On the recognition of neighborhood inclusion posets

Pablo De Caria

CONICET/ Departamento de Matemática, Universidad Nacional de La Plata

IX LAGOS, Marseille, September 2017

Posets

A poset is a pair $(X, \leq)$ consisting of a set X and a partial order $\leq$ defined on X.

Inclusion Poset

The elements of X are sets, ordered by inclusion.

Posets

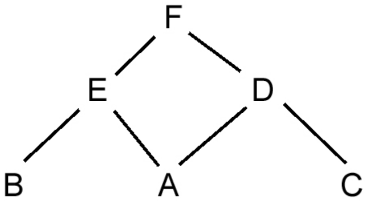
A poset is a pair $(X, \leq)$ consisting of a set X and a partial order $\leq$ defined on X.

Inclusion Poset

The elements of X are sets, ordered by inclusion.

Example

$A = \{1, 2\}$, $B = \{2, 3, 4\}$, $C = \{1, 5\}$, $D = \{1, 2, 5\}$,
$E = \{1, 2, 3, 4\}$, $F = \{1, 2, 3, 4, 5\}$

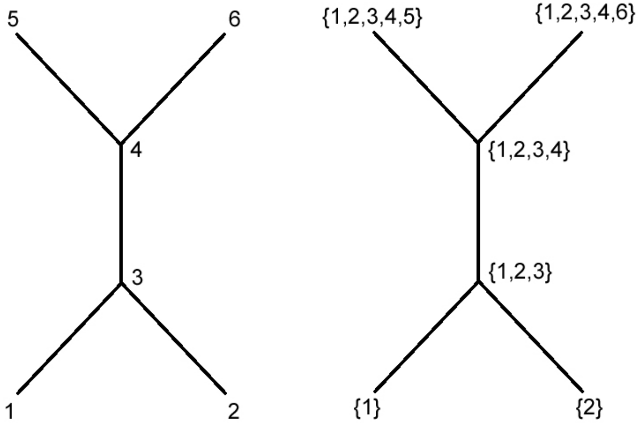


Every poset is an inclusion poset

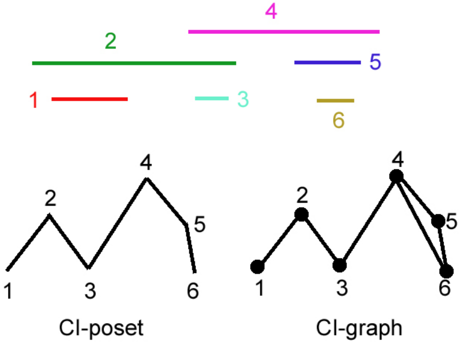
Make every $x \in X$ correspond to the set $\{y \in X : \ y \leq x\}$

Every poset is an inclusion poset

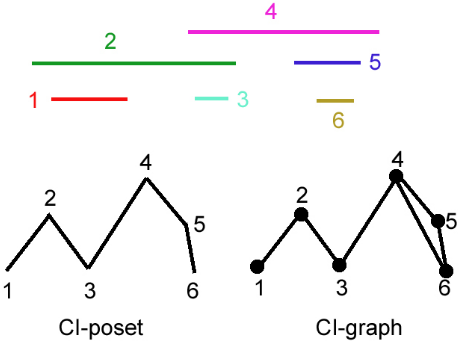
Make every $x \in X$ correspond to the set $\{y \in X : \; y \leq x\}$



Interval inclusion posets



Interval inclusion posets



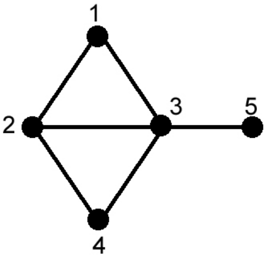
Theorem

The following are equivalent:

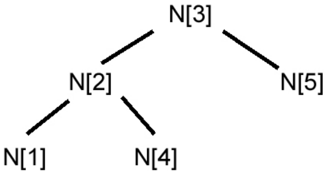
- G is a CI-graph.
- G and its complement are comparability graphs.
- G is a permutation graph.

Neighborhood inclusion posets

Its elements are the different closed neighborhoods of a graph.



N[1]={1,2,3}
N[2]={1,2,3,4}
N[3]={1,2,3,4,5}
N[4]={2,3,4}
N[5]={3,5}



Neighborhood inclusion poset recognition problem

Given a poset P, is it the neighborhood inclusion poset of some graph?

Neighborhood inclusion poset recognition problem

Given a poset P, is it the neighborhood inclusion poset of some graph?

Not always the answer is yes.

If a graph had this poset, then it would be complete. But the neighborhood inclusion poset of a complete graph has only one neighborhood.

Neighborhood Inclusion Poset Recognition is NP-complete

Set basis problem

Given: a collection $\mathcal{C}$ of subsets of a set S and an integer $k \leq |\mathcal{C}|$.
Question: Is there a collection $\mathcal{B}$ of subsets of S with $|\mathcal{B}| = k$ such that, for every $C \in \mathcal{C}$, there is a subcollection of $\mathcal{B}$ whose union is exactly $\mathcal{C}$?

The problem was proved to be NP-complete by Stockmeyer in 1975.

Neighborhood Inclusion Poset Recognition is NP-complete

Set basis problem

Given: a collection $\mathcal{C}$ of subsets of a set S and an integer $k \leq |\mathcal{C}|$.
Question: Is there a collection $\mathcal{B}$ of subsets of S with $|\mathcal{B}| = k$ such that, for every $C \in \mathcal{C}$, there is a subcollection of $\mathcal{B}$ whose union is exactly $\mathcal{C}$?

The problem was proved to be NP-complete by Stockmeyer in 1975.

Set basis, intersection version

Given: a collection $\mathcal{C}$ of subsets of a set S and an integer $k \leq |\mathcal{C}|$.
Question: Is there a collection $\mathcal{B}$ of subsets of S with $|\mathcal{B}| = k$ such that, for every $C \in \mathcal{C}$, there is a subcollection of $\mathcal{B}$ whose intersection is exactly $\mathcal{C}$?

Note: $B_1 \cap B_2 \cap \ldots \cap B_n = C \Longrightarrow \overline{B_1} \cup \overline{B_2} \cup \ldots \cup \overline{B_n} = \overline{C}$

A polynomial reduction

$S = \{1,2,3,4,5\}$, $C_1 = \{1,3\}$, $C_2 = \{2,3\}$, $C_3 = \{1,2,3,5\}$, $C_4 = \{2,3,4,5\}$, $k = 2$

A polynomial reduction

$S = \{1, 2, 3, 4, 5\}$, $C_1 = \{1, 3\}$, $C_2 = \{2, 3\}$, $C_3 = \{1, 2, 3, 5\}$, $C_4 = \{2, 3, 4, 5\}$, $k = 2$

Check whether there is an element that is in all the sets. If not, add it.

A polynomial reduction

$S = \{1, 2, 3, 4, 5\}$, $C_1 = \{1, \mathbf{3}\}$, $C_2 = \{2, \mathbf{3}\}$, $C_3 = \{1, 2, \mathbf{3}, 5\}$, $C_4 = \{2, \mathbf{3}, 4, 5\}$, $k = 2$

Check whether there is an element that is in all the sets. If not, add it.

A polynomial reduction

$S = \{1, 2, 3, 4, 5\}$, $C_1 = \{1, \mathbf{3}\}$, $C_2 = \{2, \mathbf{3}\}$, $C_3 = \{1, 2, \mathbf{3}, 5\}$, $C_4 = \{2, \mathbf{3}, 4, 5\}$, $k = 2$

Check whether there is an element that is in all the sets. If not, add it.

Add new elements x_1, x_2, x_3, x_4 (as many as sets the collection has) and, for every i, add to C_i the elements x_j such that $C_j \subseteq C_i$.

A polynomial reduction

$S = \{1, 2, 3, 4, 5, x_1, x_2, x_3, x_4\}$, $C_1 = \{1, 3, x_1\}$, $C_2 = \{2, 3, x_2\}$, $C_3 = \{1, 2, 3, 5, x_1, x_2, x_3\}$, $C_4 = \{2, 3, 4, 5, x_2, x_4\}$, $k = 2$

Check whether there is an element that is in all the sets. If not, add it.

Add new elements x_1, x_2, x_3, x_4 (as many as sets the collection has) and, for every i, add to C_i the elements x_j such that $C_j \subseteq C_i$.

A polynomial reduction

$S = \{1, 2, 3, 4, 5, x_1, x_2, x_3, x_4\}$, $C_1 = \{1, 3, x_1\}$, $C_2 = \{2, 3, x_2\}$, $C_3 = \{1, 2, 3, 5, x_1, x_2, x_3\}$, $C_4 = \{2, 3, 4, 5, x_2, x_4\}$, $k = 2$

Check whether there is an element that is in all the sets. If not, add it.

Add new elements x_1, x_2, x_3, x_4 (as many as sets the collection has) and, for every i, add to C_i the elements x_j such that $C_j \subseteq C_i$.

If there are elements that appear in exactly the same sets, leave only one of them.

A polynomial reduction

$S = \{1, 2, 3, 4, 5, x_1, x_2, x_3, x_4\}$, $C_1 = \{1, 3, x_1\}$, $C_2 = \{2, 3, x_2\}$, $C_3 = \{1, 2, 3, 5, x_1, x_2, x_3\}$, $C_4 = \{2, 3, 4, 5, x_2, x_4\}$, $k = 2$

Check whether there is an element that is in all the sets. If not, add it.

Add new elements x_1, x_2, x_3, x_4 (as many as sets the collection has) and, for every i, add to C_i the elements x_j such that $C_j \subseteq C_i$.

If there are elements that appear in exactly the same sets, leave only one of them.

A polynomial reduction

$S = \{1, 2, 3, 4, 5, x_3\}$, $C_1 = \{1, 3\}$, $C_2 = \{2, 3\}$,
$C_3 = \{1, 2, 3, 5, x_3\}$, $C_4 = \{2, 3, 4, 5\}$, $k = 2$

Check whether there is an element that is in all the sets. If not, add it.

Add new elements x_1, x_2, x_3, x_4 (as many as sets the collection has) and, for every i, add to C_i the elements x_j such that $C_j \subseteq C_i$.

If there are elements that appear in exactly the same sets, leave only one of them.

A polynomial reduction

$S = \{1, 2, 3, 4, 5, x_3\}$, $C_1 = \{1, 3\}$, $C_2 = \{2, 3\}$, $C_3 = \{1, 2, 3, 5, x_3\}$, $C_4 = \{2, 3, 4, 5\}$, $k = 2$

Check whether there is an element that is in all the sets. If not, add it.

Add new elements x_1, x_2, x_3, x_4 (as many as sets the collection has) and, for every i, add to C_i the elements x_j such that $C_j \subseteq C_i$.

If there are elements that appear in exactly the same sets, leave only one of them.

The solution to the question is still the same.

A polynomial reduction

$S = \{1, 2, 3, 4, 5, x_3\}$, $C_1 = \{1, 3\}$, $C_2 = \{2, 3\}$,
$C_3 = \{1, 2, 3, 5, x_3\}$, $C_4 = \{2, 3, 4, 5\}$, $k = 2$

Check whether there is an element that is in all the sets. If not, add it.

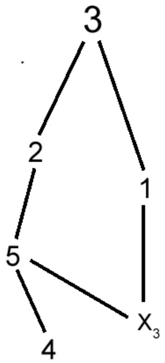
Add new elements x_1, x_2, x_3, x_4 (as many as sets the collection has) and, for every i, add to C_i the elements x_j such that $C_j \subseteq C_i$.

If there are elements that appear in exactly the same sets, leave only one of them.

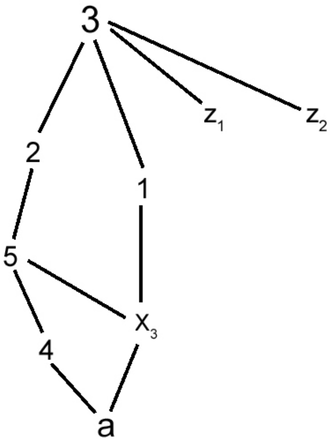
The solution to the question is still the same.

Consider the partial order on S where $a \leq b$ when every set of the collection that has a also has b.

$S = \{1,2,3,4,5,x_3\}$, $C_1 = \{1,3\}$, $C_2 = \{2,3\}$,
$C_3 = \{1,2,3,5,x_3\}$, $C_4 = \{2,3,4,5\}$, $k = 2$

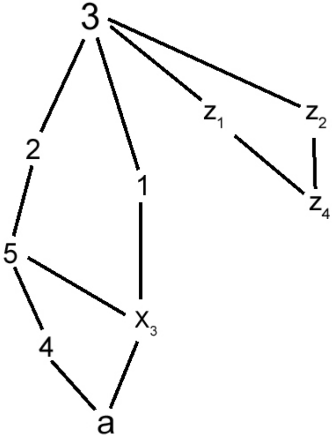


$S = \{1, 2, 3, 4, 5, x_3\}$, $C_1 = \{1, 3\}$, $C_2 = \{2, 3\}$,
$C_3 = \{1, 2, 3, 5, x_3\}$, $C_4 = \{2, 3, 4, 5\}$, $k = 2$



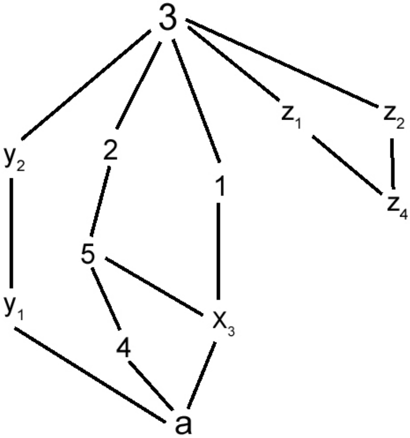
Reason: In case the poset corresponds to a graph, the sets $N[z_1]$ and $N[z_2]$ will tell us about the solution to the set basis problem.

$S = \{1, 2, 3, 4, 5, x_3\}$, $C_1 = \{1, 3\}$, $C_2 = \{2, 3\}$,
$C_3 = \{1, 2, 3, 5, x_3\}$, $C_4 = \{2, 3, 4, 5\}$, $k = 2$



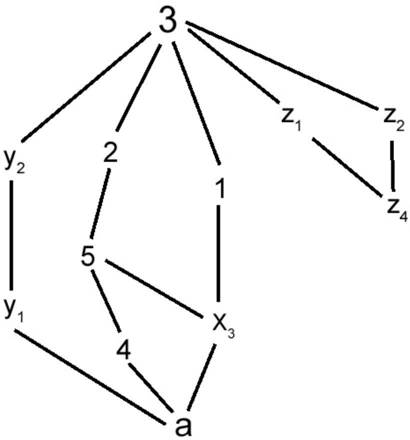
Reason: to facilitate that the neighborhood of no z_i is contained in the neighborhood of a vertex in S different from 3.

$S = \{1, 2, 3, 4, 5, x_3\}$, $C_1 = \{1, 3\}$, $C_2 = \{2, 3\}$,
$C_3 = \{1, 2, 3, 5, x_3\}$, $C_4 = \{2, 3, 4, 5\}$, $k = 2$



Reason: to avoid that the neighborhood of z_1 is contained in the neighborhood of z_2

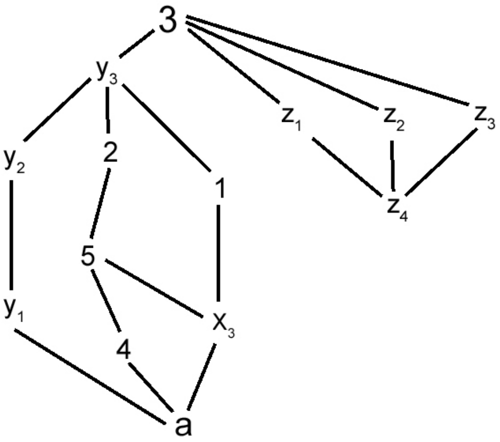
$S = \{1, 2, 3, 4, 5, x_3\},\ C_1 = \{1, 3\},\ C_2 = \{2, 3\},$
$C_3 = \{1, 2, 3, 5, x_3\},\ C_4 = \{2, 3, 4, 5\},\ k = 2$



Reason: to avoid that the neighborhood of z_1 is contained in the neighborhood of z_2

Problem: If z_1 is a neighbor of y_1 and z_1,z_2 are neighbors of y_2, then any vertex in S\{3} adjacent to z_1 can contain the neighborhood of y_1.

$S = \{1,2,3,4,5,x_3\}$, $C_1 = \{1,3\}$, $C_2 = \{2,3\}$,
$C_3 = \{1,2,3,5,x_3\}$, $C_4 = \{2,3,4,5\}$, $k = 2$



Solution: $N[y_1]$ has z_1
$N[y_2]$ has z_1, z_2
$N[y_3]$ has z_1, z_2, z_3

No vertex of S\{3} can be adjacent to z_1

A motivation for the problem

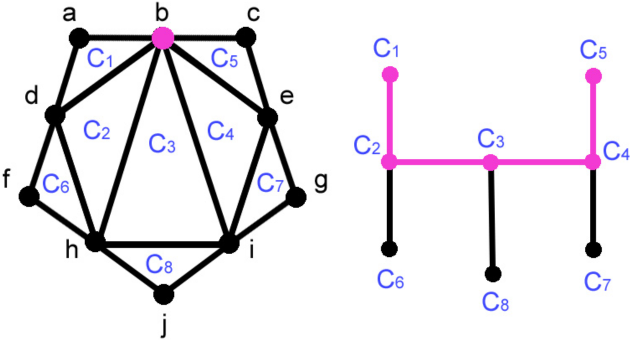
Chordal graphs and the clique tree

The vertices of a clique tree of a graph G are all its maximal cliques and, for every $v \in V(G)$, the set $\mathcal{C}_v$ of maximal cliques of G that contain v induces a subtree.

A motivation for the problem

Chordal graphs and the clique tree

The vertices of a clique tree of a graph G are all its maximal cliques and, for every $v \in V(G)$, the set $\mathcal{C}_v$ of maximal cliques of G that contain v induces a subtree.



A motivation for the problem

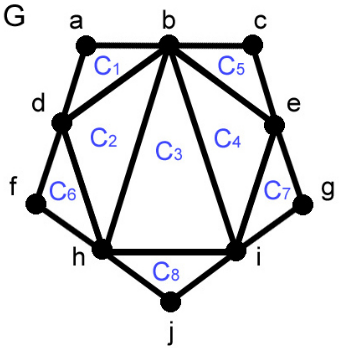
Dually chordal graphs and the compatible tree

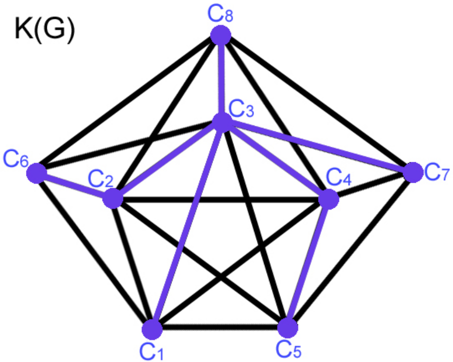
The vertices of a compatible tree are those of the graph G and every maximal clique C induces a subtree.

A motivation for the problem

Dually chordal graphs and the compatible tree

The vertices of a compatible tree are those of the graph G and every maximal clique C induces a subtree.





Recognizing families of clique trees and compatible trees

Fact 1: It can be decided in polynomial time whether a family of trees is the family of all the clique trees of some chordal graph.

Fact 2: If G is a dually chordal and $\mathcal{F}$ is the family that has all the maximal cliques of G and the sets of the form $\{v\}$, where $v \in V(G)$, then the intersection graph of $\mathcal{F}$ is chordal and its clique trees are exactly the compatible trees of G.
Therefore, a family of compatible trees is always a family of clique trees.

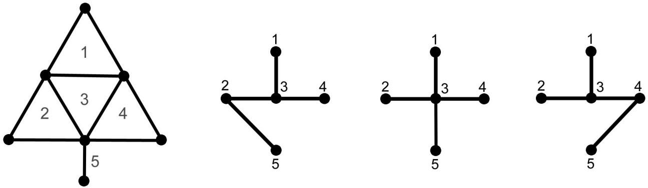
Fact 3: Not every family of clique trees of a chordal graph is the family of compatible trees of a dually chordal graph.

Recognizing families of clique trees and compatible trees

Fact 1: It can be decided in polynomial time whether a family of trees is the family of all the clique trees of some chordal graph.

Fact 2: If G is a dually chordal and $\mathcal{F}$ is the family that has all the maximal cliques of G and the sets of the form $\{v\}$, where $v \in V(G)$, then the intersection graph of $\mathcal{F}$ is chordal and its clique trees are exactly the compatible trees of G.
Therefore, a family of compatible trees is always a family of clique trees.

Fact 3: Not every family of clique trees of a chordal graph is the family of compatible trees of a dually chordal graph.



Proposition

Let G be a dually chordal graph with universal vertex u. Then a spanning tree T of G is a compatible tree if and only if, for every $v \in V(G)$, the set $\{w \in V(G) : N[v] \subseteq N[w]\}$ induces a subtree of T.

Idea of proof:

For every maximal clique C, $C = \bigcup_{v \in C} \{w \in V(G) : N[v] \subseteq N[w]\}$

For every $v \in V(G)$, $\{w \in V(G) : N[v] \subseteq N[w]\} = \bigcap_{v \in C} C$

Proposition

Let G be a dually chordal graph with universal vertex u. Then a spanning tree T of G is a compatible tree if and only if, for every $v \in V(G)$, the set $\{w \in V(G): \ N[v] \subseteq N[w]\}$ induces a subtree of T.

Idea of proof:

For every maximal clique C, $C = \bigcup_{v \in C} \{w \in V(G): \ N[v] \subseteq N[w]\}$

For every $v \in V(G)$, $\{w \in V(G): \ N[v] \subseteq N[w]\} = \bigcap_{v \in C} C$

Proposition

Let $(X, \leq)$ be a poset with unique maximal element. Let $\mathcal{F}$ be the family that contains, for every $x \in X$, the sets $\{x\}$ and $\{y : \ x \leq y\}$. Then the intersection graph of $\mathcal{F}$ is chordal and every tree T whose vertices are the maximal cliques of this graph is a clique tree if and only if all the sets of the form $\{y : \ x \leq y\}$ induce a subtree of T.

Theorem

Given a chordal graph G, the problem of deciding whether the family of clique trees of G is also the family of compatible trees of some dually chordal graph is NP-complete.

Theorem

Given a chordal graph G, the problem of deciding whether the family of clique trees of G is also the family of compatible trees of some dually chordal graph is NP-complete.

Proof: We find a polynomial reduction of Neighborhood Inclusion Poset Recognition to this problem.

Let $P = (X, \leq)$ be a poset. Let P' be another poset such that $P' = P$ if P has a unique maximal element u.
Otherwise, let $P' = (X \cup \{u\}, \preceq)$ where $\preceq$ is an extension of $\leq$ such that $x \preceq u$ for every $x \in X$.

Theorem

Given a chordal graph G, the problem of deciding whether the family of clique trees of G is also the family of compatible trees of some dually chordal graph is NP-complete.

Proof: We find a polynomial reduction of Neighborhood Inclusion Poset Recognition to this problem.

Let $P = (X, \leq)$ be a poset. Let P' be another poset such that $P' = P$ if P has a unique maximal element u.
Otherwise, let $P' = (X \cup \{u\}, \preceq)$ where $\preceq$ is an extension of $\leq$ such that $x \preceq u$ for every $x \in X$.

P is a neighborhood inclusion poset if and only if P' is a neighborhood inclusion poset.

Let $\mathcal{F}$ be the family that has for every $x \in X \cup \{u\}$ the sets $\{x\}$ and $\{y : \ x \preceq y\}$.

Let G be the intersection graph of $\mathcal{F}$, which is chordal.

Let $\mathcal{F}$ be the family that has for every $x \in X \cup \{u\}$ the sets $\{x\}$ and $\{y : \ x \preceq y\}$.

Let G be the intersection graph of $\mathcal{F}$, which is chordal.

Let us see that P' is a neighborhood inclusion poset if and only if the family of clique trees of G is also the family of compatible trees of a dually chordal graph.

Let $\mathcal{F}$ be the family that has for every $x \in X \cup \{u\}$ the sets $\{x\}$ and $\{y : \; x \preceq y\}$.

Let G be the intersection graph of $\mathcal{F}$, which is chordal.

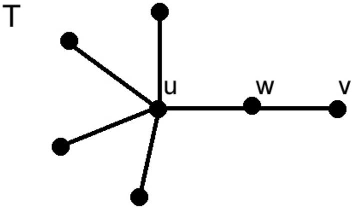
Let us see that P' is a neighborhood inclusion poset if and only if the family of clique trees of G is also the family of compatible trees of a dually chordal graph.

If P' is the Neighborhood inclusion poset of a graph H, then H is dually chordal because it has a universal vertex.

Both the clique trees of G and the compatible trees of H are those for which every set of the form $\{y : \; x \preceq y\}$ induce a subtree, so they are the same.

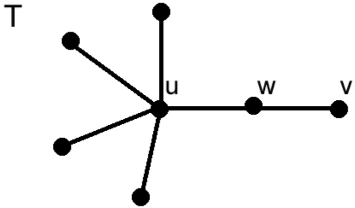
Conversely, let the family of clique trees of G be the family of compatible trees of a dually chordal graph H.

Given two elements v and w, different from u, and the following tree T



Conversely, let the family of clique trees of G be the family of compatible trees of a dually chordal graph H.

Given two elements v and w, different from u, and the following tree T

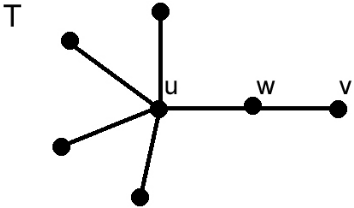


T is a clique tree of G if and only if $v \preceq w$.

T is a compatible tree of H if and only $N_H[v] \subseteq N_H[w]$.

Conversely, let the family of clique trees of G be the family of compatible trees of a dually chordal graph H.

Given two elements v and w, different from u, and the following tree T



T is a clique tree of G if and only if $v \preceq w$.

T is a compatible tree of H if and only $N_H[v] \subseteq N_H[w]$.

Hence $v \preceq w$ if and only if $N_H[v] \subseteq N_H[w]$.

We conclude that P' is the neighborhood inclusion poset of H.