# On Efficient Domination for Some Classes of *H*–Free Chordal Graphs

Andreas Brandstädt
*University of Rostock, Germany*
and
Raffaele Mosca
*University of Pescara, Italy*

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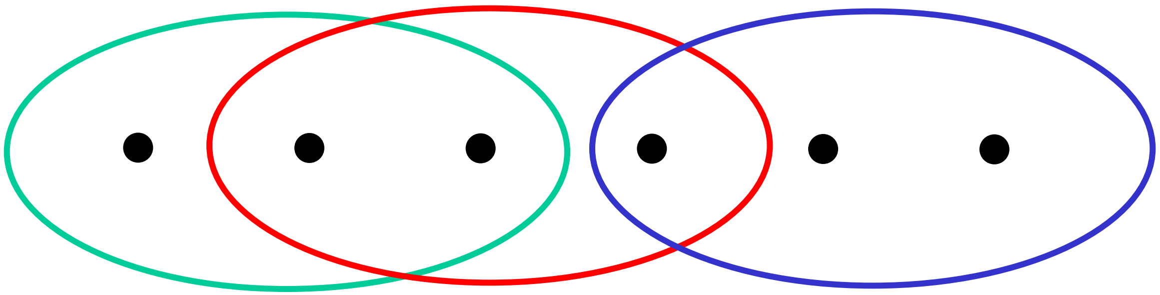
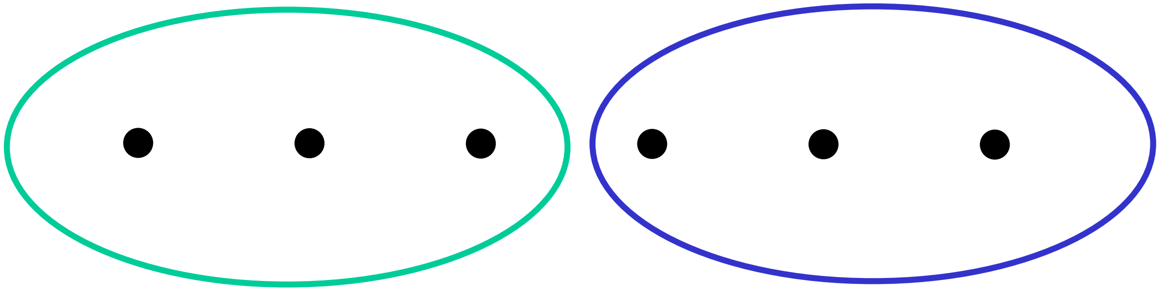
# Exact Cover by 3-Sets (X3C)

Problem X3C [SP2] of [Garey, Johnson, Computers and Intractability, 1979]:

INSTANCE: Set $X$ with $|X| = 3q$ and a collection $C$ of 3-element subsets of $X$.

QUESTION: Does $C$ contain an *exact cover* for $X$, i.e., a subcollection $D \subseteq C$ such that every element of $X$ occurs in exactly one member of $D$?





# Exact Cover by 3-Sets (X3C)

**Theorem** [Karp 1972]

X3C is NP-complete.

(transformation from 3DM)

**Remark.** Exact Cover for 2-element subsets (i.e., X2C) corresponds to Perfect Matching.

# Efficient domination

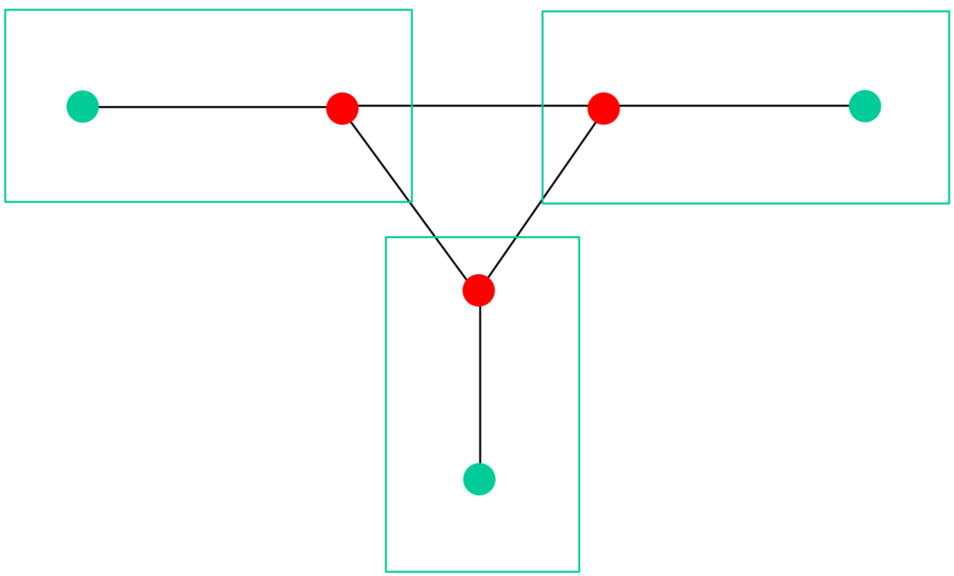
Let $G = (V, E)$ be a finite undirected graph.

A vertex $v$ *dominates* itself and its neighbors, i.e., $v$ *dominates* $N[v]$.

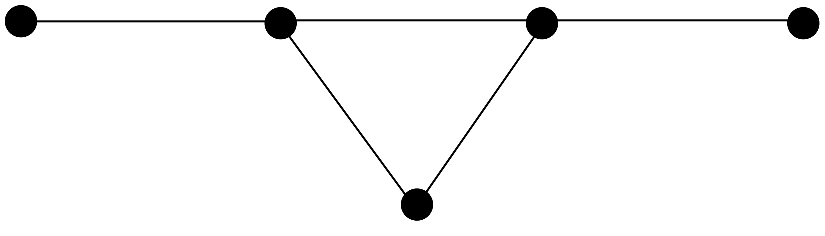
[Biggs 1973, Bange, Barkauskas, Slater 1988]:

$D$ is an *efficient dominating set* (*e.d.s.*) in $G$ if

- $D$ is dominating in $G$, and
- each $v \in V$ is dominated exactly once by $D$.



Not every graph has an e.d.s.!



# Efficient domination

Efficient dominating sets were introduced as *perfect codes* by Biggs in 1973; they also appear as *independent perfect dominating sets.*

Let $\boldsymbol{N}(G)$ = closed neighborhood hypergraph of $G$.

**Fact.** $G$ has an e.d.s. $\Leftrightarrow$ $\boldsymbol{N}(G)$ has an exact cover for $V$.

# Efficient domination

Efficient Domination (ED) problem:

INSTANCE: A finite graph $G = (V, E)$.
QUESTION: Does $G$ have an e.d.s.?

**Theorem** [Bange, Barkauskas, Slater 1988]

ED is NP–complete.

The Weighted Efficient Domination (WED) problem asks for an e.d.s. of minimum vertex weight.

# Efficient edge domination

[Grinstead, Slater, Sherwani, Holmes, 1993]:

$M \subseteq E$ is an *efficient edge dominating set* (*e.e.d.s.*) in $G$ (also called *dominating induced matching*) if

- $M$ is dominating in $L(G)$, and
- each $e \in E$ is dominated exactly once in $L(G)$ by $M$,

that is, $M$ is an e.d.s. in $L(G)$.

# Efficient edge domination

Efficient Edge Domination (EED) problem:

INSTANCE: A finite graph $G = (V, E)$.
QUESTION: Does $G$ have an e.e.d.s.?

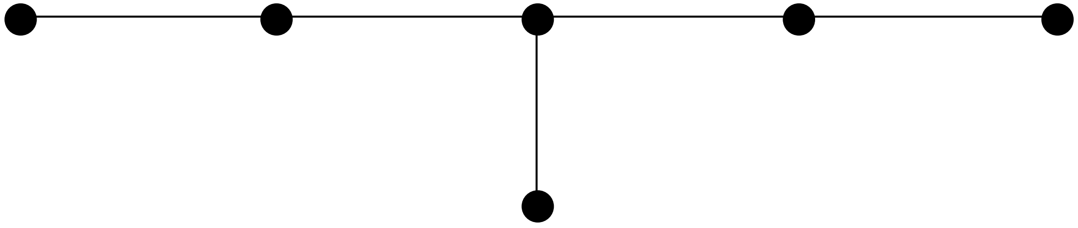
**Theorem** [Grinstead, Slater, Sherwani, Holmes 1993]

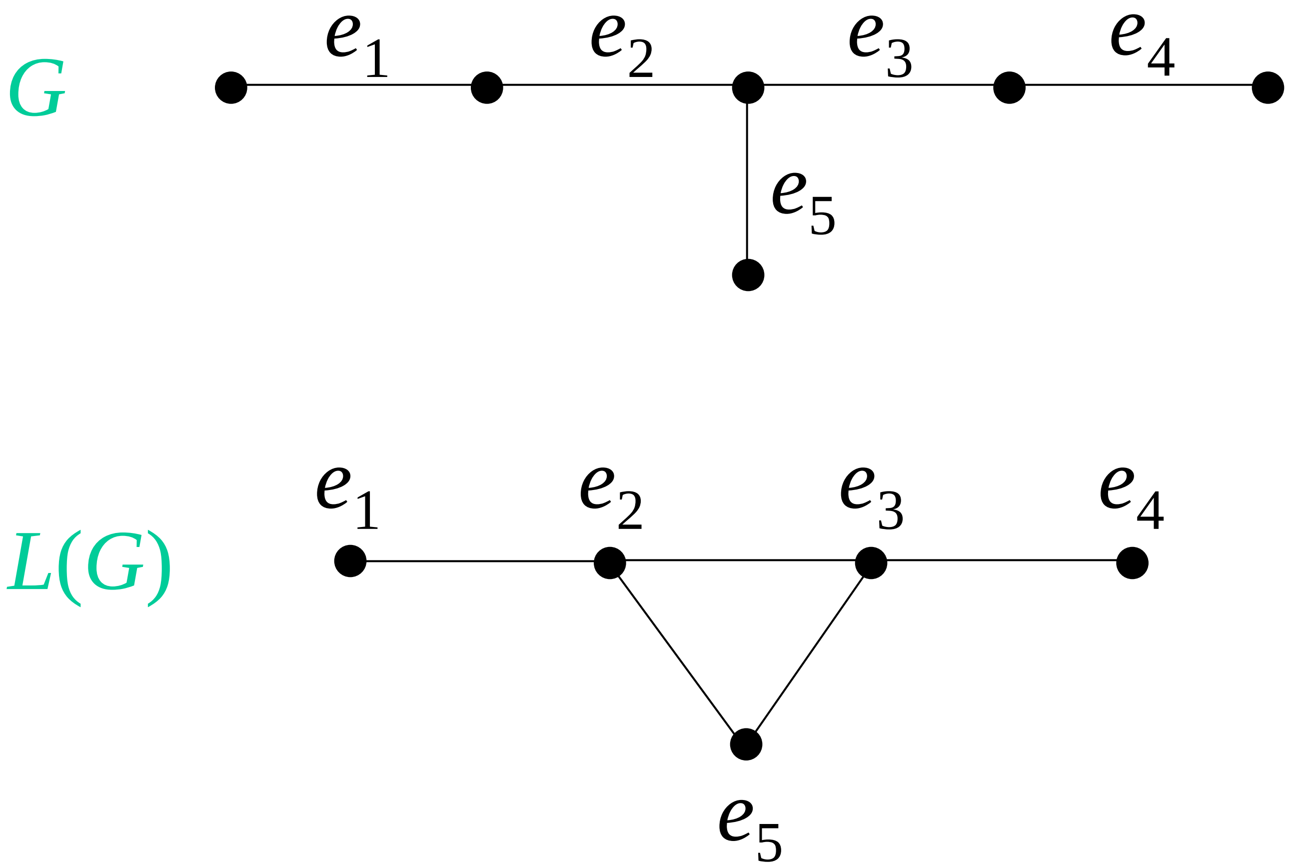
EED is NP-complete.

**Corollary.** ED is NP-complete for line graphs, and thus for claw-free graphs.

Not every graph (not every tree !)

has an e.e.d.s.:

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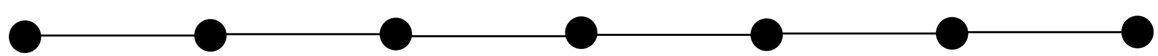
$2P_3$



$P_7$

# Efficient domination

**Theorem** [Smart, Slater 1995, Yen, Lee 1996]

ED is NP–complete for bipartite graphs and for $2P_3$–free chordal graphs.

**Corollary.** If $H$ contains a cycle or claw then ED is NP–complete on $H$–free graphs.

If $H$ is cycle– and claw–free then $H$ is a *linear forest* (= disjoint union of paths).

# Efficient domination

$G$ is a *split graph* if $V(G)$ is partitionable into a clique and an independent set. Clearly,

$G$ is a split graph $\Leftrightarrow$ $G$ and its complement are chordal.

**Theorem** [Főldes, Hammer 1977]

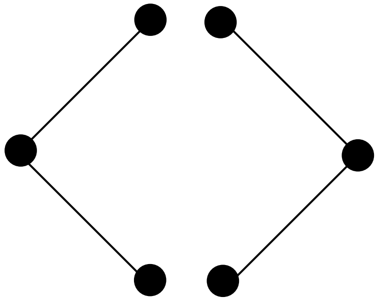
$G$ is a split graph $\Leftrightarrow$ $G$ is $(2P_2, C_4, C_5)$–free.

**Theorem** [M.-S. Chang, Liu 1993]

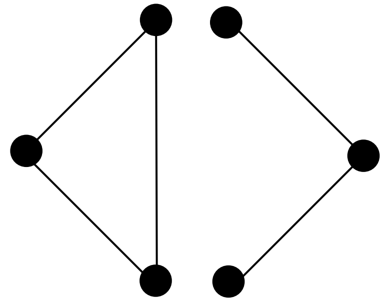
WED in time $O(n + m)$ for split graphs.

# Efficient domination

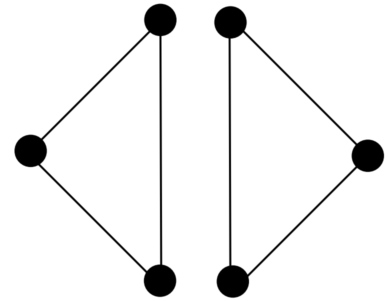
**Theorem** [B., Milanič, Nevries, MFCS 2013]
WED in linear time for $2P_2$–free graphs.
**Theorem** [B. 2015]
WED in linear time for $P_5$–free graphs.
(based on modular decomposition)



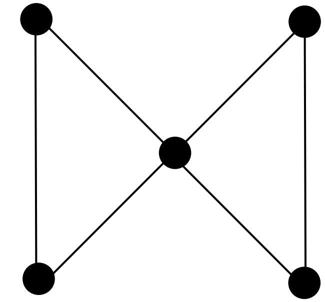
$2P_3$



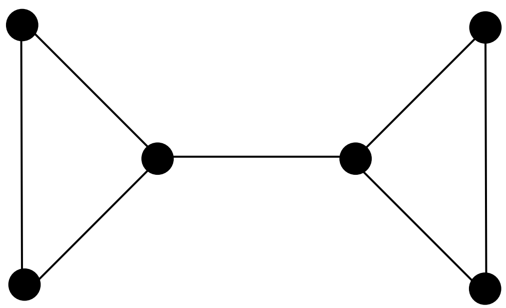
$K_3 + P_3$



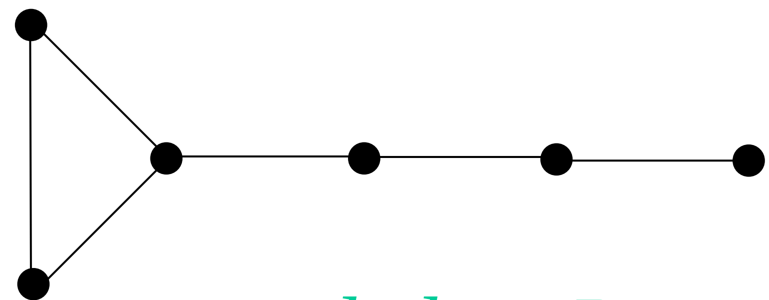
$2K_3$



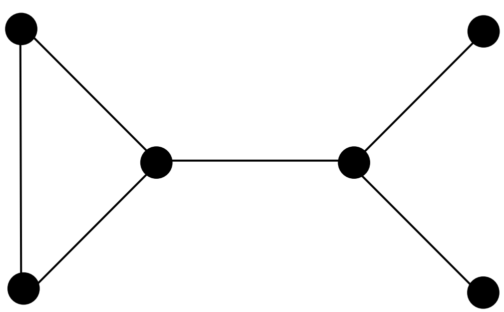
*butterfly*



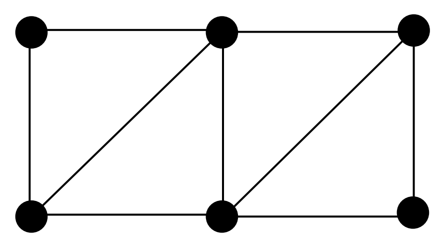
*extended butterfly*



*extended co-P*



*extended chair*



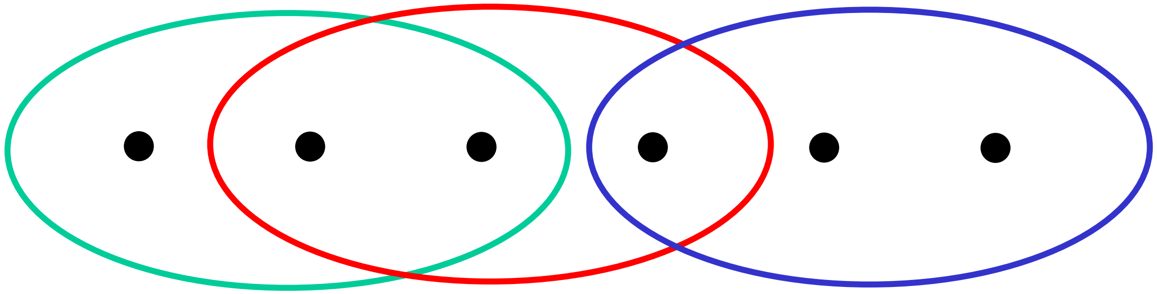
*double-gem*

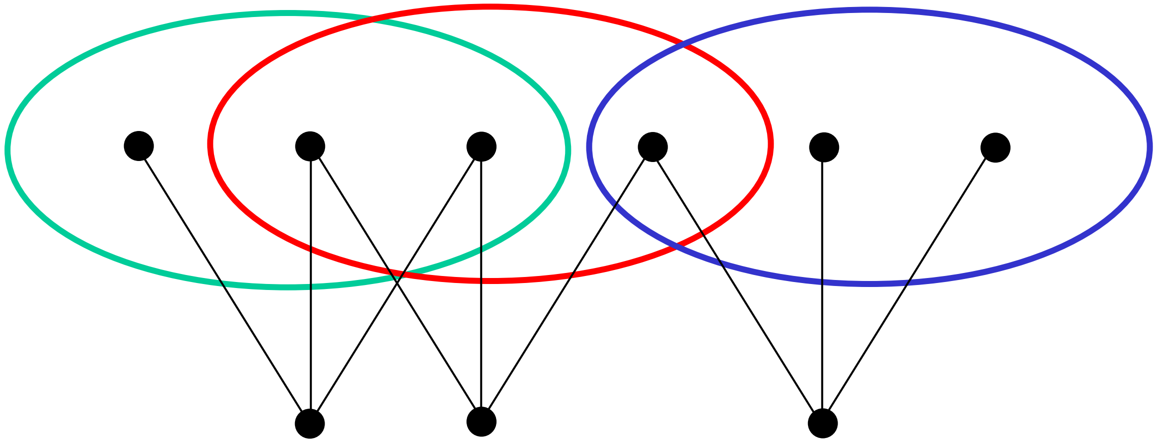
# Efficient domination

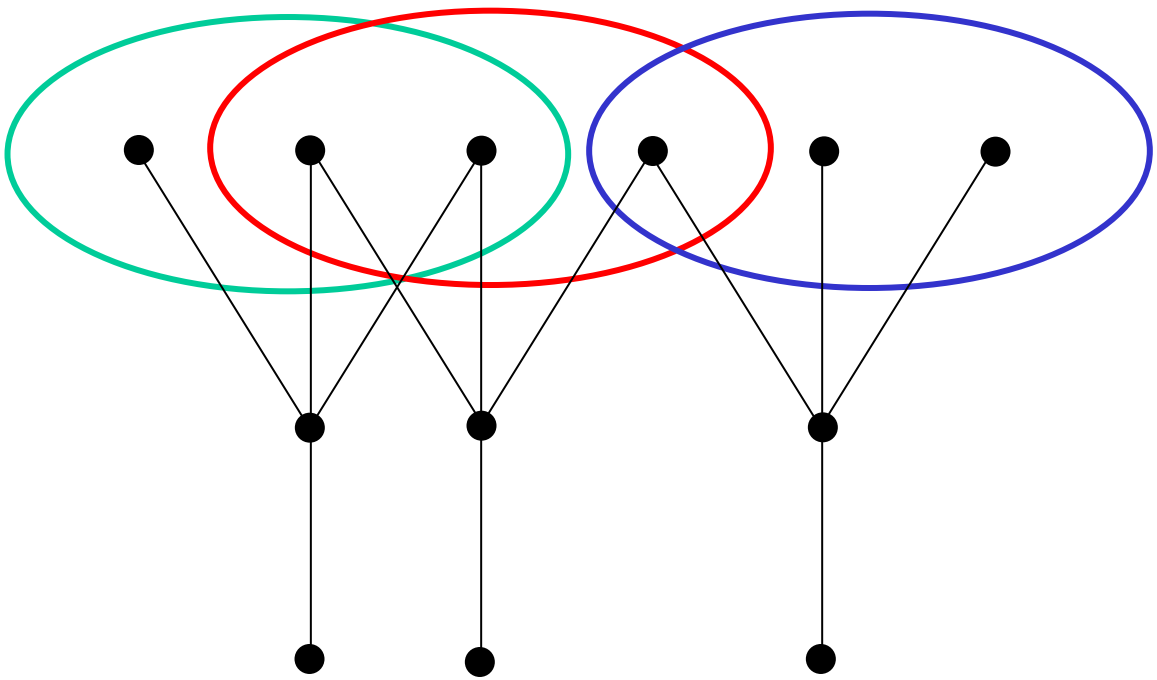
($2P_3$, $K_3$+$P_3$, $2K_3$, butterfly, extended butterfly, extended co-$P$, extended chair, double-gem)–free chordal graphs represent a generalized version of split graphs called *satgraphs* in [Zverovich 2006].

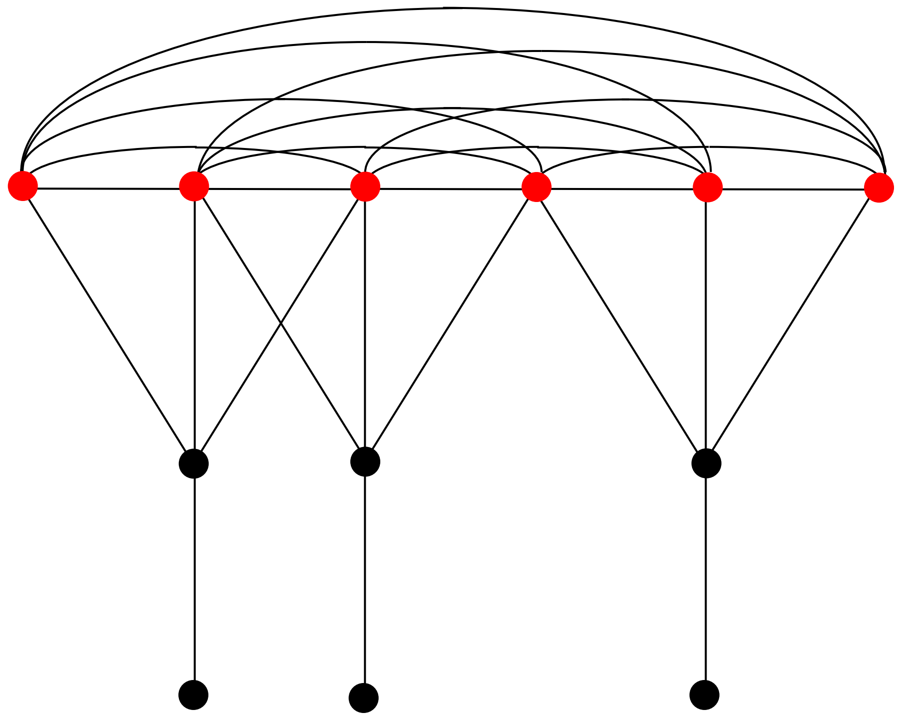
**Theorem.** ED is NP–complete for satgraphs.

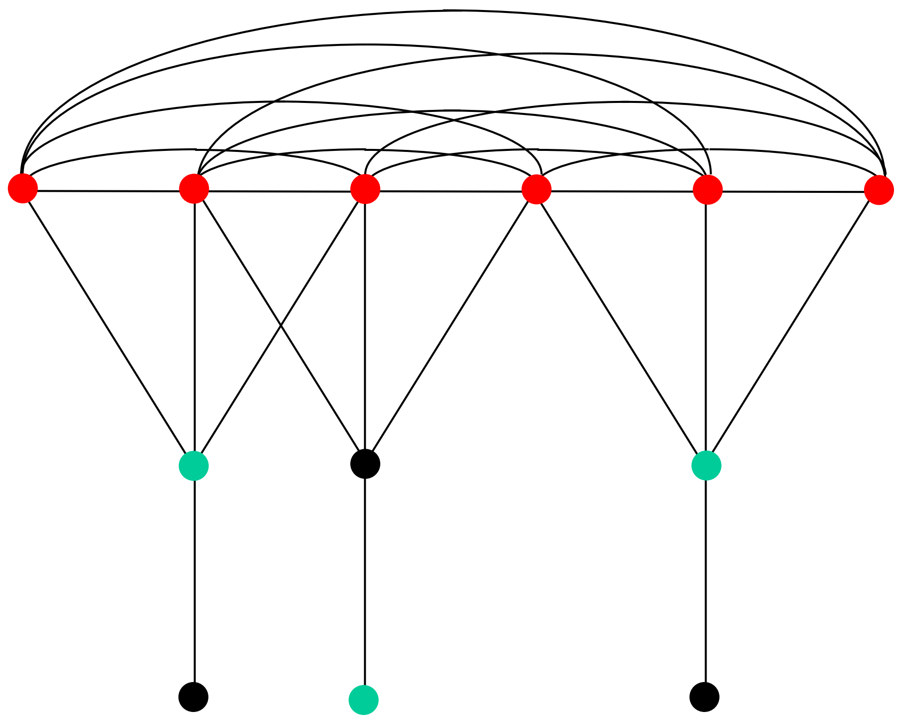
(simple standard reduction from X3C)











# Efficient domination and clique-width

ED based on *clique-width*:

**Fact.** ED can be formulated in (a special kind of) Monadic Second Order Logic.

**Theorem** [Golumbic, Rotics 2000]

Clique-width of gem–free chordal graphs is at most 3.

**Corollary.** ED in linear time for gem–free chordal graphs.

# Efficient domination and clique-width

**Theorem** [B., Dabrowski, Huang, Paulusma, Bounding the clique-width of *H*-free chordal graphs, MFCS 2015] … There are only two open cases for clique-width of *H*–free chordal graphs …

For ED, we focus on *H*–free chordal graphs with unbounded clique-width.

# Efficient domination via $G^2$

Let $G^2 = (V, E^2)$ with $xy \in E^2$ if $d_G(x,y) \leq 2$.

$\boldsymbol{N}(G)$ = closed neighborhood hypergraph of $G$.

**Fact.**

- $G^2 = L(\boldsymbol{N}(G))$.
- $D$ is an e.d.s. in $G \Leftrightarrow D$ is dominating in $G$ and independent in $G^2$.

# Efficient domination via $G^2$

Let $w(v) := |N[v]|$.

**Fact** [Leitert; Milanič 2012]

$D$ is an e.d.s. in $G \Leftrightarrow D$ is a maximum weight independent set in $G^2$ with $w(D) = |V|$.

**Theorem** [Friese 2013]

$G$ $P_6$–free with e.d.s. $\Rightarrow G^2$ hole–free.

# Efficient domination via $G^2$

**Conjecture** [Friese 2013]

$G$ $P_6$–free with e.d.s. $\Rightarrow$ $G^2$ odd–antihole–free (and thus, by the Strong Perfect Graph Theorem, $G^2$ would be perfect).

**Theorem** [B., Eschen, Friese WG 2015]

$G$ $P_6$–free chordal with e.d.s. $\Rightarrow$ $G^2$ chordal.

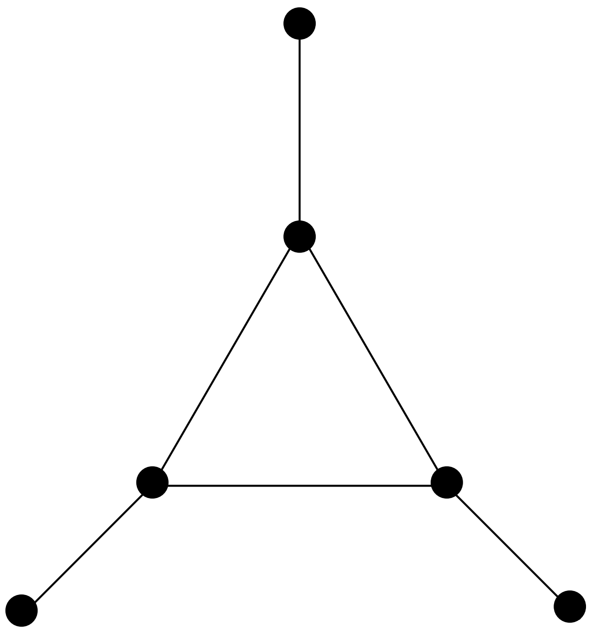
# Efficient domination

**Corollary.** ED in polynomial time for $P_6$–free chordal graphs (but NP–complete for $P_7$–free chordal graphs).

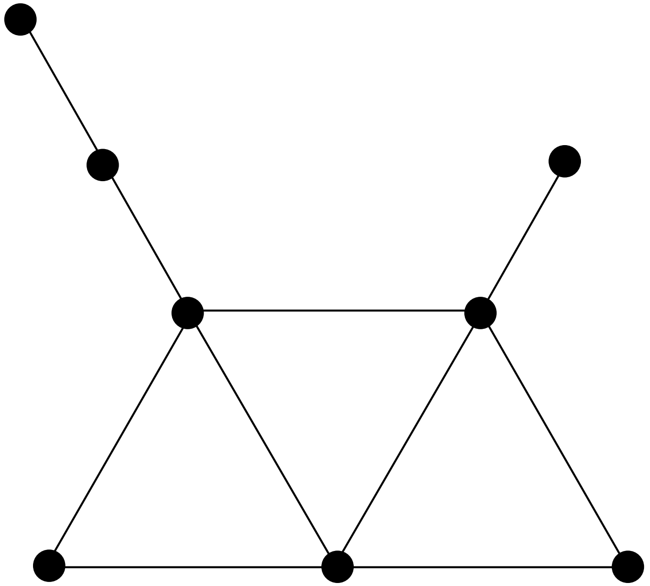
**Theorem** [B., Mosca WG 2016]

WED in polynomial time for $P_6$–free graphs.

(direct approach)



*net*



*extended gem*

# Efficient domination via $G^2$

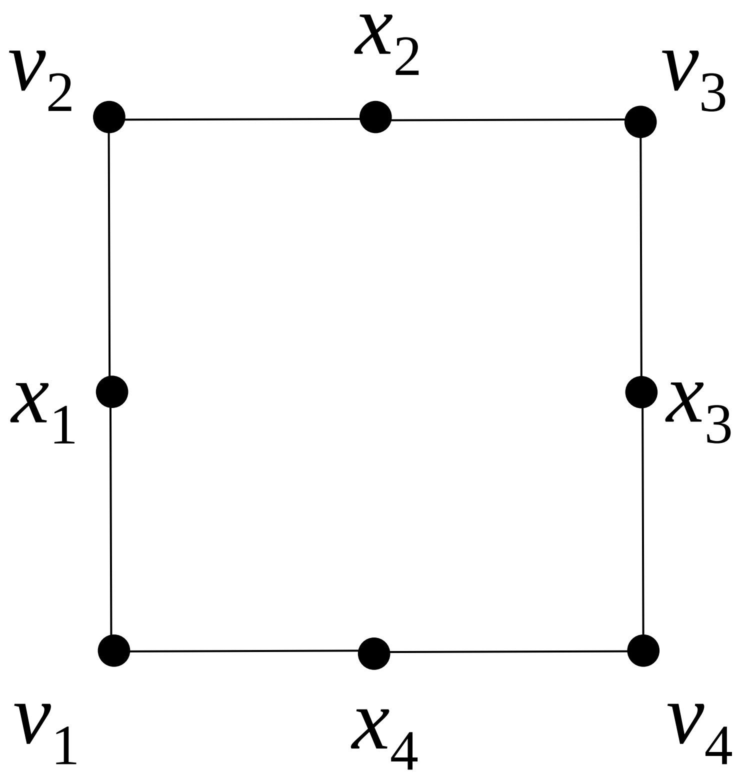
**Theorem** [B., Mosca 2017]

If $G$ is net–free chordal or extended-gem-free chordal with e.d.s. then $G^2$ is chordal.

(interval graphs $\subset$ net–free chordal)

# Efficient domination via $G^2$

**Proof.** Let $G$ be net–free chordal with e.d.s. First suppose to the contrary that $G^2$ contains a $C_4$, say with vertices $v_1, v_2, v_3, v_4$ such that $d_G(v_i,v_{i+1}) \leq 2$ and $d_G(v_i,v_{i+2}) \geq 3$. By the chordality of $G$, $d_G(v_i,v_{i+1}) = 2$. Let $x_i$ be the common neighbor of $v_i$ and $v_{i+1}$. Clearly, $x_i \neq x_j$ for $i \neq j$.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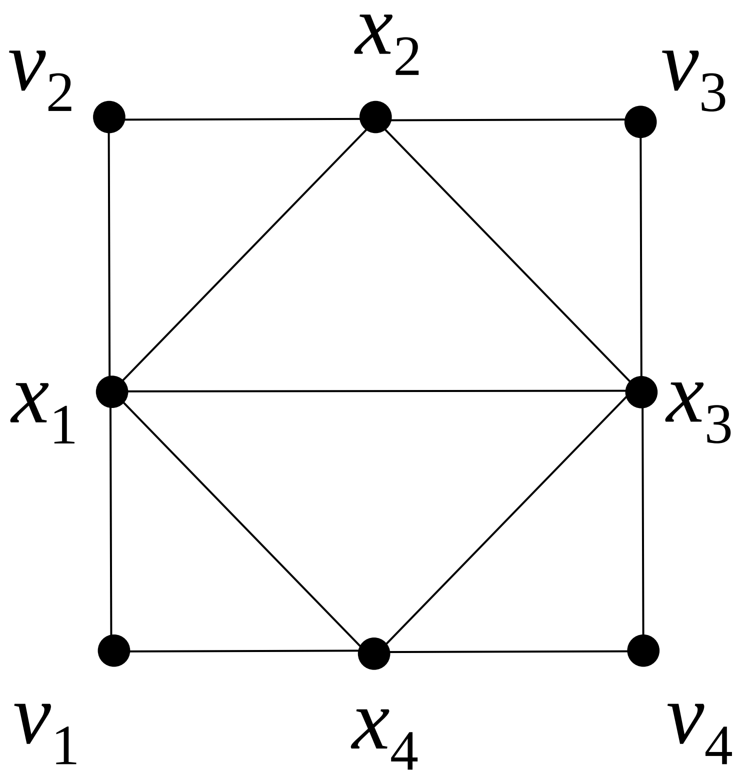

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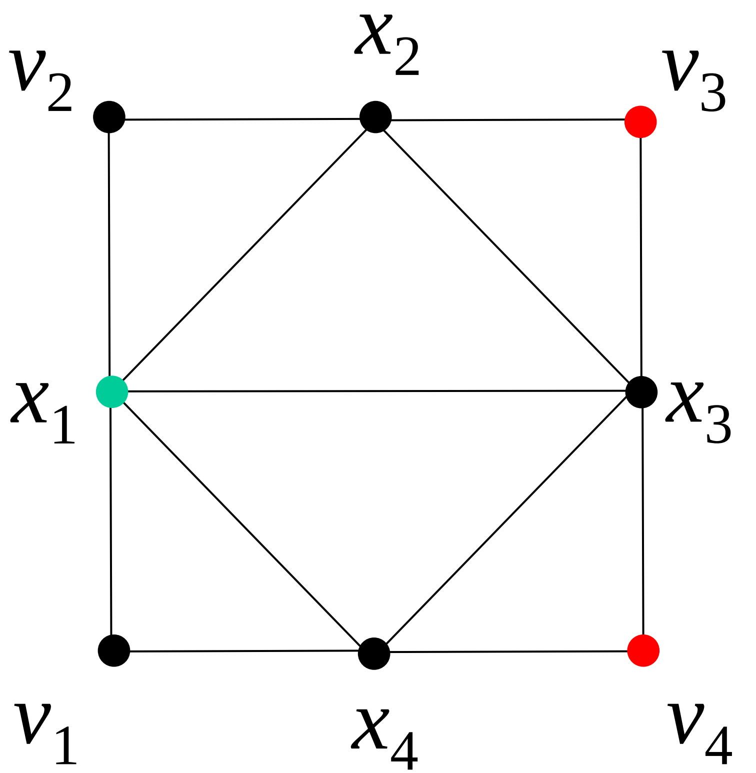

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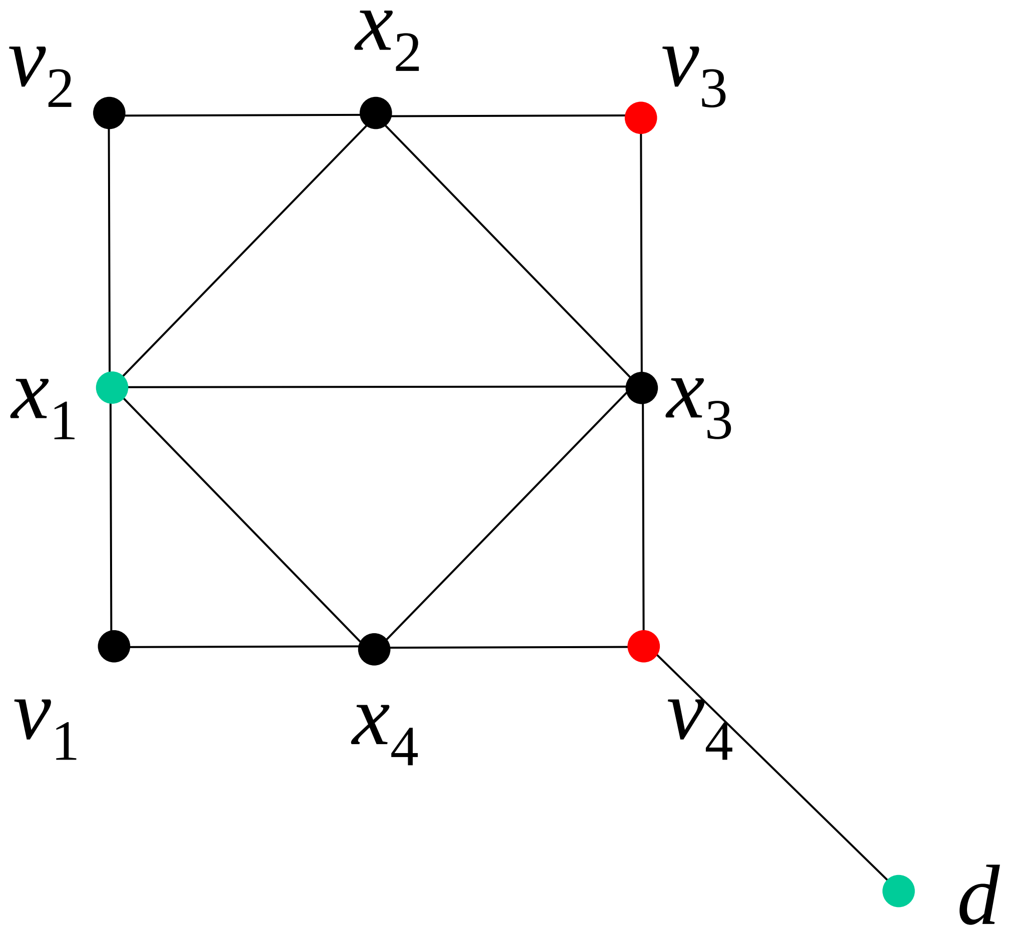

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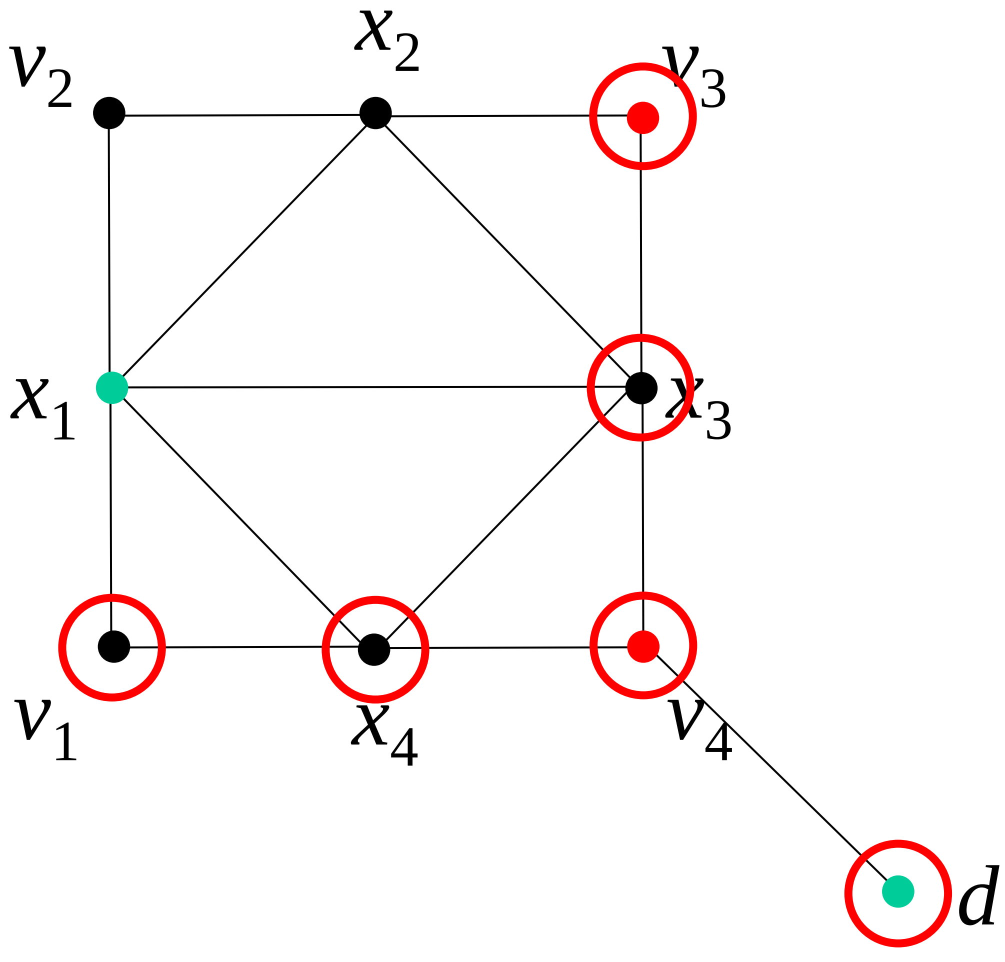

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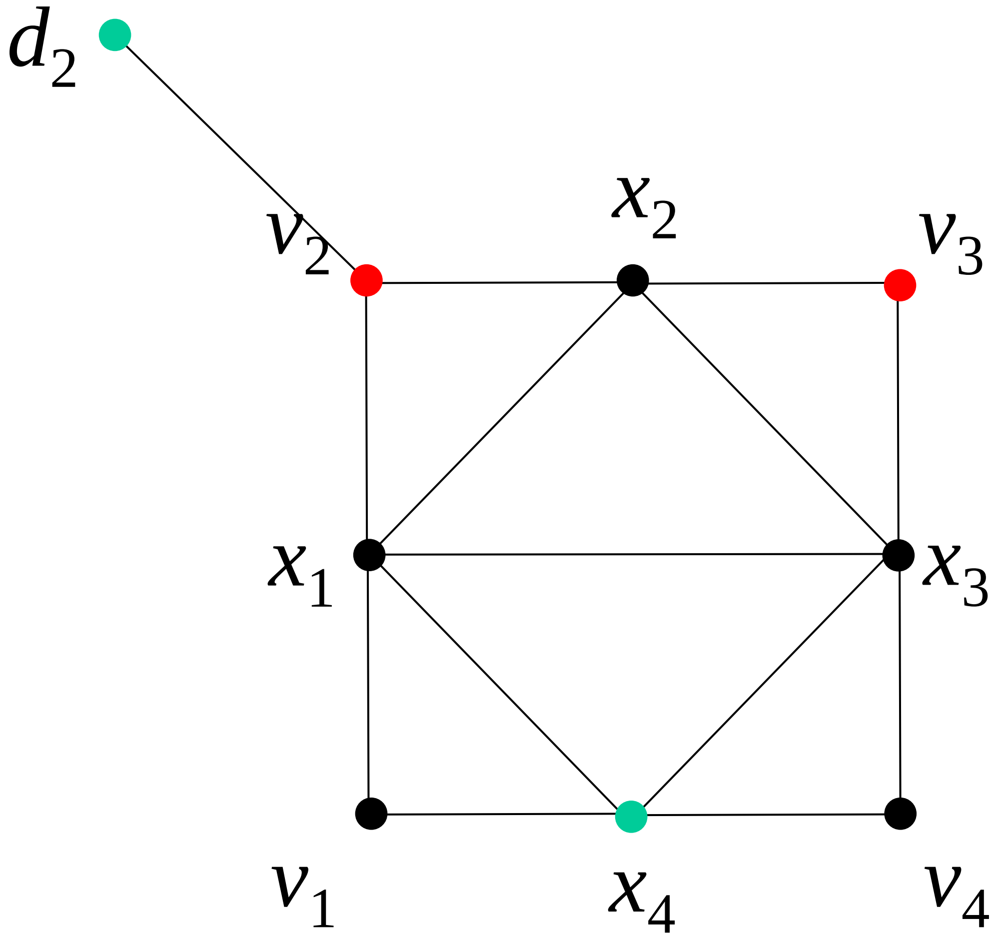

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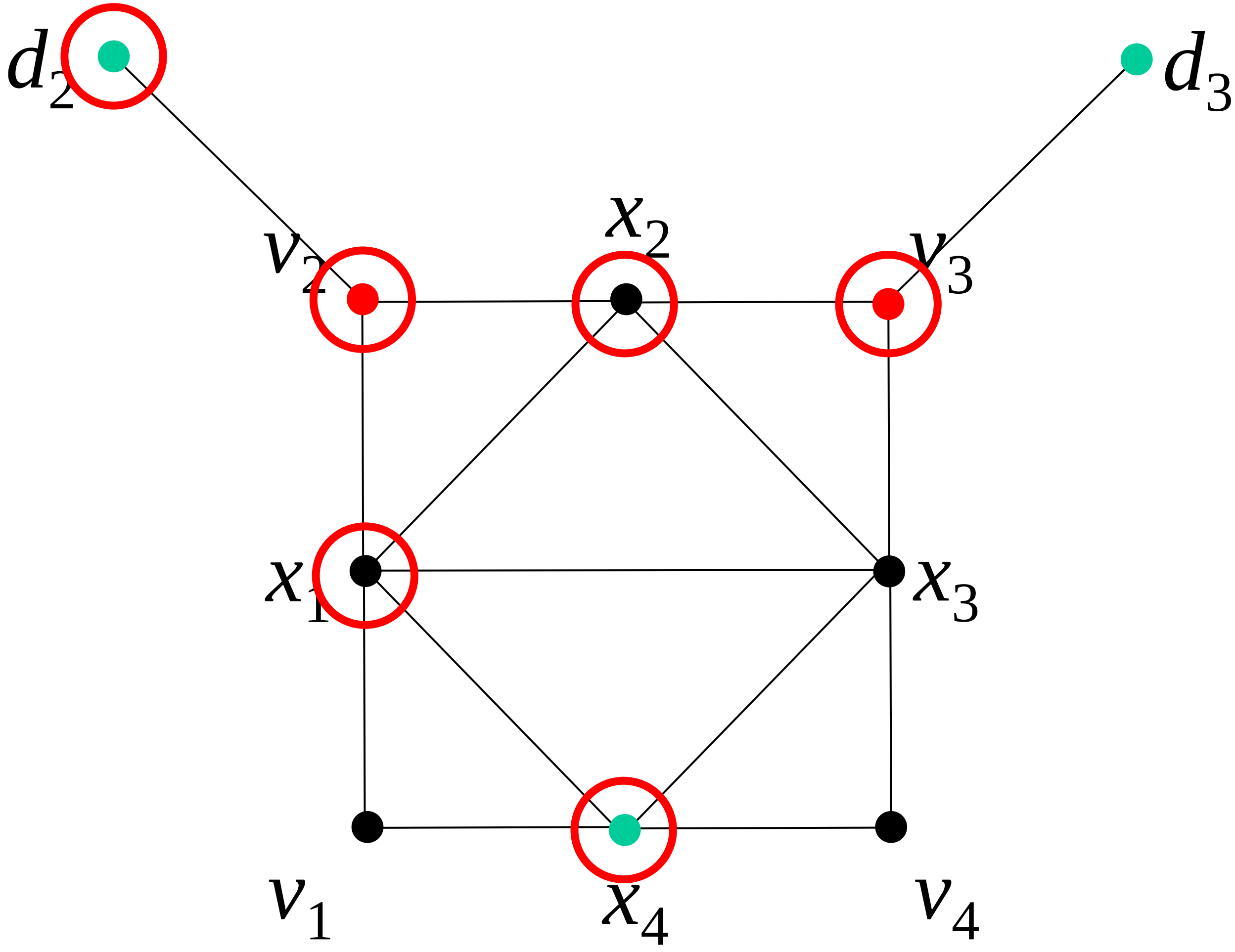
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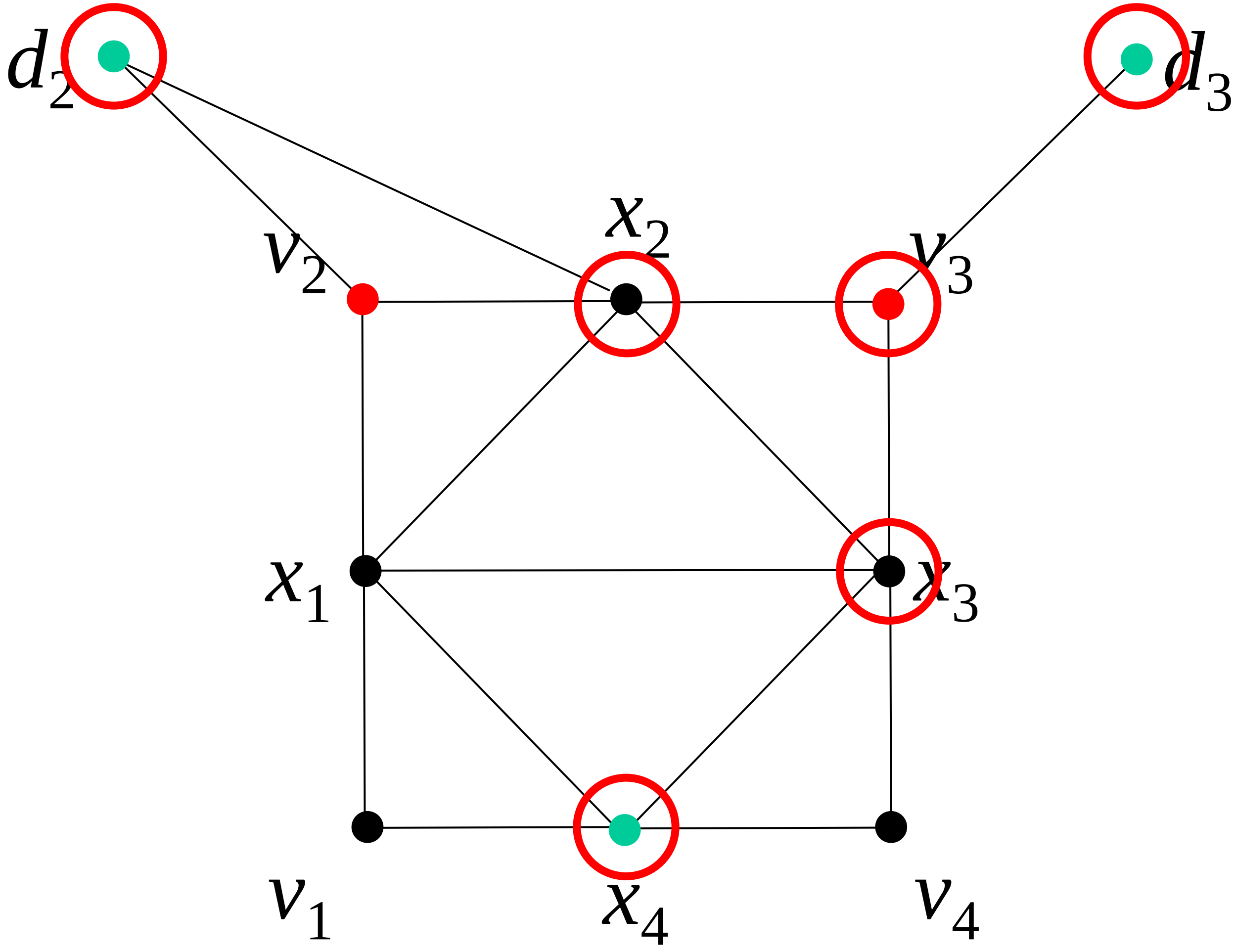

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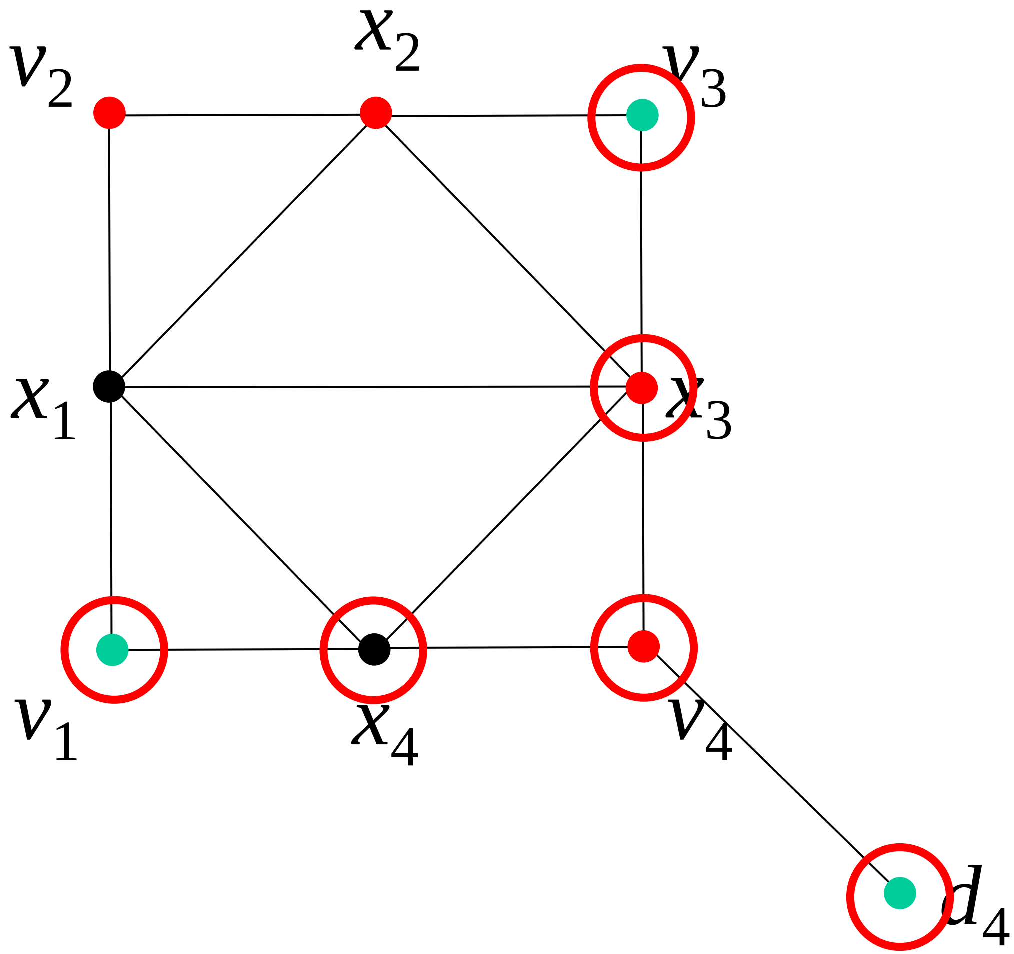

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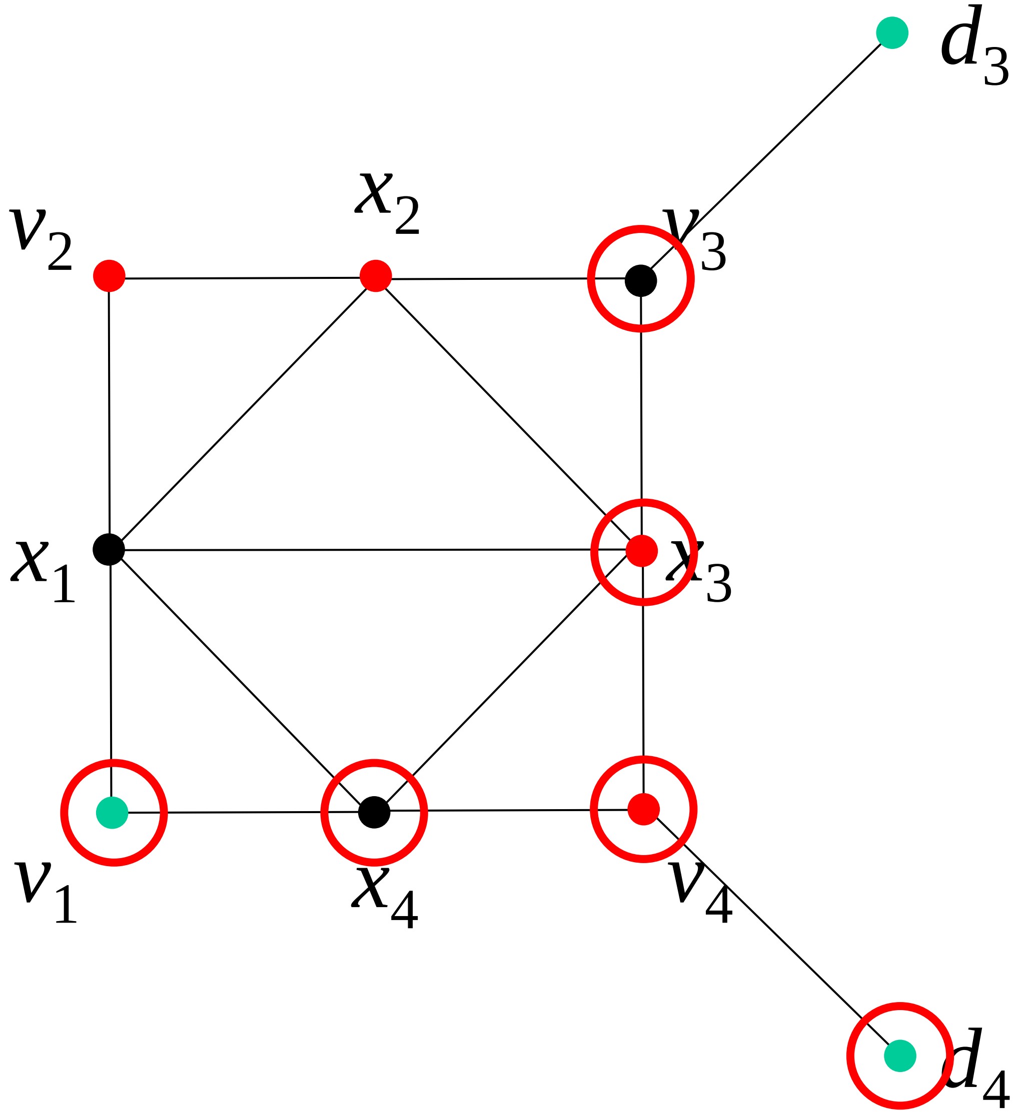

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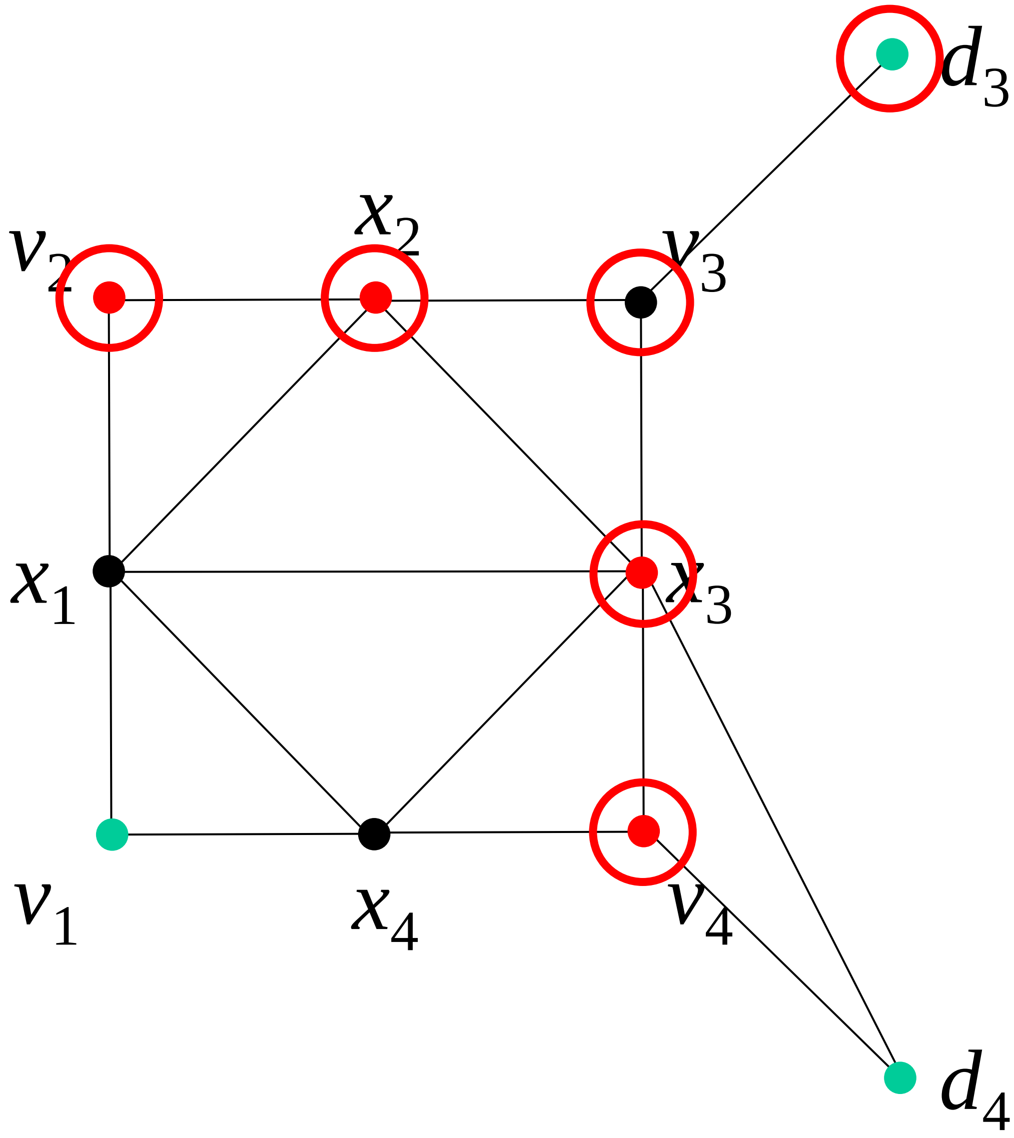

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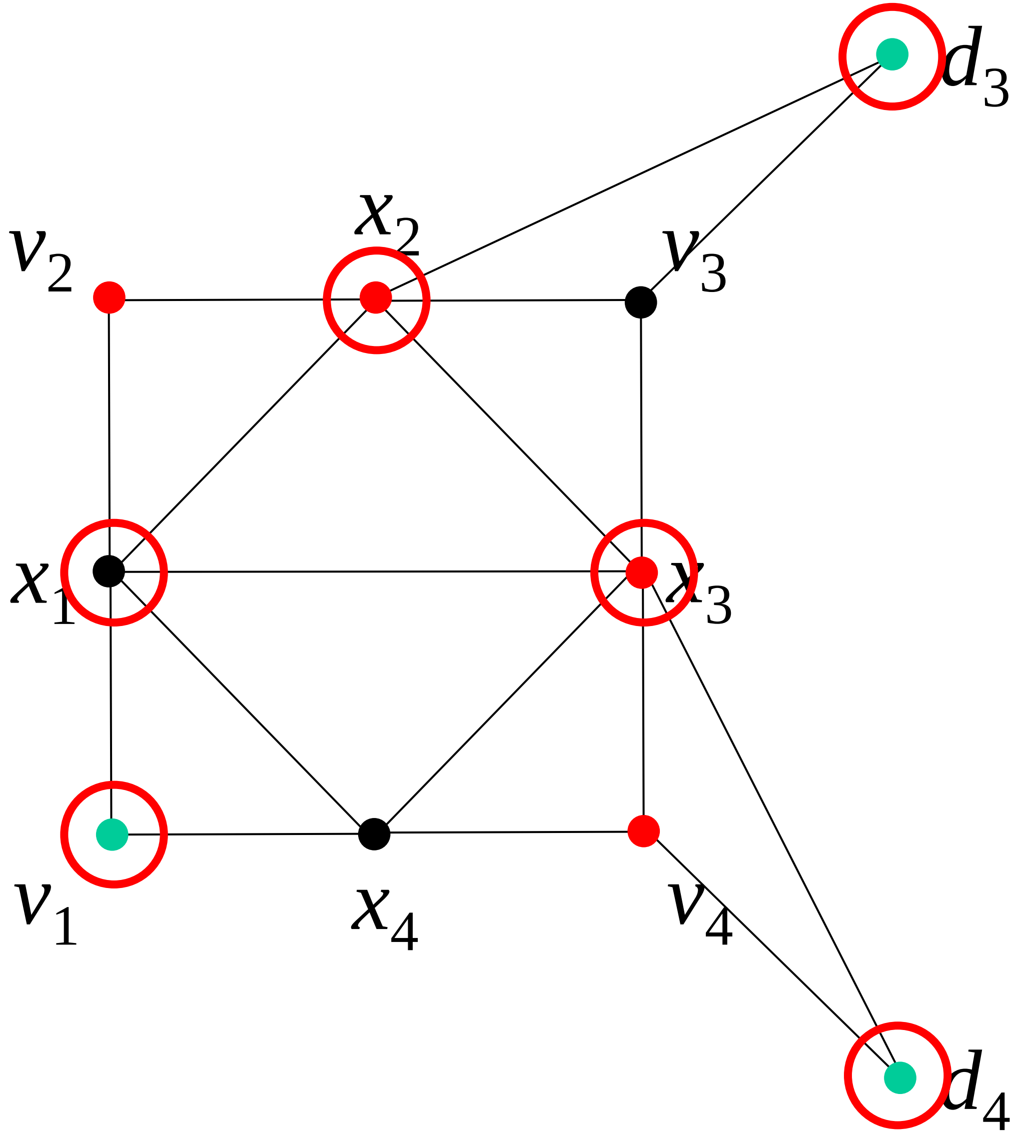

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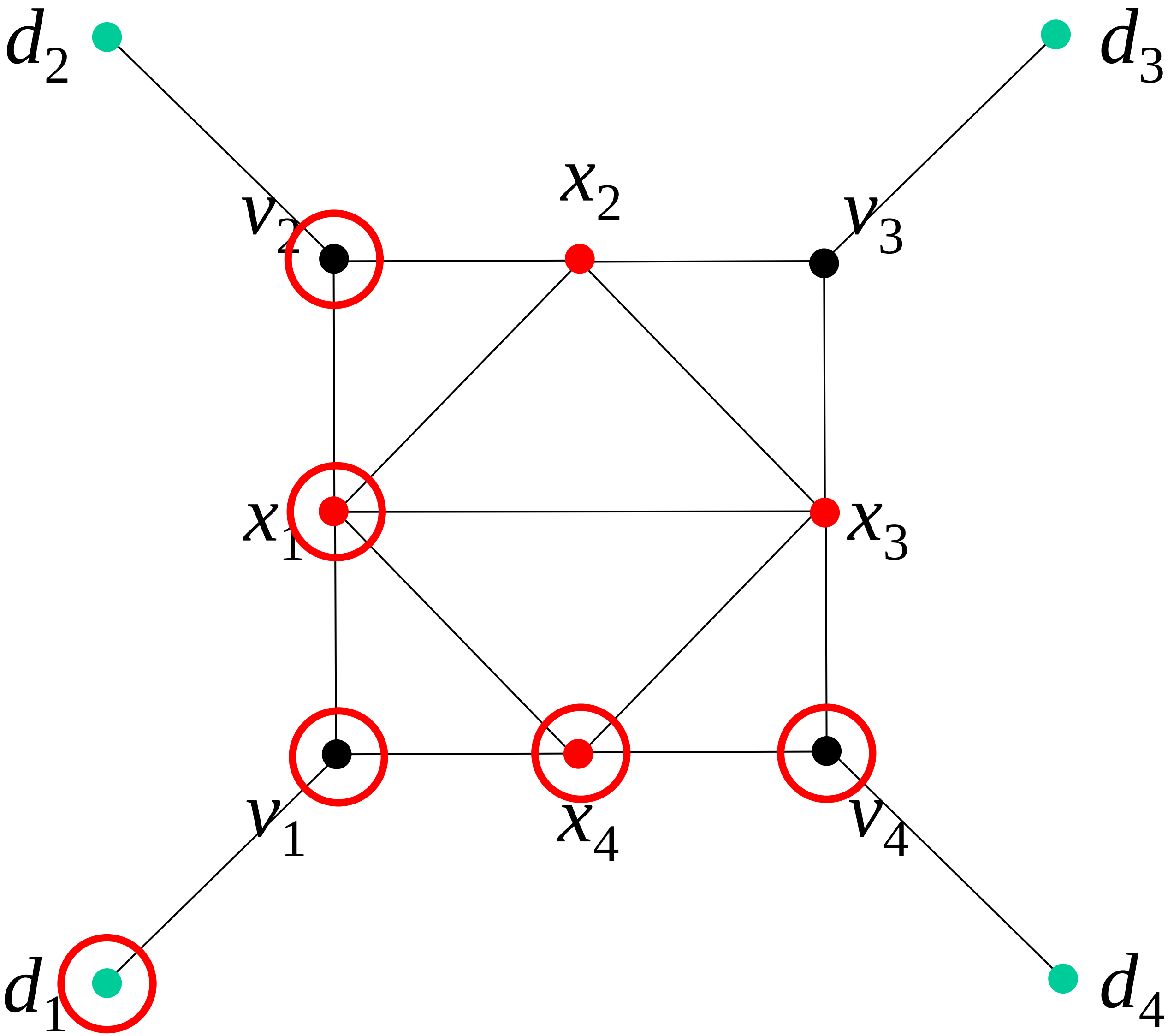
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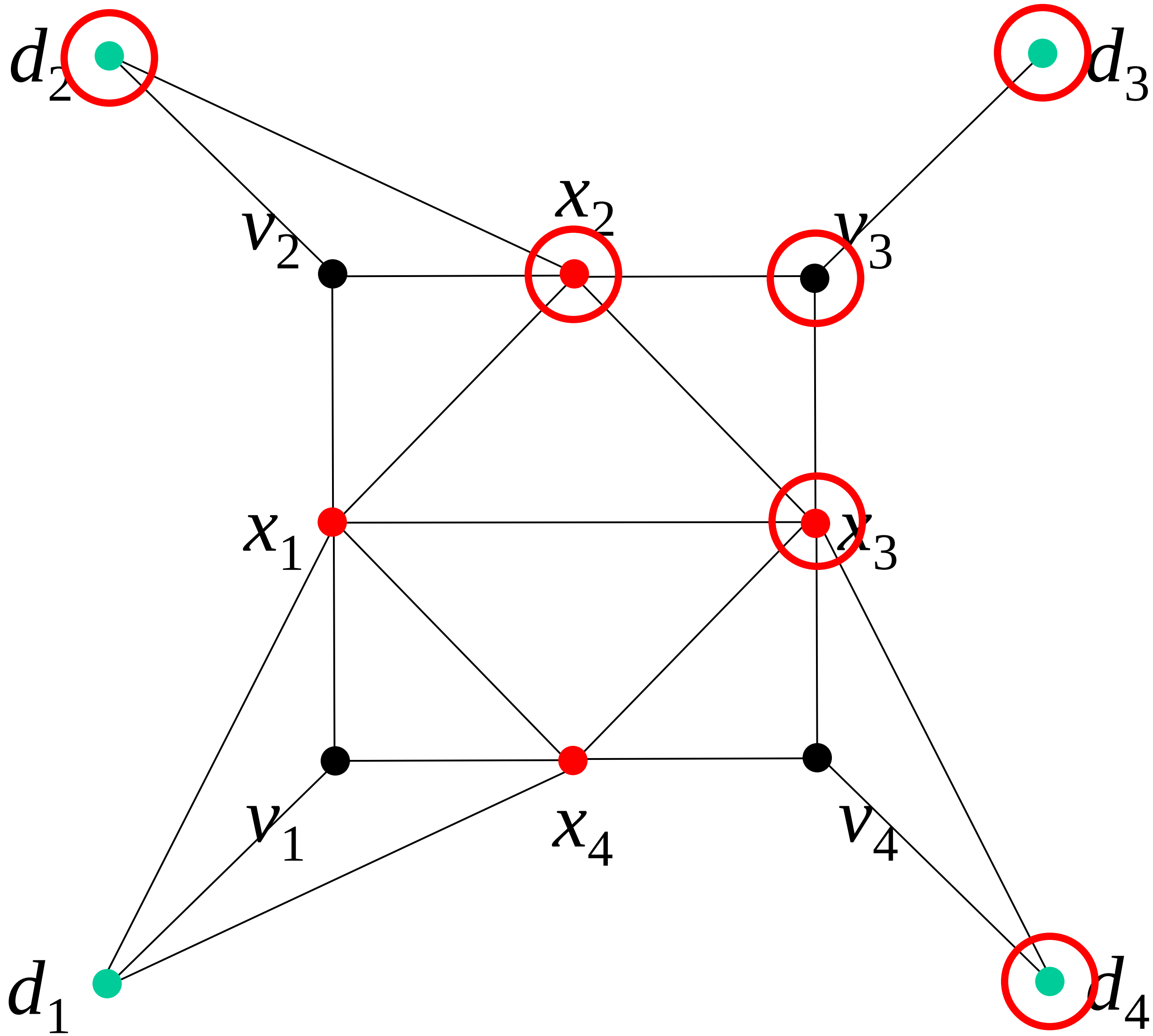
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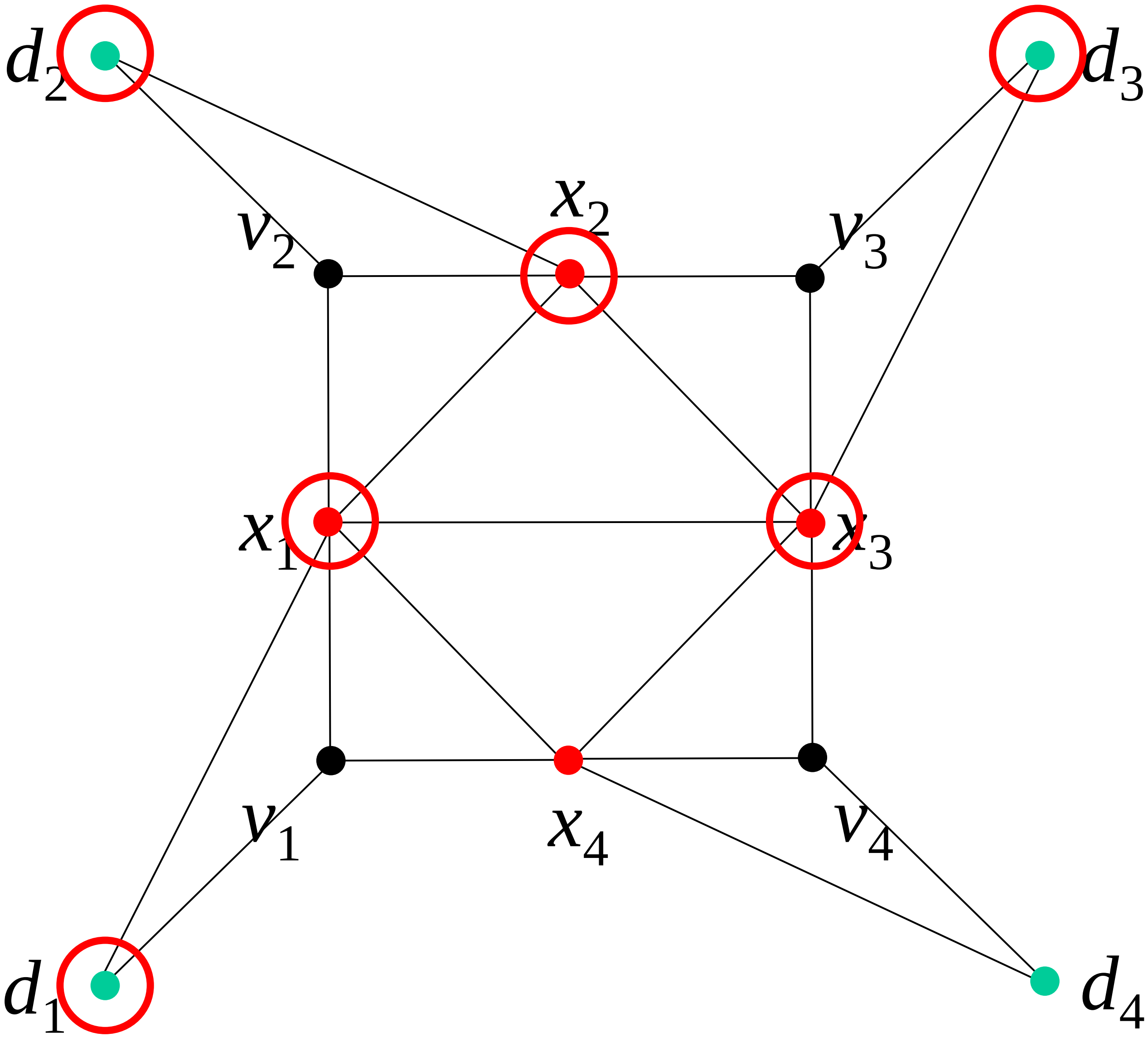

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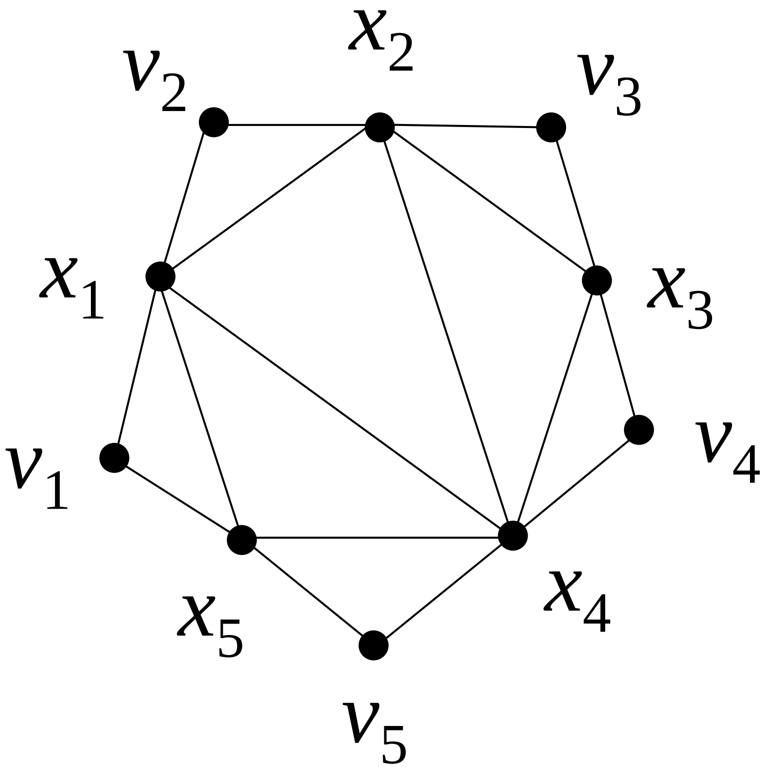
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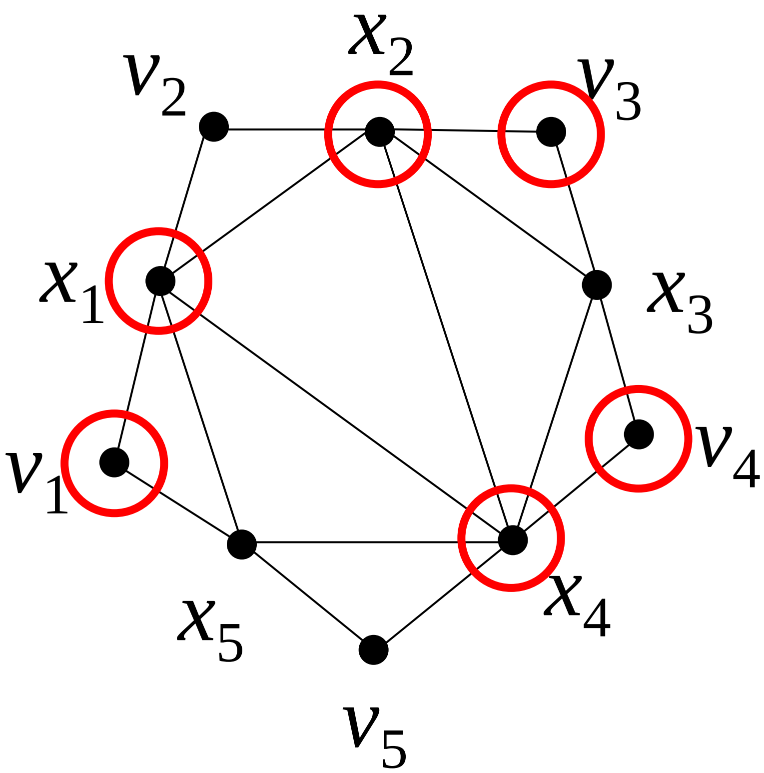

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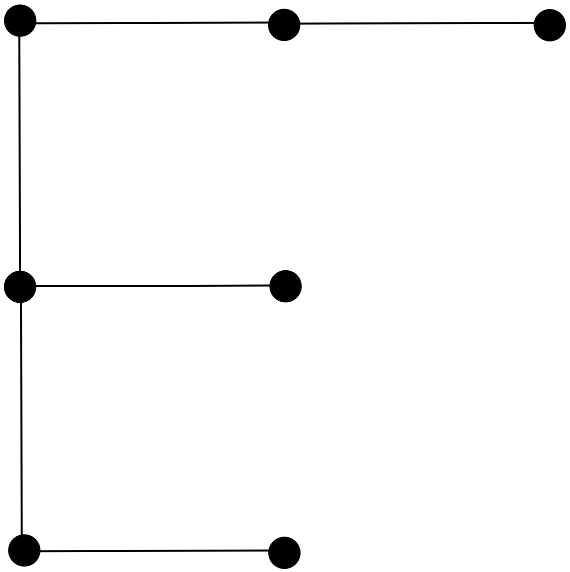


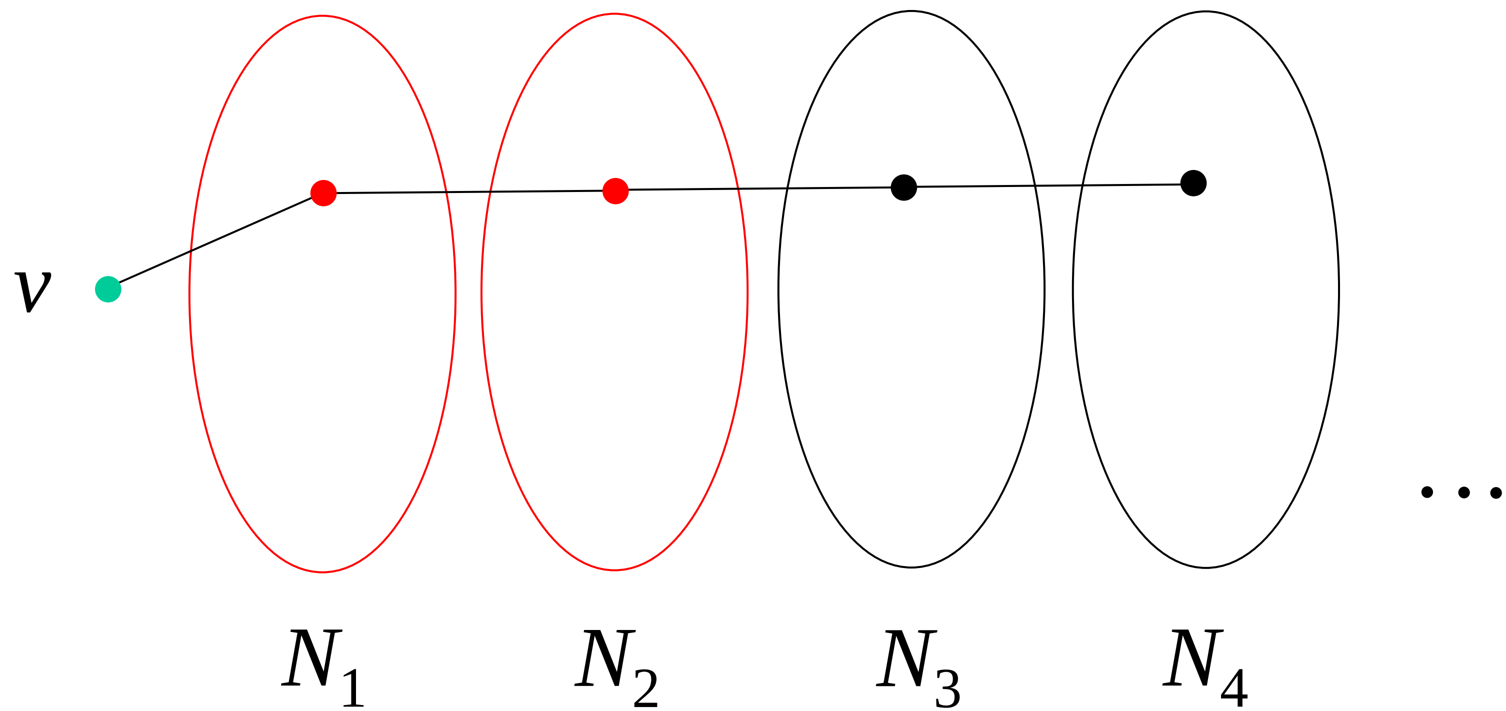
$S_{1,2,3}$

# Efficient domination – direct approach

**Theorem** [B., Mosca 2017]

WED in polynomial time for $S_{1,2,3}$–free chordal graphs.

(This generalizes the result for $P_6$–free chordal graphs.)



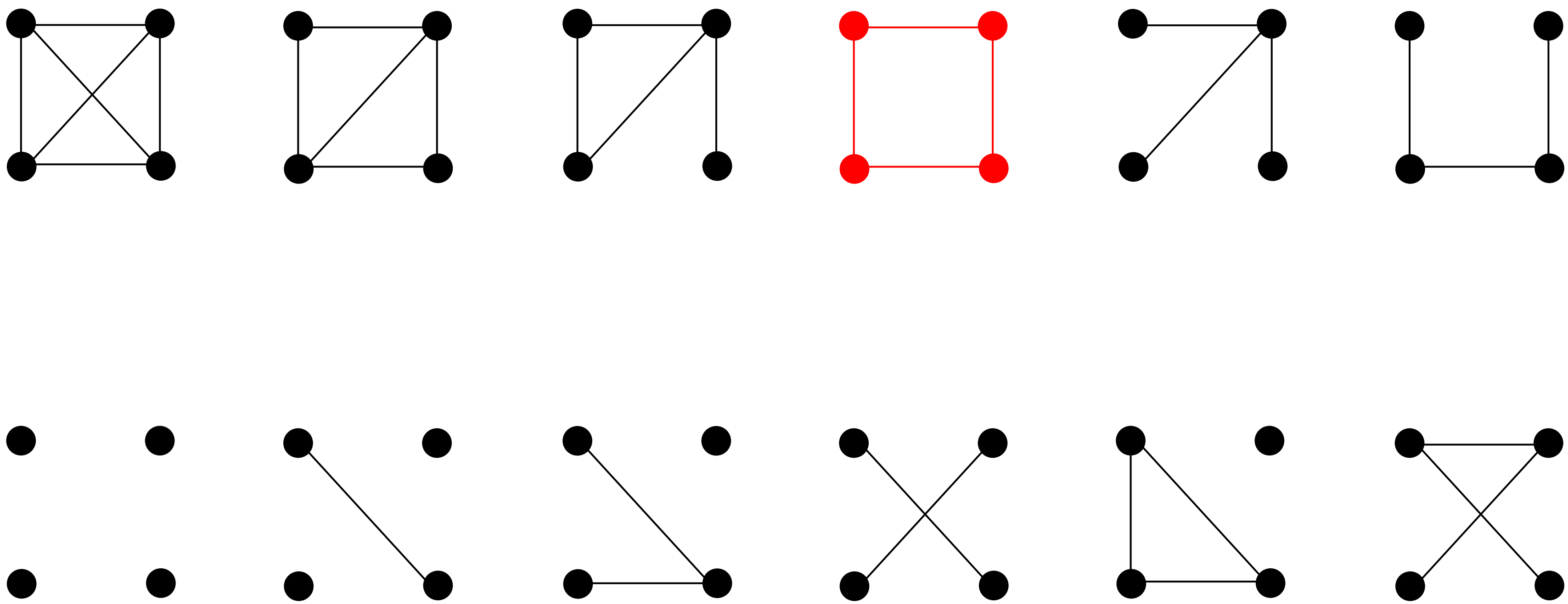
v
N1
N2
N3
N4

# Efficient domination – direct approach

**Theorem** [B., Giakoumakis 2014]

If WED is solvable in polynomial time for $H$-free graphs then WED is solvable in polynomial time for $(H + kP_2)$–free graphs for every fixed $k$.

# All graphs with four vertices:

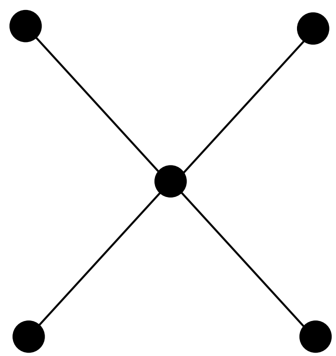


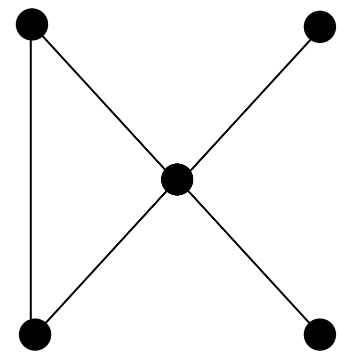
# Efficient domination

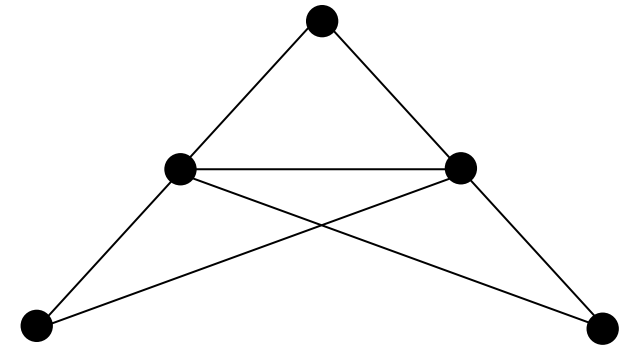
**Theorem** [B., Mosca 2017]

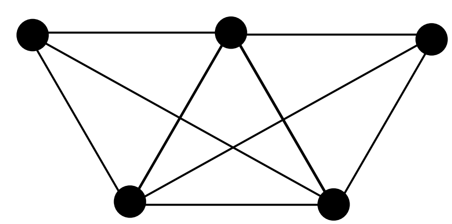
(i) For every chordal graph $H$ with at most four vertices, WED is solvable in polynomial time for $H$–free chordal graphs.

(ii) For chordal graphs $H$ with exactly five vertices, there are exactly four open cases for the complexity of WED on $H$–free chordal graphs:







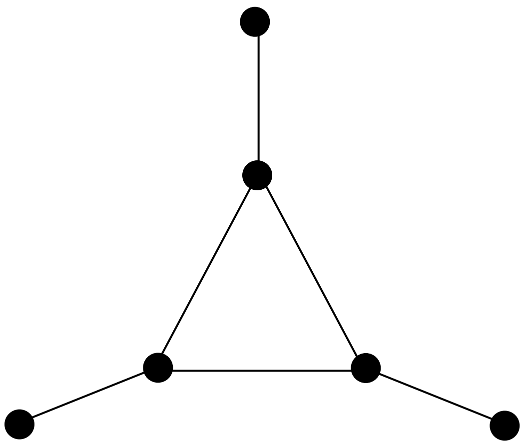


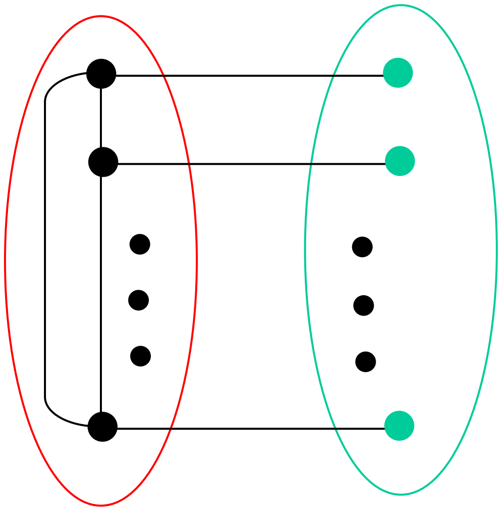
Thank you for your attention!

Thank you for your attention!

Thank you for your attention!

# *thin spider*





# Efficient domination

**Lemma** [B., Milanič, Nevries MFCS 2013]

A prime $2P_2$-free graph has an e.d.s. $\Leftrightarrow$ it is a *thin spider*.

**Thm.** [B., Milanič, Nevries MFCS 2013]

ED in linear time for $2P_2$-free graphs.

**Thm.** [B. 2015] ED in linear time for $P_5$-free graphs.