

COLORING GRAPHS WITH FORBIDDEN INDUCED PATHS

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①

COLORING IS NPC

3-COLORING IS NPC

BEST KNOWN APPROX:
(THORUP & KAWARABAYASHI)

AN n -VERTEX 3-COLORABLE
GRAPH CAN BE $n^{4/11}$ -COLORED
IN POLYNOMIAL TIME

②
COLORING GRAPHS WITH CERTAIN
INDUCED S.G.'S EXCLUDED

THM (KAMINSKI, LOZIN)

$\forall k \geq 3, g \geq 3$ THE k -COLORING
PROBLEM IS NPC FOR GRAPHS OF
GIRTH $\geq g$

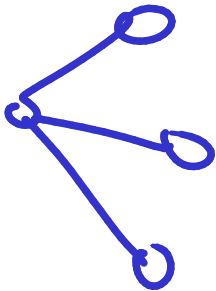
COR $\forall k \geq 3$, k -COLORING IS
NPC FOR H -FREE GRAPHS IF
 H HAS A CYCLE

(3)

THM (HOLZER)

K -EDGE-COLORING IS NPC FOR
 $K \geq 3$

COR K -COLORING CLAW-FREE
GRAPHS IS NPC FOR $K \geq 3$



CLAW

⑤

REMAINS OPEN:

- H IS THE DISJOINT UNION OF PATHS
- G IS H -FREE
- $k \geq 3$
- ? k -COLOR G

FOCUS ON: H IS A PATH

P_t t -VERTEX PATH

5

	$k=3$	$k=4$	$k=5$	$k \geq 6$
$t \leq 5$	P	P	P	P
$t=6$	P	?	NPC	NPC
$t=7$	P*	NPC	NPC	NPC
$t \geq 8$?	NPC	NPC	NPC

FROM: THREE COMPLEXITY RESULTS
ON COLORING P_k -FREE GRAPHS

(BRUERSMA, FOMIN, GOLOVA CH,
PAULUSMÁ)

IMPROVED COMPLEXITY RESULTS
ON k -COLORING P_t -FREE GRAPHS

(HUANG)

* (BONOMO, C., MACELI, SCHAUDT,
STEIN, ZHONG)

⑥

OPEN CASES:

- k -COLORING P_t -FREE GRAPHS
 $k=3, t \geq 8$
- 4-COLORING P_6 -FREE GRAPHS
 - LIST VERSION IS NPC
(GOLOVACH, PAULUSMA, SONG)
 - 4-COLORING (P_6, C_5) -FREE GRAPHS IS POLY
(C., MACELI, STACHO, ZHONG)
 - 4-COLORING (P_6, C_3) -FREE GRAPHS IS POLY
(RANDERATH, SCHIERMEYER, TEWESB)

RECENTLY: COMPLETE STRUCTURE THM
(C., SEYMOUR, SPIRKL, ZHONG)

THM (BONOMO, C., MACELI,
SCHAUDT, STEIN, ZHONG)

⑦

THERE IS A POLY TIME ALG
TO TEST IF A P_7 -FREE GRAPH
IS 3-COLORABLE

8

LIST COLORING

G GRAPHS

$L = \{L(v)\}_{v \in V(G)}$ LISTS OF COLOR

(G, L) IS COLORABLE IF \exists COLORING c OF $V(G)$, S.T. $\forall v c(v) \in L(v)$

IDEA: REDUCE COLORING G TO $\{(G_1, L_1), \dots, (G_k, L_k)\}$ S.T.

G IS COLORABLE IFF (G_i, L_i) IS COLORABLE FOR SOME i

(9)

IMPORTANT TOOL 1

THM (EDWARDS)

- G GRAPH
- $L = \{ L(v) \}_{v \in V(G)}$ LISTS
- $\forall v \quad |L(v)| \leq 2$

THERE IS A POLY-TIME ALG TO
TEST IF (G, L) IS COLORABLE

PROOF: REDUCE TO 2-SAT

(10)

IMPORTANT TOOL 2

THM (CAMBI & SCHAUDT)

IF G IS P_t -FREE AND
CONNECTED, THEN $\exists H < G$

S.T.

① $V(H)$ IS DOMINATING, AND

② $H \approx P_{t-2}$ OR H IS P_{t-2} -FREE,
AND

③ H IS CONNECTED.

USING THE TOOLS

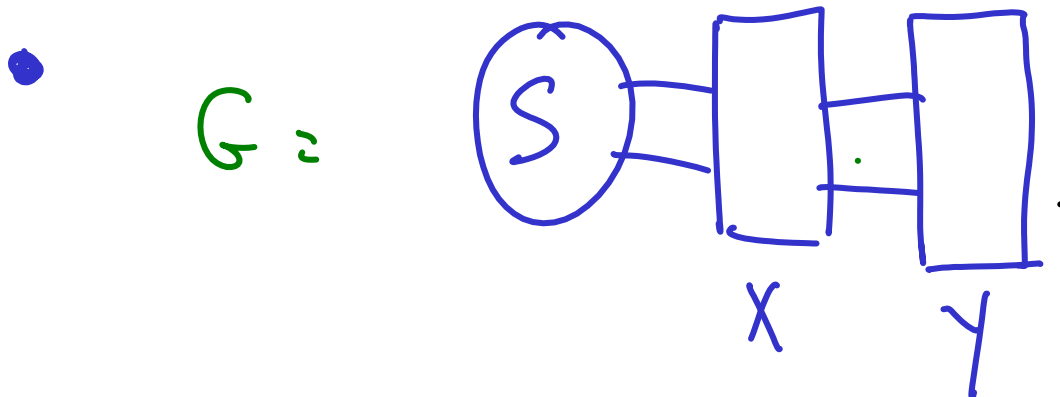
(11)

- IF $H \approx P_{t-2}$ TRY ALL POSSIBLE 3-COLORINGS OF H ;
REDUCE TO $3^{|V(H)|}$ 2-SAT PROBLEMS

- $t=7$

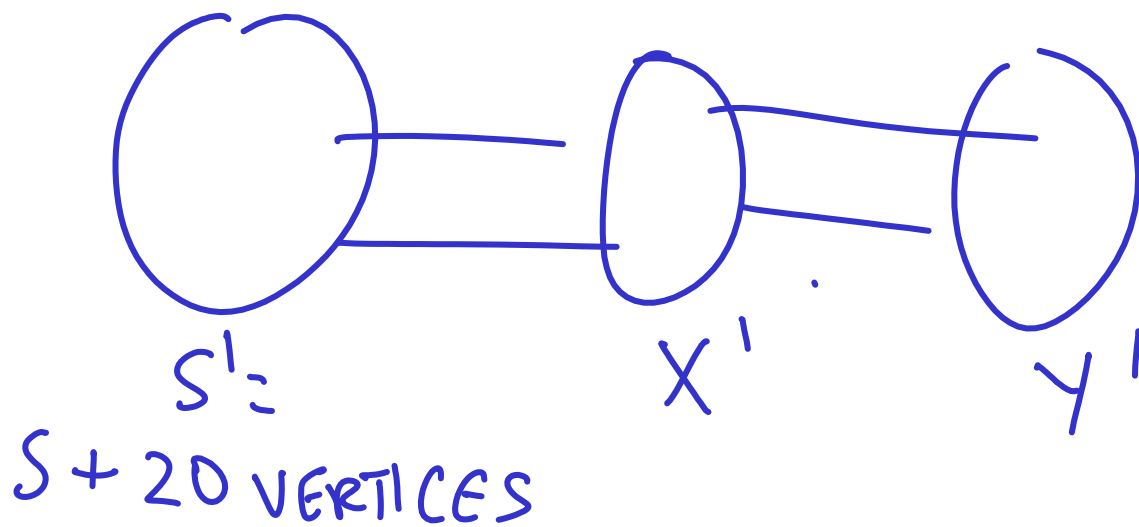
H IS P_5 -FREE

$\Rightarrow H$ HAS A BOUNDED SIZE DOMINATING SUBGRAPH S

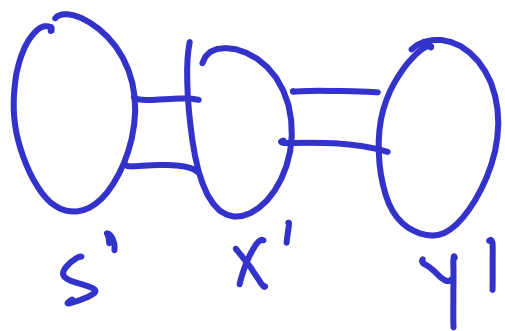


- "GUESS" (TRY ALL POSSIBILITIES)

~20 VERTICES WITH CERTAIN PROPERTIES TO ARRANGE

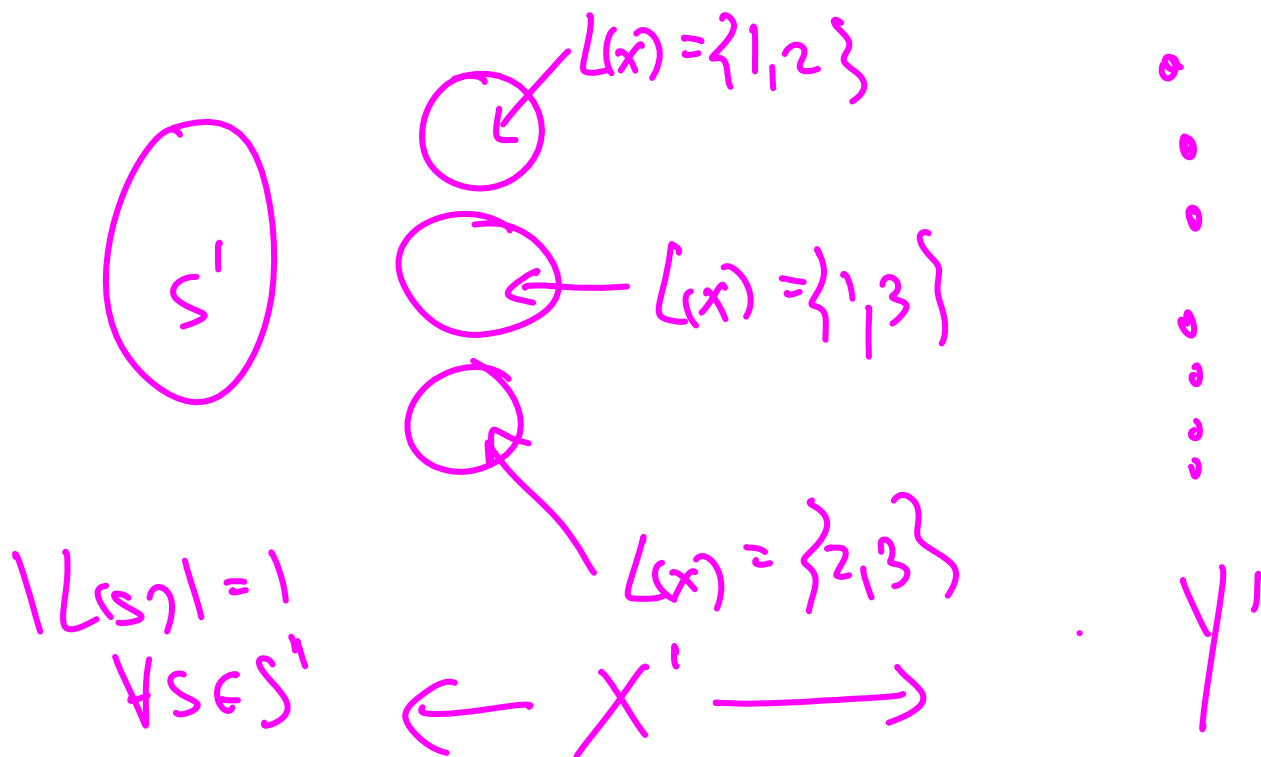


Y' STABLE



TRY ALL POSSIBLE COLORING OF S'

- FIX A COLORING OF S' , AND UPDATE LISTS
- $|L(x)| \leq 2 \quad \forall x \in X'$



• GUESS THE COLORS OF
~5 MORE VERTICES TO ARRANGE
THAT $|L(y)| \leq 2 \quad \forall y \in Y'$

NOW WIN BY EDWARD'S THM
(2-SAT)

15

RELATED QUESTION (SEYMOUR)

IS THERE A BOUNDED NUMBER
OF P_t -FREE OBSTRUCTIONS TO
3-COLORING?

THM (C., GOEDGEBEUR, SCHAUDT, ZHONG)
YES IF $t \leq 6$

THM (POKROVSKIY)
NO IF $t \geq 7$

MORE GENERALLY

(16)

(GOLOVACH, JOHNSON, PAULUSMA, SONG)

FOR WHICH H ARE THERE

FINITELY MANY H -FREE

4 -CRITICAL GRAPHS?

H IS POWERFUL IF THERE

ARE FINITELY MANY H -FREE

VERTEX- 4 -CRITICAL GRAPHS

GRAPHS THAT ARE NOT POWERFUL (17)

- IF  $\leq H$ OR $C_k \leq H$


3-COLORING IS NPC

(HOLZER; KAMINSKI & LOZIN)



H IS NOT POWERFUL

A CONSTRUCTION:

INFINITE FAMILY OF
 -FREE 4-CRITICAL
GRAPHS (POKROUSKIY)

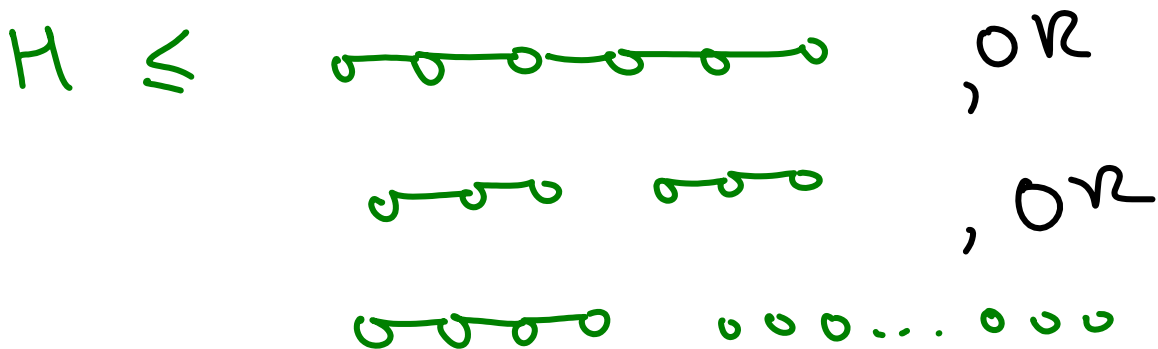


 NOT POWERFUL

P_7 NOT POWERFUL

THM (C., GOEDGEBEER, SCHAUDT, ZHONG)

H IS POWERFUL IFF



A CLOSELY RELATED PROBLEM:

20

LIST-3-COLORING

$(G, L) \quad L(v) \subseteq \{1, 2, 3\} \quad \forall v$

FIND A PROPER COLORING c
OF G S.T. $c(v) \in L(v) \quad \forall v$



FOR WHICH H ARE THERE
FINITELY MANY H -FREE
MINIMAL OBSTRUCTIONS TO
LIST-3-COLORING?

THE ANSWER IS DIFFERENT!



LIST-POWERFUL

THM (C., GOEDGEBEUR, SCHAUDT, ZHONG)

H IS LIST-POWERFUL IFF

$H \leq$  , OR


A CONSTRUCTION:

INFINITE FAMILY OF
  -FREE OBSTRUCTIONS
 WITH $|L(v)|=2 \forall v$

(SCHAUDT)

SO  IS POWERFUL,

BUT NOT LIST-POWERFUL

(22)

PROOFS

ENOUGH TO SHOW:

P_6 , $2P_3$, $P_1 + kP_1$

ARE POWERFUL

STEP 1 SHOW THAT IF G IS
A MINIMAL OBSTRUCTION THEN

$\exists G', L'$ s.t.

- $G' \leq G$
- $|E(G')| \leq 10^6 |V(G)|$
- $|L'(v)| \leq 2 \quad \forall v \in V(G')$
- * IF $i \notin L'(v)$, THEN $\exists u$
s.t. $uv \in E(G')$ & $L(u) = \{i\}$
- (G', L') IS A MINIMAL
OBSTRUCTION

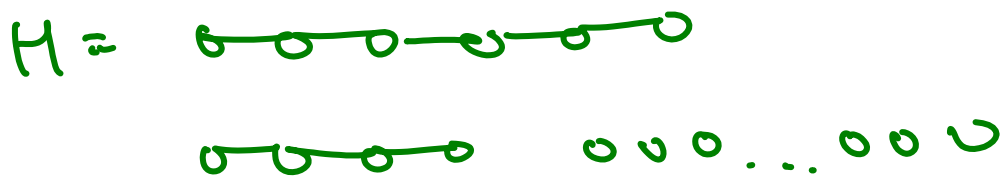
STEP 2

SHOW THAT IF G' IS H -FREE AND
AND

- $|L'(v)| \leq 2 \quad \forall v \in V(G')$.
- (G', L') IS A MINIMAL OBSTRUCTION

THEN $|V(G')| \leq 10^6$

- WORKS FOR



- FOR $H =$

WORKS UNDER ASSUMPTION*

(25)

MORE DETAILS FOR STEP 1

STRUCTURAL RESULT:

LET G BE A MINIMAL H -FREE
OBSTRUCTION. THEN $\exists R \subseteq G$
S.T.

① R HAS ≤ 1000 3-COLORINGS

② PRECOLORING R
& UPDATING 3 TIMES
GIVES $|L(v)| \leq 2 \quad \forall v \in V(G)$

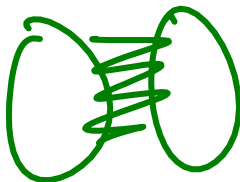
"PROOFS"

- $H = P_4 + kP_1$

TAKE $R = P_4 + (k-1)P_1$

- $H = P_6$

CAMBI - SCHAUDT + TRICKS

R IS 

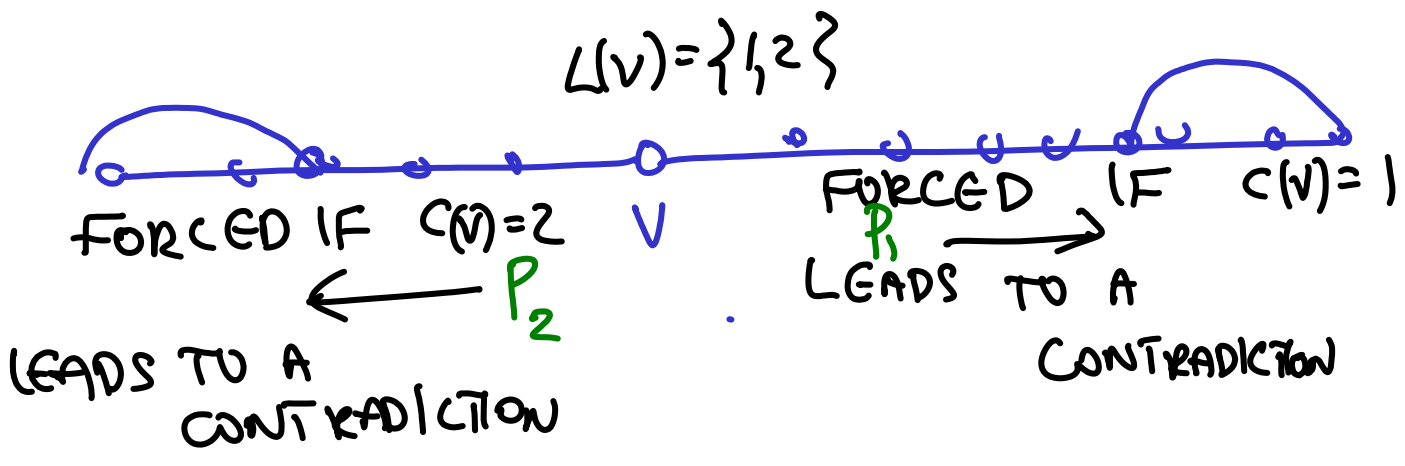
- $H = 2P_3$ HARD

- \forall PRECOLORING c_i OF R (27)
 G CONTAINS AN OBSTRUCTION
 (G_i, L_i)
- BY STEP 2 $|V(G_i)| \leq 10^2$
- $G' = R \cup \bigcup_i G_i \cup$ updating paths
 (IS NOT 3-COLORABLE)
- $G' = G$ (BY MINIMALITY OF G)
- $|V(G') \setminus R|$ BOUNDED
 \Downarrow
 BOUNDED # "TYPES" OF
 VERTICES IN R
- USE STRUCTURE OF R TO
 DEDUCE THAT $|V(R)|$ BOUNDED

MORE DETAILS OF STEP 2

ORIGINALLY DONE BY COMPUTER,
NOW HAVE A COMPUTER-FREE
PROOF

PROPAGATION PATH:



$$V(G) = V(P_1) \cup V(P_2)$$

STUDY THE STRUCTURE OF
 P_1, P_2 ; FIND H

