On Type 2 Snarks and Dot Products

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Definition 1

A **total colouring** of a graph is an attribution of colours to its vertices and edges such that two adjacent or incident elements do not have the same colour. The **total chromatic number** ${}_T(G)$ is the least n for which G admits a total colouring with n colours.

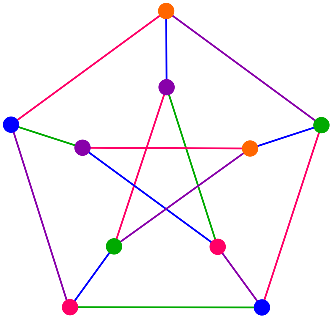

Figure 1: A total colouring

Question 2

Total Colouring Conjecture (Behzad, 1964 - Vizing, 1967)
Every simple graph admits total colouring using at most $\Delta + 2$ colours.

It was proved for **cubic graphs** in 1971, independently, by Rosenfeld and Vijayaditya.
Cubic graphs with $_{T} = 4$ are said to be **Type 1** and cubic graphs with $_{T} = 5$ are said to be **Type 2**.

Definition 3

A **snark** is a cubic cyclically 4-edge connected graph that admits no 3-edge colouring (Class 2).

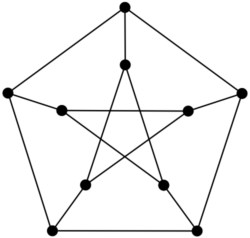

Figure 2: The Petersen graph is the smallest snark

The importance of these graphs arise from the fact that snarks are counterexamples for many conjectures in Graph Theory.

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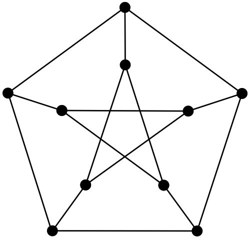

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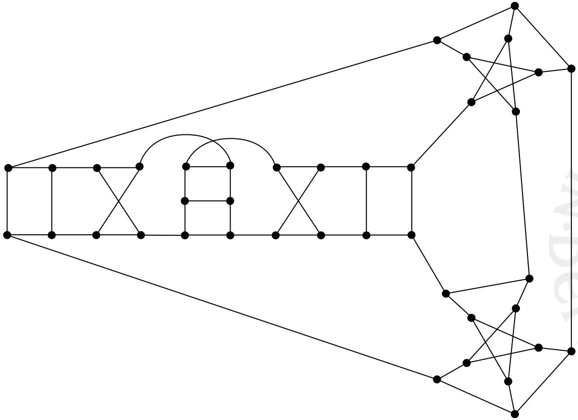
The importance of these graphs arise from the fact that snarks are counterexamples for many conjectures in Graph Theory.

In 2003, Cavicchioli et al. [3] verified that all snarks with girth at least 5 and fewer than 30 vertices are Type 1. In the same paper, they also proposed the question of finding the smallest Type 2 snark with girth at least 5.

In 2015, Brinkmann, Preissmann and Sasaki [2] constructed an infinite family of Type 2 snarks with girth 4 for all orders larger than 40.



Based on the construction of Type 2 snarks given in [2] and inspired by the dot product, we determine new ways of constructing Type 2 snarks by investigating the dot product between Type 1 snarks. We also determine infinite families of Class 2 blocks that can be used in this kind of construction.

Definition 4

A **dot product** between two snarks is constructed by deleting two non-adjacent edges in the first one and two adjacent vertices not in a triangle in the other, and then joining the four exposed vertices obtained, pair to pair. A dot product between two snarks is still a snark.

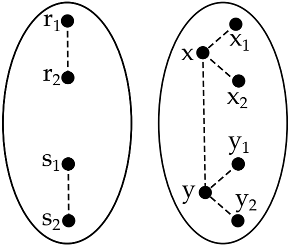


Figure 3: Dot product: deleted vertices and edges

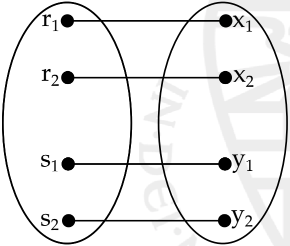


Figure 4: The resulting graph

Definition 5

A *brick* B^* is a cubic semigraph with exactly four semiedges, pairwise non-adjacent, such that its underlying graph B, the graph formed by its vertices and edges, is subgraph of some cubic cyclically 4-edge connected graph.

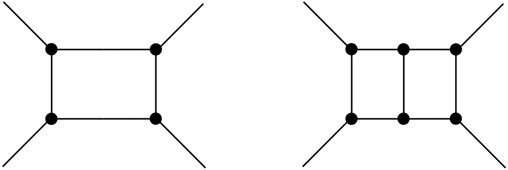


Figure 5: Two bricks: s-square (left) and s-domino (right).

Definition 6

A *junction* of semigraphs B' and B'' with the same number k of semiedges is a graph with all the vertices in both B' and B'' plus k disjoint edges (x, y) such that $(x, \cdot)$ is a semiedge of B' and $(y, \cdot)$ is a semiedge of B''.

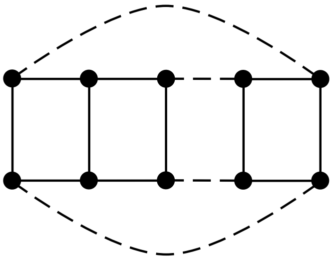


Figure 6: A junction of an s-square and an s-domino.

Remark 7

If there exists a Class 2 (resp. Type 2) subgraph of G H and $\Delta(G)$ $= \Delta(H)$, *then G is Class 2 (resp. Type 2).*

Lemma 8 (Brinkmann, Preissmann and Sasaki [2])

Any junction of two bricks is a cyclically 4-edge connected graph.

Remark 7

If there exists a Class 2 (resp. Type 2) subgraph of G H and $\Delta(G) = \Delta(H)$, then G is Class 2 (resp. Type 2).

Lemma 8 (Brinkmann, Preissmann and Sasaki [2])

Any junction of two bricks is a cyclically 4-edge connected graph.

Brinkmann, Preissmann and Sasaki [2] proved that P^* is a Class 2 brick and B^* is a Type 2 brick and any of their junctions results in a Type 2 snark. Also, from B^* and P^* it is possible to build Type 2 snarks for any even order $n \geq 40$.

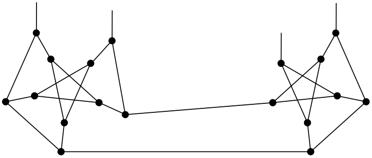


Figure 7: P^*

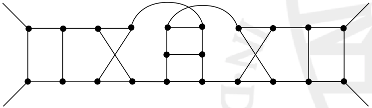


Figure 8: B^*

The family of snarks obtained this way will be called here S^*.

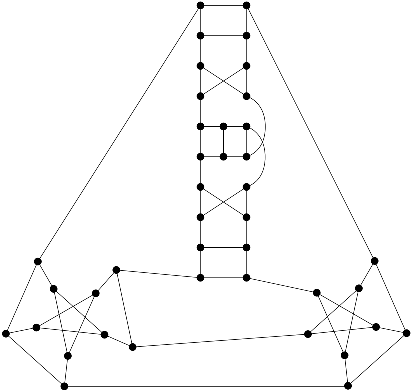
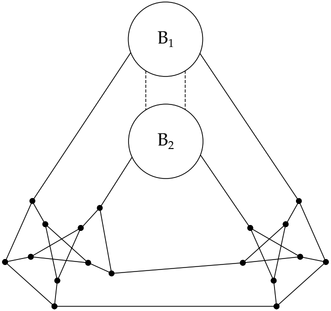


Figure 9: One of the smallest elements in S^*.

Theorem 9

The graphs in family S^* cannot be obtained from the product of two Type 1 snarks.



Proposition 10

The dot product of two Type 1 snarks can be a Type 2 snark.

S_1 and S_2 are Type 1 cubic graphs

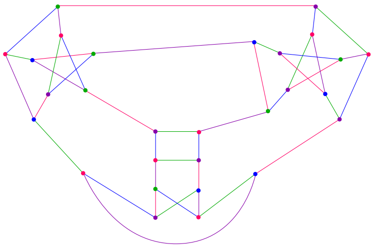


Figure 10: An equitable 4-total colouring for S_1

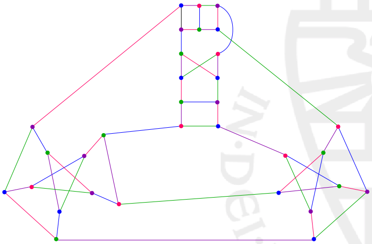


Figure 11: An equitable 4-total colouring for S_2

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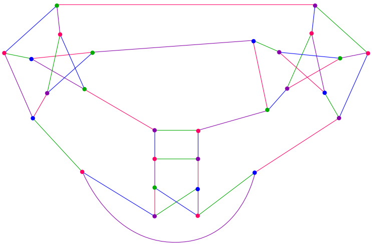

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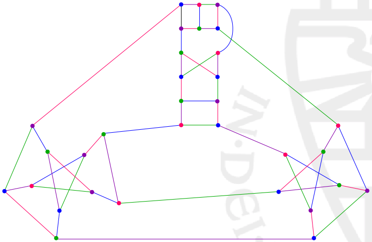

Figure 11: An equitable 4-total colouring for S_2

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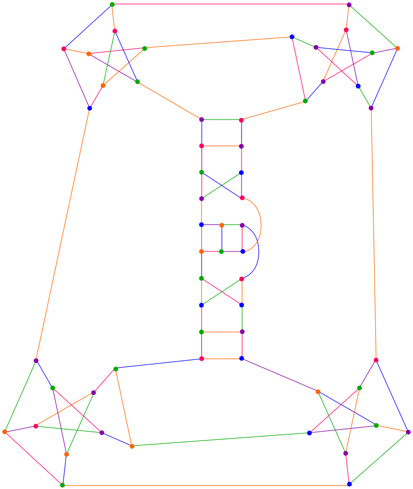

Figure 12: An equitable 5-total colouring for $S_1^{-v_1,v_2} \cdot S_2^{e_1,e_2}$

Proposition 12

Semigraph L^* depicted in the figure below is a Class 2 brick.

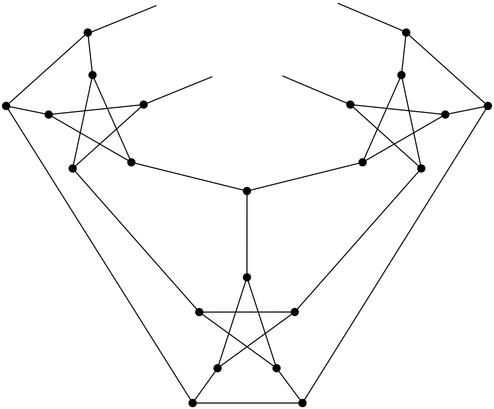

Figure 13: Brick L^*.

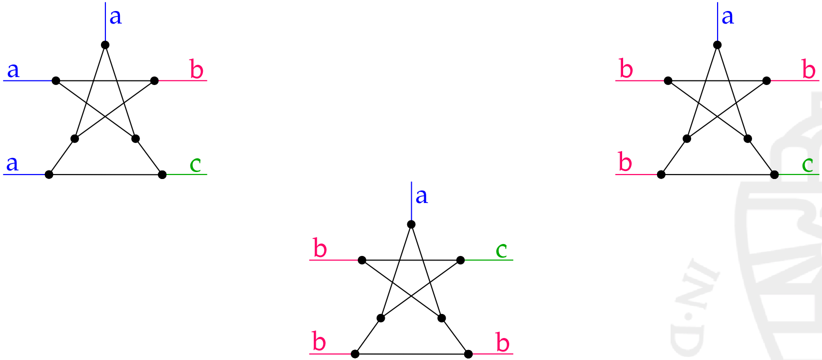

Figure 14: Representation of the 3-edge colourings for blocks L'.

Corollary 13

Graphs T_1 and T_2 shown in Figure 15 are snarks.

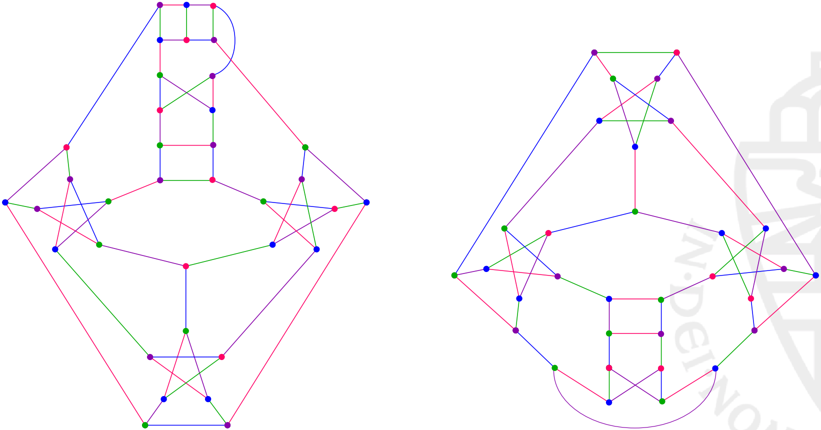

Figure 15: Graphs T_1 and T_2.

Theorem 14

Let $T = T_1 \cdot T_2$ be the graph depicted in Figure 16. Graph T is a Type 2 snark obtained from a dot product of two Type 1 snarks.

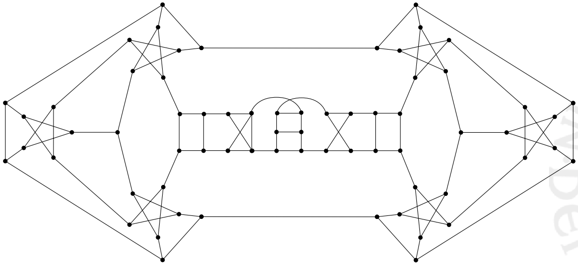

Figure 16: Type 2 snark T.

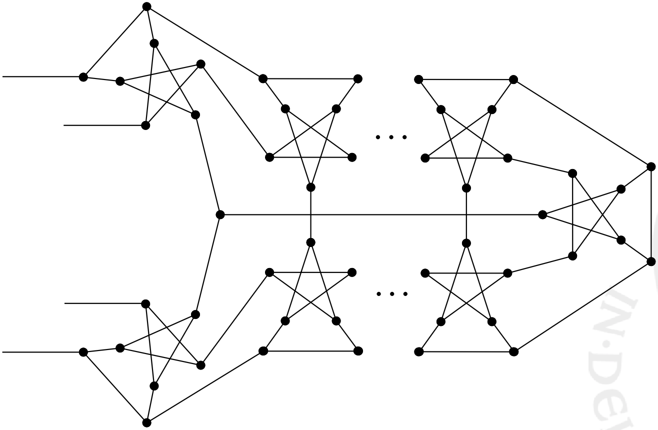

Figure 17: A general Loupekine-based brick.

Thank you!

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