Sufficient conditions for hyper-Hamiltonicity in graphs

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A graph G is said to be hyper-Hamiltonian when G is Hamiltonian and $G - v$ is also Hamiltonian for any vertex v of G.

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In this work, we present some sufficient conditions to ensure that an arbitrary graph is hyper-Hamiltonian, in analogy to results on Hamiltonicity.

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Theorem 1 *Let G be a graph with $n \geq 3$ vertices, such that for every pair of nonadjacent vertices u and v, $d_G(u)+d_G(v) \geq n+1$. Then G is hyper-Hamiltonian.*

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uw and $vw \in E(G)$;

uw and $vw \notin E(G)$;

$uw \in E(G)$ and $vw \notin E(G)$.

As an immediate consequence we also have an analogous to Dirac's theorem.

Corollary 2 *If* $\delta(G) \geq \frac{n+1}{2}$ *then* G *is hyper-Hamiltonian.*

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◇ $\mathbb{P}_n$: the graph obtained from the complete graph on n vertices by adding a pendent vertex;

◇ $\mathbb{P}_n + e$ the graph obtained from $\mathbb{P}_n$ by inserting an edge.

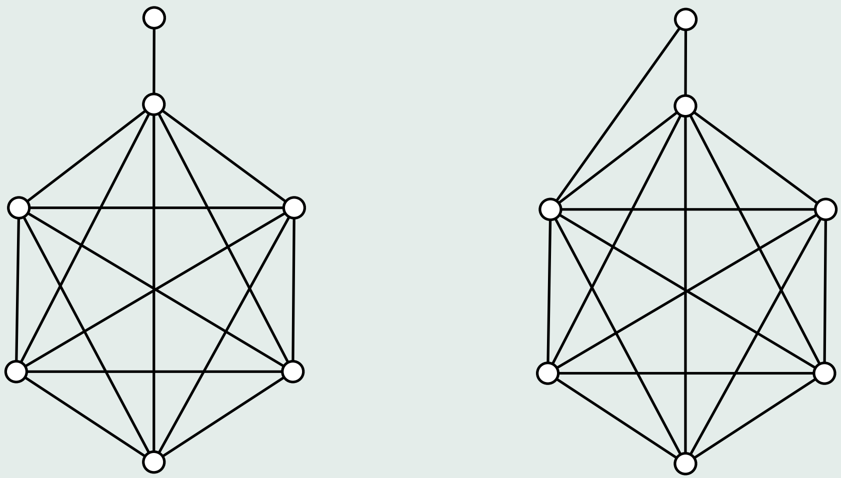

Figure 1: $\mathbb{P}_6$ **and** $\mathbb{P}_6 + e$

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Theorem 3 *Let G be a graph with $n \geq 3$ vertices and m edges. If $m \geq \frac{n^2-3n+6}{2}$ then G is hyper-Hamiltonian or $G = \mathbb{P}_{n-1} + e$.*

Definition 4 ([2]) *For an integer $k > 0$, the k-closure of the graph G is a graph obtained from G by successively joining pairs of nonadjacent vertices whose degree sum is at least k until no such pair remains.*

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The k-closure of a graph allows to state the following proposition, analogous to one found in [2], for hamiltonian graphs.

Proposition. 5 *A graph G on n vertices is hyper-Hamiltonian if, and only if, the $(n+1)$-closure of G is hyper-Hamiltonian.*

2. Spectral conditions for hyper-Hamiltonicity

2.1. Conditions based on spectral radius of adjacency matrix

- Let $G = (V, E)$ be a simple undirected graph on n vertices;

- Adjacency matrix of G: $A(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \sim v_j \\ 0, & \text{otherwise.} \end{cases}$$

- spectral radius of $A(G)$: $\lambda(G)$, (the largest eigenvalue of $A(G)$).

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These results motivated many other spectral conditions for Hamiltonicity, as in [13] and [12], for instance.

Theorem 6 *Let* G *be a graph with* n *vertices. If* $\lambda(G) > -\frac{1}{2} + \sqrt{\left(n - \frac{3}{2}\right)^2 + 2}$ *then* G *is hyper-Hamiltonian or* $G = \mathbb{P}_{n-1} + e$.

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By Stanley's inequality: $\lambda(G) \leq -\frac{1}{2} + \sqrt{2m + \frac{1}{4}}$, where m is the number of edges in G)

$$m \geq \frac{n^2 - 3n + 6}{2},$$

which allows the use of Theorem 3, concluding the proof.

Theorem 7 *Let G be a graph with n vertices and $\lambda(\overline{G})$ be the spectral radius of its complement $\overline{G}$. If $\lambda(\overline{G}) \leq \sqrt{\binom{n-2}{2} - \binom{n-2}{n}}$ then G is hyper-Hamiltonian or $G = \mathbb{P}_{n-1} + e$.*

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Furthermore, $\forall u, v \in I, u \nsim v$, $d_I(u) + d_I(v) \leq n$.
Applying Hofmeister's inequality to the complement $\overline{I}$, we have:

$$\sqrt{\frac{1}{n}(d^2(v_1) + \ldots + d^2(v_n))} \leq \lambda(\overline{I})$$

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As $\overline{I} \subseteq \overline{G}$, we have $\lambda(\overline{I}) \leq \lambda(\overline{G}) \leq \sqrt{\binom{n-2}{2} - \binom{n-2}{n}}$.
This implies $m(\overline{I}) \leq \frac{n}{2} - 1$.
After some algebraic manipulation,if $G \neq \mathbb{P}_{n-1} + e$, we achieve a contradiction.

2.2. Conditions based on spectral radius of the Signless Laplacian matrix

- $Deg(G)$: the diagonal matrix whose entries are the vertex degrees of G;

- signless Laplacian matrix of G: $Q(G) = Deg(G) + A(G)$;

- spectral radius of $Q(G)$: $q_1(G)$, (the largest eigenvalue of $Q(G)$).

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Similar to what is done in [13], we obtain a condition for hyper-Hamiltonicity, based on this parameter.

Consider the set $\mathcal{E}_n$ of graphs G on n vertices such that:

- $G = P_2 \vee (K_a \cup K_{n-a-2})$, $a < n - 2$;
- G is $n/2$-regular , n even;
- $G = H \vee F$, H is $\left(\frac{n}{2} - r\right)$ -regular and $\mid F \mid = r < \frac{n}{2}$

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Theorem 8 *Let G be a graph with n vertices, for $n \geq 3$. If $q_1(\overline{G}) \leq n-2$ and $G \notin \mathcal{E}_n$ then G is hyper-Hamiltonian.*

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Theorem 9 *Let G be a connected graph with $n \geq 2$ vertices. If $\rho(G) < \dfrac{(n-1)(n+2)-2}{n}$ then G is hyper-Hamiltonian or $G = \mathbb{P}_{n-1} + e$.*

Theorem 10 *Let G be a graph with $n \geq 2$ vertices, such that $\overline{G}$ is connected. If $\rho(\overline{G}) > n - \frac{5}{2} + 3\sqrt{\left(n - \frac{3}{2}\right)^2 + 2}$ then G is hyper-Hamiltonian or $G = \mathbb{P}_{n-1} + e$.*

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Sketch of the proof: In [12] it is proved that: If $d(u) + d(v) \geq n$ for every pair of adjacent vertices u and v of a graph, then this graph has diameter no greater than 4.

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As $d_{\overline{G}}(u) + d_{\overline{G}}(v) \geq n$, we obtain that $D(\overline{G}) \leq J_n - I_n + 3A(G)$, where J_n is the $n \times n$ all one matrix and I_n, the identity matrix.

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So, $\rho(\overline{G}) < n - 1 + 3\lambda(G)$. Then we can apply Hofmeister's inequality to $\lambda(G)$ and get the result.

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- $L(G) = Deg(G) - A(G)$.
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$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0.$$

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$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0.$$

- μ_{n-1} is called the algebraic connectivity of G and denoted $a(G)$.

Threshold graphs are graphs free of P_4, C_4 and $2K_2$.

Hamiltonicity in threshold graphs is studied in [6] under a non spectral approach.

In [10] it is shown that Laplacian eigenvalues of a threshold graph can be obtained from its degree sequence. This result and Theorem 1 imply the following theorem.

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Theorem 11 *Let G be a threshold graph with n vertices. If $\mu_{n-1} + \mu_{n-2} \geq n + 1$ then G is hyper-Hamiltonian.*

An immediate consequence is the next corollary.

Corollary 12 *Let G be a threshold graph with n vertices. If $a(G) \geq \frac{n+1}{2}$ then G is hyper-Hamiltonian.*

We may note that different matrices do not produce the same conclusion considering hyperhmiltonicity of graphs as can be seen in the following example:

Exemplo 13 *Hyper-Hamiltonian graph* G_1 *with 10 vertices and non connected complement.*

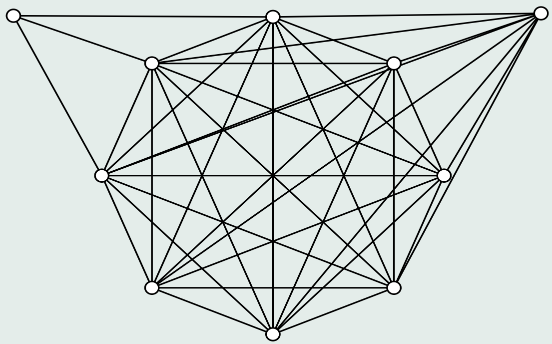

Figure 2: G_1.

G_1 has $m = 39$, $\lambda(G_1) = 8,126$, $\lambda(\overline{G_1}) = 2,44$, $q_1(\overline{G_1}) = 7$, $\rho(G_1) = 10,43$, $\mu_{n-1}(G_1) = 3$ and $\mu_{n-2}(G_1) = 9$.

The graph G_1 satisfies the conditions based on:

- number of edges;
- $\lambda(G_1)$;
- $q_1(G_1)$;
- $\rho(G_1)$;
- $\mu_{n-1}(G_1) + \mu_{n-2}(G_1)$.

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but it does not satisfy conditions:

- $\lambda(\overline{G_1})$;
- algebraic connectivity.

Exemplo 14 *Hyper-Hamiltonian graph G_2 with 10 vertices and non connected complement.*

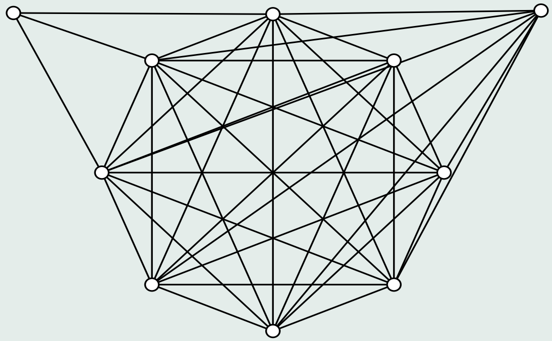

Figure 3: G_2.

G_2 has $m = 38$, $\lambda(G_2) = 7,93$, $\lambda(\overline{G_2}) = 2,68$, $q_1(\overline{G_2}) = 7,13$, $\rho(G_2) = 10,64$, $\mu_{n-1}(G_2) = 3$ and $\mu_{n-2}(G_2) = 7$.

The graph G_2 satisfies the conditions based on:

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- number of edges;
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but it does not satisfy conditions:

- $\lambda(G_2)$;
- $\lambda(\overline{G_2})$;
- $\rho(G_2)$;
- $\mu_{n-1}(G_2) + \mu_{n-2}(G_2)$

Thank you

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