5) = 9 [Chadder-Deza 2017]

Lattice polytopes with large diameter

δ(*d*,*k*)		*k*								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7	9	10+	11+	12+	13+
	4	4	6	8	10+	12+	14+	16+	17+	18+
	5	5	7	10+	12+	15+	17+	20+	22+	25+

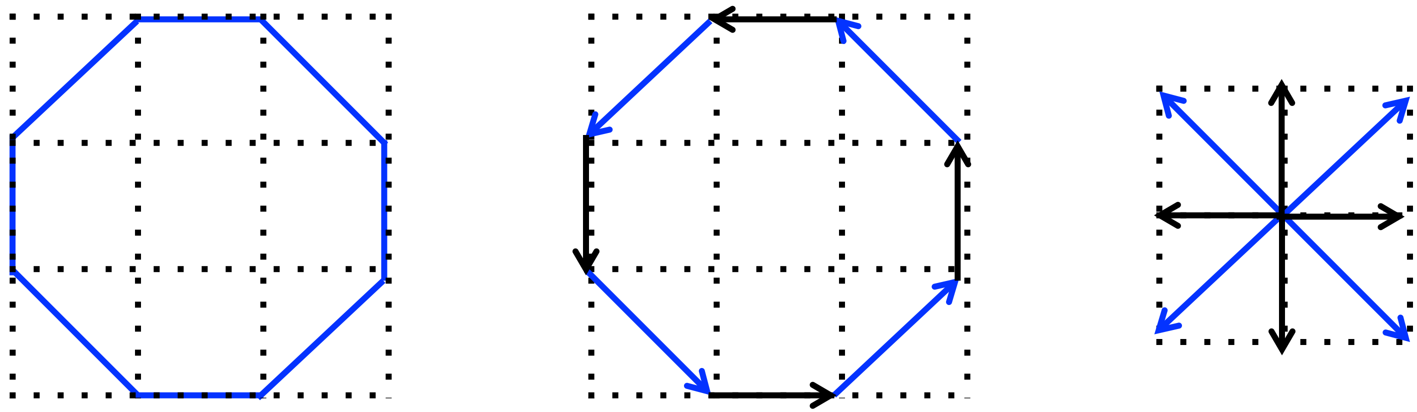
- Conjecture [Deza-Manoussakis-Onn 2017] $\delta(d,k) \le \lfloor (k+1)d/2 \rfloor$

and δ(*d*,k) is achieved, up to translation, by a *Minkowski sum of primitive lattice vectors*. The conjecture holds for all known entries of δ(*d*,*k*)

Lattice polygons with many vertices

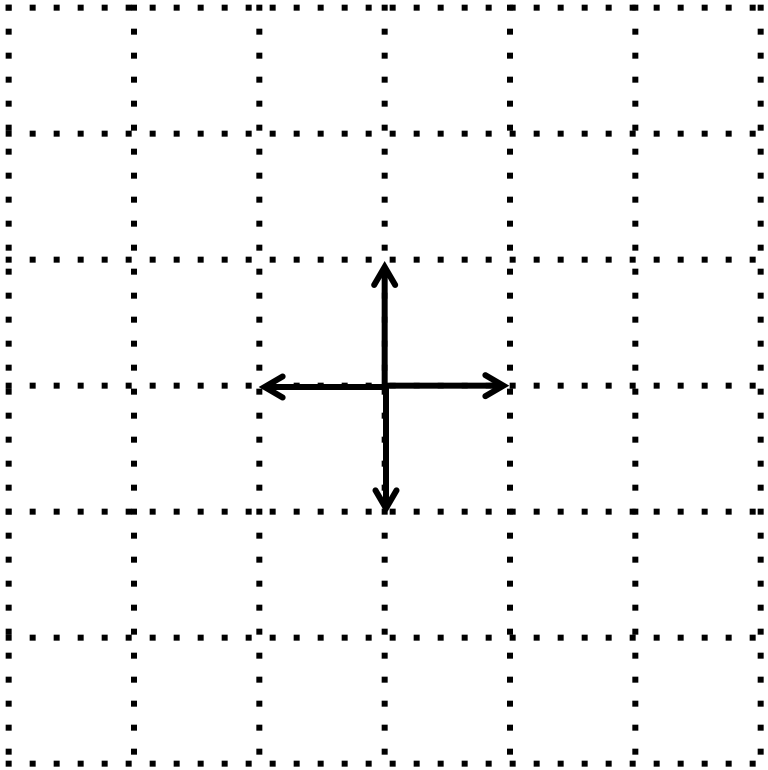
Q. What is δ(2,***k***) : largest diameter of a polygon which vertices are drawn form the ***k*** x ***k*** grid?

A polygon can be associated to a set of vectors (*edges*) *summing up to zero*, and *without a pair of positively multiple vectors*



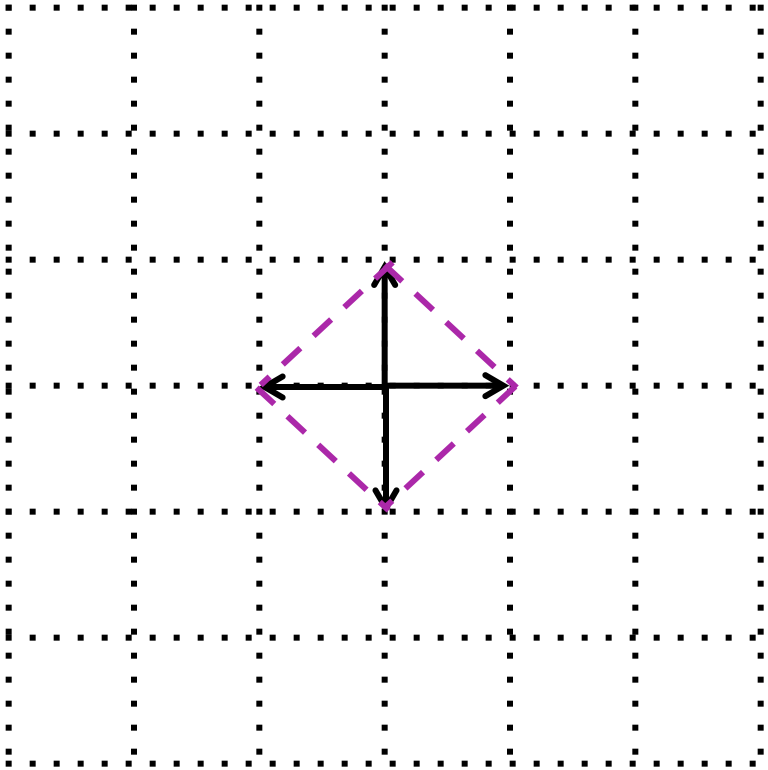
δ(2,3) = 4 is achieved by the 8 vectors : (±1,0), (0,±1), (±1,±1)

Lattice polygons with many vertices



$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

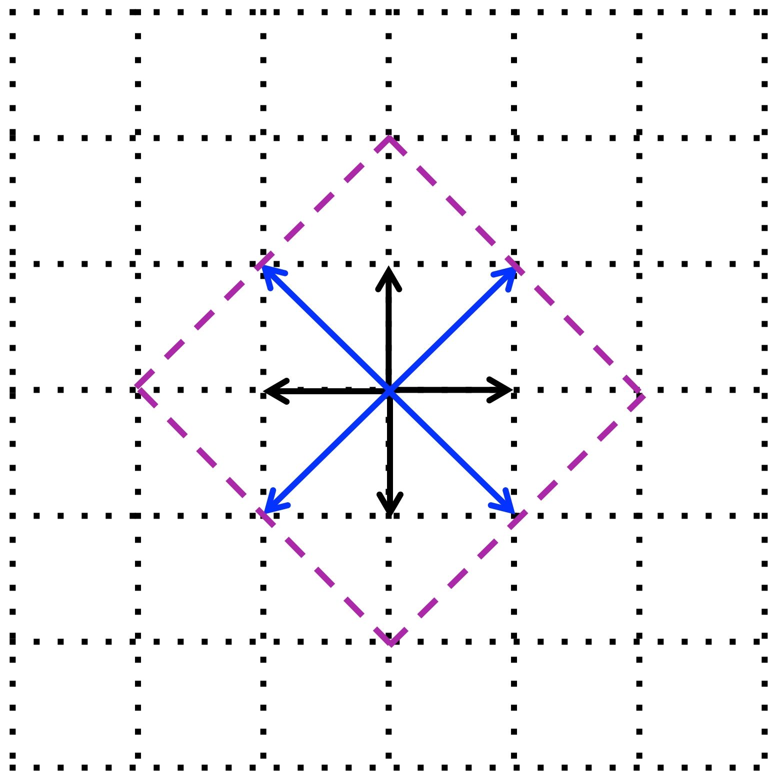
Lattice polygons with many vertices



$||x||_1 \leq 1$

$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

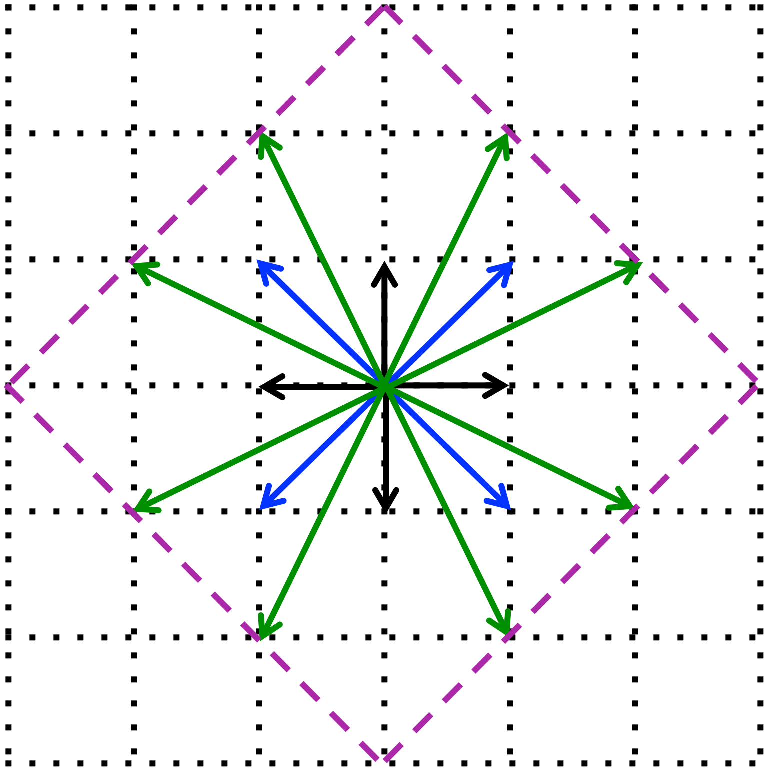
Lattice polygons with many vertices



$||x||_1 \leq 2$

δ(2,2) = 2 ; vectors : (±1,0), (0,±1)
δ(2,3) = 4 ; vectors : (±1,0), (0,±1), (±1,±1)

Lattice polygons with many vertices



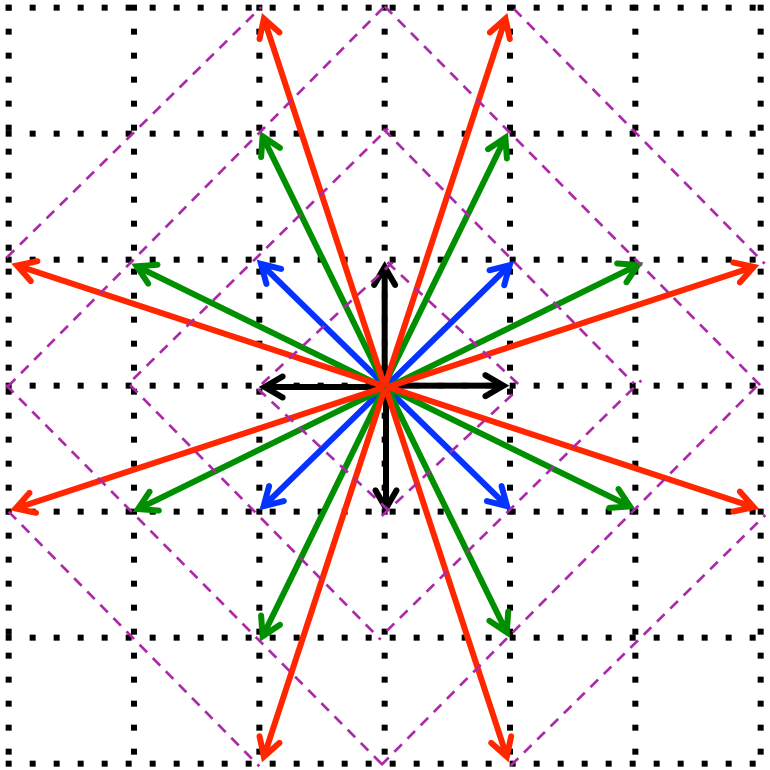
$||x||_1 \leq 3$

$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

$\delta(2,3) = 4$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$

$\delta(2,9) = 8$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$, $(\pm 1,\pm 2)$, $(\pm 2,\pm 1)$

Lattice polygons with many vertices



$||x||_1 \leq p$

$$\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i) \text{ for } k = \sum_{i=1}^{p} i\varphi(i)$$

φ(*p*) : ***Euler totient function*** counting positive integers less or equal to *p* relatively prime with *p*

φ(1) = φ(2) = 1, φ(3) = φ(4) = 2,…

Lattice polygons

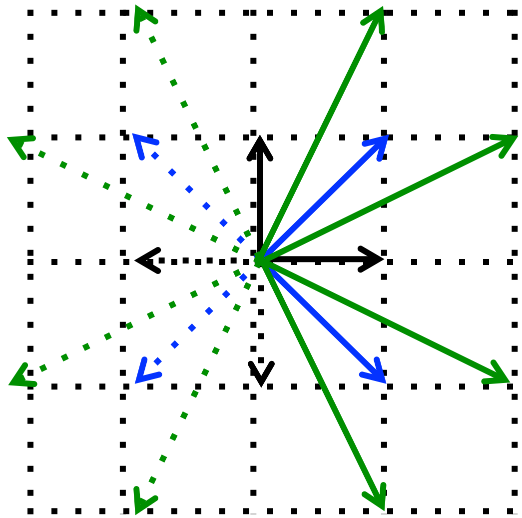
δ(2,*k*)		*k*								
		1	2	3	4	5	6	7	8	9
	p	1		2						3
	v	4	6	8	8	10	12	12	14	16
	δ	2	3	4	4	5	6	6	7	8

$$\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i) \quad \text{for} \quad k = \sum_{i=1}^{p} i\varphi(i)$$

φ(*p*) : ***Euler totient function*** counting positive integers less or equal to *p* relatively prime with *p*

φ(1) = φ(2) = 1, φ(3) = φ(4) = 2,…

Primitive polygons



$||x||_1 \leq p$

$H_1(2,p)$: Minkowski sum generated by $\{x \in \mathbb{Z}^2 : ||x||_1 \leq p, \gcd(x)=1, x \succeq 0\}$

$H_1(2,p)$ has diameter $\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

Ex. $H_1(2,2)$ generated by (1,0), (0,1), (1,1), (1,-1) (fits, *up to translation*, in 3x3 grid)

$x \succeq 0$: first nonzero coordinate of x is nonnegative

Primitive zonotopes

(generalization of the permutahedron of type B_d)

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

Given a set G of m vectors (generators)

Minkowski (G) : convex hull of the 2^m sums of the m vectors in G
Zonotope (G) : convex hull of the 2^m ***signed*** sums of the m vectors in G

up to translation $Z(G)$ is the image of $H(G)$ by an homothety of factor 2

- ***Primitive zonotopes***: zonotopes generated by ***short integer*** vectors which are ***pairwise linearly independent***

Primitive zonotopes

(generalization of the permutahedron of type B_d)

$H_q(d,p)$: Minkowski ($x \in \mathbb{Z}^d$: $||x||_q \leq p$, gcd(x)=1, $x \succeq 0$)

$Z_q(d,p)$: Zonotope ($x \in \mathbb{Z}^d$: $||x||_q \leq p$, gcd(x)=1, $x \succeq 0$)

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $H_q(d, 1)$: $[0, 1]^d$ cube for $q \neq \infty$

Primitive zonotopes

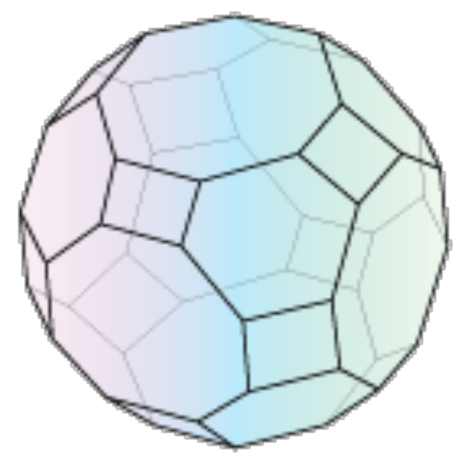
(generalization of the permutahedron of type B_d)

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $Z_1(d,2)$: permutahedron of type B_d



Primitive zonotopes

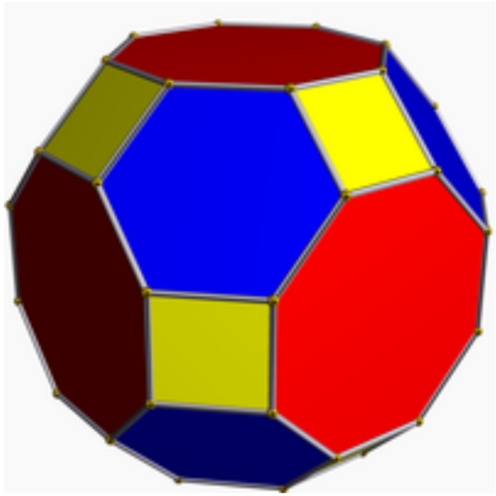
(generalization of the permutahedron of type B_d)

$H_q(d,p)$: Minkowski ($x \in \mathbb{Z}^d$: $||x||_q \leq p$, gcd(x)=1, $x \succeq 0$)

$Z_q(d,p)$: Zonotope ($x \in \mathbb{Z}^d$: $||x||_q \leq p$, gcd(x)=1, $x \succeq 0$)

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $H_1(3,2)$: truncated cuboctahedron (great rhombicuboctahedron)



Primitive zonotopes

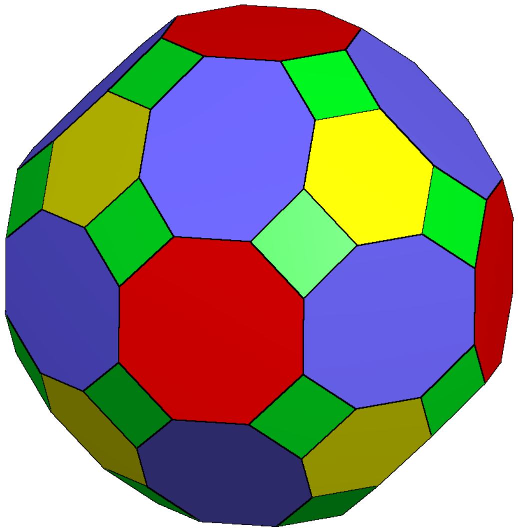
(generalization of the permutahedron of type B_d)

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $H_\infty(3,1)$: truncated small rhombicuboctahedron



Primitive zonotopes

(generalization of the permutahedron of type B_d)

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative
H^+ / Z^+: ***positive*** primitive lattice polytope $x \in \mathbb{Z}^d_+$

- $H_1(d,2)^+$: Minkowski sum of the permutahedron with the $\{0,1\}^d$, i.e., graphical zonotope obtained by the d-clique with a loop at each node

graphical zonotope Z_G: Minkowski sum of segments $[e_i,e_j]$ for all *edges* $\{i,j\}$ of a given graph G

Primitive zonotopes

(generalization of the permutahedron of type B_d)

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative
H^+ / Z^+: ***positive*** primitive lattice polytope $x \in \mathbb{Z}^d_+$

- For $k < 2d$, Minkowski sum of a subset of the generators of $H_1(d,2$ is, up to translation, a lattice (d,k)-polytope with diameter $\lfloor (k+1)d/2 \rfloor$

Lattice polytopes with large diameter

$\delta(d,k)$		*k*								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7	9	10+	11+	12+	13+
	4	4	6	8	10+	12+	14+	16+	17+	18+
	5	5	7	10+	12+	15+	17+	20+	22+	25+

- Conjecture [Deza-Manoussakis-Onn 2017] $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$

and $\delta(d,k)$ is achieved, up to translation, by a *Minkowski sum of primitive lattice vectors*. The conjecture holds for all known entries of $\delta(d,k)$

Lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7	9	10	11	12	13
	4	4	6	8	10	12	14	16	17	18
	5	5	7	10	12	15	17	20	22	25

- Conjecture [Deza-Manoussakis-Onn 2017] $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$

and $\delta(d,k)$ is achieved, up to translation, by a *Minkowski sum of primitive lattice vectors*. The conjecture holds for all known entries of $\delta(d,k)$

Computational determination of $\delta(d,k)$

Given a lattice (d,k)-polytope P, two vertices u and v such that $\delta(P) = d(u,v)$, then $d(u,v) \leq \delta(d-1,k) + k$ and $d(u,v) < \delta(d-1,k) + k$ unless:

- $u+v = (k,k,\ldots,k)$,
- any edge of P with u or v as vertex is $\{-1,0,1\}$-valued,
- any intersection of P with a facet of the cube $[0,k]^d$ is a $(d-1)$-dimensional face of P of diameter $\delta(d-1,k)$.

These conditions, combined with combinatorial properties, drastically reduce the search space for a lattice (d,k)-polytope P such that $\delta(P) = \delta(d-1,k) + k$

Computationally ruling out $\delta(d,k) = \delta(d-1,k) + k$ and using $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$ for $k < 2d$ yields : $\delta(3,4) = 7$ and $\delta(3,5) = 9$

i.e. : δ(great rhombicuboctahedron) = $\delta(3,5)$

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A034997 Number of Generalized Retarded Functions in Quantum Field Theory. 1

2, 6, 32, 370, 11292, 1066044, 347326352, 419172756930 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS a(d) is the number of parts into which d-dimensional space (x_1,...,x_d) is split by a set of (2^d - 1) hyperplanes c_1 x_1 + c_2 x_2 + ...+ c_d x_d =0 where c_j are 0 or +1 and we exclude the case with all c=0.

Also, a(d) is the number of independent real-time Green functions of Quantum Field Theory produced when analytically continuing from euclidean time/energy (d+1 = number of energy/time variables). These are also known as Generalized Retarded Functions.

The numbers up to d=6 were first produced by T. S. Evans using a Pascal program, strictly as upper bounds only. M. van Eijck wrote a C program using a direct enumeration of hyperplanes which confirmed these and produced the value for d=7. Kamiya et al. showed how to find these numbers and some associated polynomials using more sophisticated methods, giving results up to d=7. T. S. Evans added the last number on Aug 01 2011 using an updated version of van Eijck's program, which took 7 days on a standard desktop computer.

REFERENCES

Björner, Anders. "Positive Sum Systems", in Bruno Benedetti, Emanuele Delucchi, and Luca Moci, editors, Combinatorial Methods in Topology and

370, 11292, 1066044, 347326352, 419172756930 (list; graph; refs; listen; history; text; internal format)

1,1

a(d) is the number of parts into which d-dimensional space (x_1,...,x_d) is split by a set of (2^d - 1) hyperplanes c_1 x_1 + c_2 x_2 + ...+ c_d x_d =0 where c_j are 0 or +1 and we exclude the case with all c=0.

Also, a(d) is the number of independent real-time Green functions of Quantum Field Theory produced when analytically continuing from euclidean time/energy (d+1 = number of energy/time variables). These are also known as Generalized Retarded Functions.

The numbers up to d=6 were first produced by T. S. Evans using a Pascal program, strictly as upper bounds only. M. van Eijck wrote a C program using a direct enumeration of hyperplanes which confirmed these and produced the value for d=7. Kamiya et al. showed how to find these numbers and some associated polynomials using more sophisticated methods, giving results up to d=7. T. S. Evans added the last number on Aug 01 2011 using an updated version of van Eijck's program, which took 7 days on a standard desktop computer.

Björner, Anders. "Positive Sum Systems", in Bruno Benedetti, Emanuele Delucchi, and Luca Moci, editors, Combinatorial Methods in Topology and Algebra. Springer International Publishing, 2015. 157-171.

T. S. Evans, N-point finite temperature expectation values at real times, Nuclear Physics B 374 (1992) 340-370.

H. Kamiya, A. Takemura and H. Terao, Ranking patterns of unfolding models of codimension one, Advances in Applied Mathematics 47 (2011) 379 - 400.

M. van Eijck, Thermal Field Theory and Finite-Temperature Renormalisation Group, PhD thesis, Univ. Amsterdam, 4th Dec. 1995.

Table of n, a(n) for n=1..8.

L. J. Billera, J. T. Moore, C. D. Moraites, Y. Wang and K. Williams, Maximal unbalanced families, arXiv preprint arXiv:1209.2309, 2012. - From N. J. A. Sloane, Dec 26 2012

Computational determination of the number of vertices of primitive zonotopes

Sloane OEI sequences

$H_\infty(d,1)^+$ vertices : A034997 = number of generalized retarded functions in quantum Field theory (determined till d =8)

$H_\infty(d,1)$ vertices : A009997 = number of regions of hyperplane arrangements with {-1,0,1}-valued normals in dimension d (determined till d =7)

Estimating the number of vertices of $H_\infty(d,1)^+$

[Odlyzko 1988], [Zuev 1992], [Kovijanić-Vukićević 2007]

$$d^2\ (1-o(1)) \leq \log_2 |\ H_\infty(d,1)^+ \ | \leq d^2$$

Lattice polytopes with large diameter and many vertices

$\delta(d,k)$: largest diameter over all lattice (d,k)-polytopes

- **Conjecture** : $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$ and $\delta(d,k)$ is achieved, up to translation, by a Minkowski sum of primitive lattice vectors (holds for all known $\delta(d,k)$)

  $$\Rightarrow \delta(d,k) = \lfloor (k+1)d/2 \rfloor \text{ for } k < 2d$$

- determination of $\delta(3,k)$ and of $\delta(d,3)$? ($\delta(d,3) = 2d$?)

- Convex matroid optimization [Melamed-Onn 2012, Deza-Manoussakis-Onn 2016]

- Answer to [Colbourn-Kocay-Stinson 1986] question:
  Deciding if a given integer sequence is the degree sequence of a 3-hypergraph is NP-complete [Deza-Levin-Meesum-Onn 2017]

Lattice polytopes with large diameter and many vertices

$\delta(d,k)$: largest diameter over all lattice (d,k)-polytopes

- **Conjecture** : $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$ and $\delta(d,k)$ is achieved, up to translation, by a Minkowski sum of primitive lattice vectors (holds for all known $\delta(d,k)$)

  $\Rightarrow \delta(d,k) = \lfloor (k+1)d/2 \rfloor$ for $k < 2d$

- determination of $\delta(3,k)$ and of $\delta(d,3)$? ($\delta(d,3) = 2d$?)

- Convex matroid optimization [Melamed-Onn 2012, Deza-Manoussakis-Onn 2016]

- Answer to [Colbourn-Kocay-Stinson 1986] question:
  Deciding if a given integer sequence is the degree sequence of a 3-hypergraph is NP-complete [Deza-Levin-Meesum-Onn 2017]

✓ ***thank you***

Convex Matroid Optimization

The optimal solution of max { **f**(**W**x) : $x \in$ **S**} is attained at a vertex of the projection integer polytope in $\mathbb{R}^{d}$: conv(**WS**) = **W**conv(**S**)

S : set of feasible point in $\mathbb{Z}^{n}$ (in the talk **S** $\in \{0,1\}^{n}$)
W : integer d x n matrix (**W** is mostly {0,1,…, p}-valued)
f : convex function from $\mathbb{R}^{d}$ to $\mathbb{R}$

Q. What is the maximum number **v**(d,n) of vertices of conv(**WS**) when **S** $\in \{0,1\}^{n}$ and **W** is a {0,1}-valued d x n matrix ?

obviously v(d,n) $\leq$ |**WS**| = $O(n^{d})$
in particular v(2,n) = $O(n^{2})$, and v(2,n) = $\Omega(n^{0.5})$

Convex Matroid Optimization

The optimal solution of max { **f**(**W**x) : $x \in$ **S**} is attained at a vertex of the projection integer polytope in $\mathbb{R}^d$: conv(**WS**) = **W**conv(**S**)

S : set of feasible point in $\mathbb{Z}^n$ (in the talk **S** $\in \{0,1\}^n$)
W : integer d x n matrix (**W** is mostly {0,1,…, p}-valued)
f : convex function from $\mathbb{R}^d$ to $\mathbb{R}$

Q. What is the maximum number **v**(d,n) of vertices of conv(**WS**) when **S** $\in \{0,1\}^n$ and **W** is a {0,1}-valued d x n matrix ?

obviously v(d,n) $\leq$ |**WS**| = $O(n^d)$
in particular v(2,n) = $O(n^2)$, and v(2,n) = $\Omega(n^{0.5})$

[Melamed-Onn 2014] Given matroid **S** of order n and {0,1,…,p}-valued d x n matrix **W**, the maximum number **m**(d,p) of vertices of conv(**WS**) is independent of n and **S**

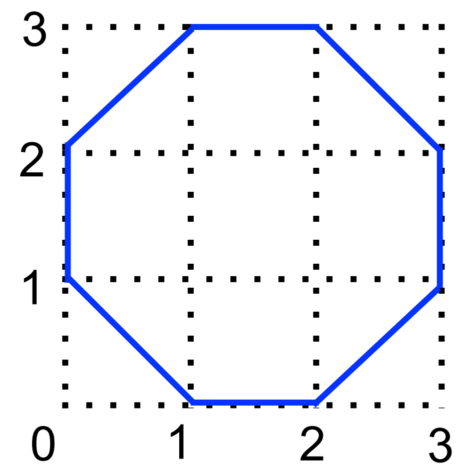
Convex Matroid Optimization

[Melamed-Onn 2014] Given matroid **S** of order ***n*** and {0,1,…,***p***}-valued ***d*** x ***n*** matrix **W**, the maximum number **m**(***d***,***p***) of vertices of conv(**WS**) is independent of ***n*** and **S**

Ex: maximum number **m**(2,1) of vertices of a planar projection conv(**WS**) of matroid **S** by a binary matrix **W** is attained by the following matrix and uniform matroid of rank 3 and order 8:

$$\mathbf{W} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{S} = U(3,8) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$



conv(**WS**)

Convex Matroid Optimization

[Melamed-Onn 2014] Given matroid **S** of order ***n*** and $\{0,1,\ldots,p\}$-valued ***d*** x ***n*** matrix **W**, the maximum number **m**(***d***,***p***) of vertices of conv(**WS**) is independent of ***n*** and **S**

[Deza-Manoussakis-Onn 2016] Given matroid **S** of order ***n***, $\{0,1,\ldots,p\}$-valued ***d*** x ***n*** matrix **W**, maximum number **m**(***d***,***p***) of vertices of conv(**WS**) is equal to the number of vertices of $H_\infty(d,p)$

$$\mathbf{m}(d,p) = | H_\infty(d,p) |$$

Convex Matroid Optimization

[Melamed-Onn 2014] Given matroid **S** of order ***n*** and {0,1,…,***p***}-valued ***d*** x ***n*** matrix **W**, the maximum number **m**(***d***,***p***) of vertices of conv(**WS**) is independent of ***n*** and **S**

[Deza-Manoussakis-Onn 2016] Given matroid **S** of order ***n***, {0,1,…,***p***}-valued ***d*** x ***n*** matrix **W**, maximum number **m**(***d***,***p***) of vertices of conv(**WS**) is equal to the number of vertices of H_∞(***d***,***p***)

$$\mathbf{m}(d,p) = |\, H_\infty(d,p) \,|$$

[Melamed-Onn 2014]

$$d\, 2^d \le \mathbf{m}(d,1) \le 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i}$$

m(2,1) = 8

24 ≤ **m**(3,1) ≤ 158

64 ≤ **m**(4,1) ≤ 19840

[Deza-Manoussakis-Onn 2016]

$$d!\, 2^d \le \mathbf{m}(d,1) \le 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i} - f(d)$$

m(3,1) = **96**

m(4,1) = **5376**

$$\mathbf{m}(2,p) = 8\sum_{i=1}^{p}\varphi(i)$$

Primitive Zonotopes

(complexity questions)

For ***fixed*** ***p*** and ***q***, linear optimization over $Z_q(d,p)$ is polynomial-time solvable, even in ***variable*** dimension ***d*** (polynomial number of generators)

⇒ for ***fixed*** positive ***integers*** ***p*** and ***q***, the following problems are polynomial time solvable:

- ***extremality***: given $x \in \mathbb{Z}^d$, decide if x is a vertex of $Z_q(d,p)$
- ***adjacency***: given $x_1, x_2 \in \mathbb{Z}^d$, decide if $[x_1, x_2]$ is an edge of $Z_q(d,p)$
- ***separation***: given rational $y \in \mathbb{R}^d$, either assert $y \in Z_q(d,p)$, or find $h \in \mathbb{Z}^d$ separating y from $Z_q(d,p)$ i.e, satisfying $h^T y > h^T x$ for all $x \in Z_q(d,p)$

Primitive Zonotopes

(complexity questions)

For ***fixed*** ***p*** and ***q***, linear optimization over $Z_q(d,p)$ is polynomial-time solvable, even in ***variable*** dimension ***d*** (polynomial number of generators)

⇒ for ***fixed*** positive ***integers*** ***p*** and ***q***, the following problems are polynomial time solvable:

- ***extremality***: given $x \in \mathbb{Z}^d$, decide if x is a vertex of $Z_q(d,p)$
- ***adjacency***: given $x_1,x_2 \in \mathbb{Z}^d$, decide if $[x_1,x_2]$ is an edge of $Z_q(d,p)$
- ***separation***: given rational $y \in \mathbb{R}^d$, either assert $y \in Z_q(d,p)$, or find $h \in \mathbb{Z}^d$ separating y from $Z_q(d,p)$ i.e, satisfying $h^T y > h^T x$ for all $x \in Z_q(d,p)$

Q. existence of a *direct* algorithm for fixed ***p*** and ***q***

existence of an algorithms for fixed ***p*** and $q = \infty$

existence of *hole* : $x \in H_q(d,p)^+ \cap \mathbb{Z}^d$ which can not be written as a sum of a subset of generators of $H_q(d,p)^+$

Primitive Zonotopes

(complexity questions)

$\boldsymbol{D_d}$: convex hull of the degree sequences of all hypergraphs on $\boldsymbol{d}$ nodes

$$\boldsymbol{D_d} = H_\infty(\boldsymbol{d},1)+$$

$\boldsymbol{D_d}(\boldsymbol{k})$: convex hull of the degree sequences of all $\boldsymbol{k}$-uniform hypergraphs on $\boldsymbol{d}$ nodes

Q: check whether $x \in \boldsymbol{D_d}(\boldsymbol{k}) \cap \mathbb{Z}^{\boldsymbol{d}}$ is the degree sequence of a $\boldsymbol{k}$-uniform hypergraph. Necessary condition: sum of the coordinates of x is multiple of $\boldsymbol{k}$.

[Erdős-Gallai 1960]: for $\boldsymbol{k} = 2$ (graphs) necessary condition is sufficient

[Liu 2013] exhibited counterexamples (holes) for $\boldsymbol{k} = 3$ (Klivans-Reiner **Q**.)

- Do $H_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})+$ have *hole* : $x \in H_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})+ \cap \mathbb{Z}^{\boldsymbol{d}}$ which can not be written as a sum of a subset of generators of $H_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})+$
- complexity of deciding whether x is a hole?

Lattice polytopes with large diameter and many vertices

$\delta(d,k)$: largest diameter over all lattice (d,k)-polytopes

- **Conjecture** : $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$ and $\delta(d,k)$ is achieved, up to translation, by a Minkowski sum of primitive lattice vectors (holds for all known $\delta(d,k)$)

  $$\Rightarrow \delta(d,k) = \lfloor (k+1)d/2 \rfloor \text{ for } k < 2d$$

- **m**$(d,p) = | H_\infty(d,p) |$

- determination of $\delta(3,k)$ and of $\delta(d,3)$? ($\delta(d,3) = 2d$?)

- complexity issues, e.g. decide whether a given point is a vertex of $Z_\infty(d,1)$

- existence of *hole* : $x \in H_q(d,p)^+ \cap \mathbb{Z}^d$ which can not be written as a sum of a subset of generators of $H_q(d,p)^+$