Gallai's Conjecture for graphs with treewidth 3.

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Joint work with M. Sambinelli, R. S. Coelho, and O. Lee

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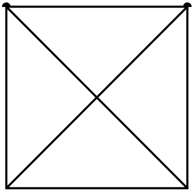

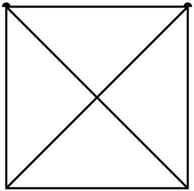

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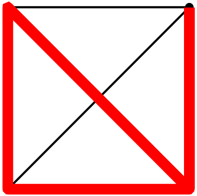

A trail
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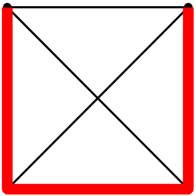

A path.

Decomposition of G:

- $\mathcal{D} = \{H_1, \ldots, H_k\}$, $H_i \subseteq G$
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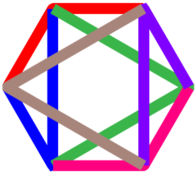

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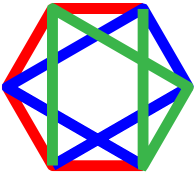

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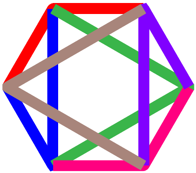

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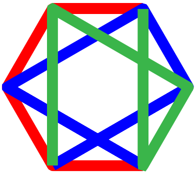

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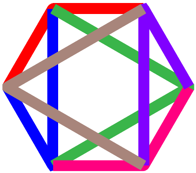

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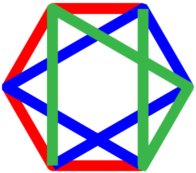

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- $\mathrm{pn(G)}$ – **path number** of G.

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Given graph G, calculate $\mathrm{pn}(\mathrm{G})$.

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- $\mathrm{pn}(\mathrm{G}) = 1$ *if and only if G is a path;*
- *If G is a cycle, then* $\mathrm{pn}(\mathrm{G}) = 2$;
- *If G is a forest with o odd degree vertices, then* $\mathrm{pn}(\mathrm{G}) = \mathrm{o}/2$.

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Conjecture (Gallai, 1966)

If G is a simple connected graph with n vertices, then $\mathrm{pn(G)} \leq \lceil \mathrm{n}/2 \rceil$.

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- If each block of G_{ev} is triangle-free and has maximum degree at most 3, then $\mathrm{pn(G)} \leq \lfloor \mathrm{n}/2 \rfloor$. (Fan, 2005)

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- If G has maximum degree at most 5, then $\mathrm{pn}(G) \leq \lceil n/2 \rceil$. (Bonamy–Perrett, 2016)

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If G has treewidth at most 3, *then* $\mathrm{pn}(\mathrm{G}) \leq \lfloor \mathrm{n}/2 \rfloor$, *or* $G \in \{K_3, K_5^-\}$.

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Reducing subgraphs

- $H \subseteq G$ is an r-**reducing subgraph** of G if $\mathrm{pn}(H) \leq r$ and $G - E(H)$ has at least $2r$ isolated vertices;

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Proof.

$$\mathrm{pn(G)} \leq \mathrm{pn(G - E(H))} + \mathrm{r} \leq \lfloor (\mathrm{n} - 2\mathrm{r})/2 \rfloor + \mathrm{r} = \lfloor \mathrm{n}/2 \rfloor$$

□

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Lemma

Let H be a reducing subgraph of G, and let K be a component of $G - E(H)$ such that $K \in \{K_3, C_4, K_5^-, K_5\}$.
Then $H + K$ is a reducing subgraph of G.

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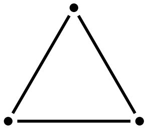

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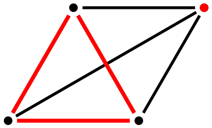

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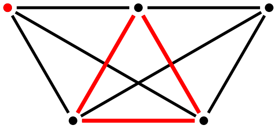

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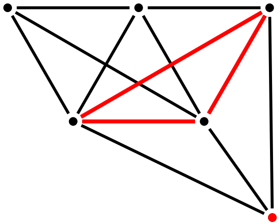

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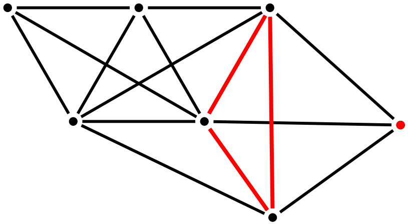

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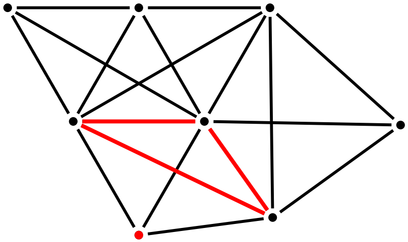

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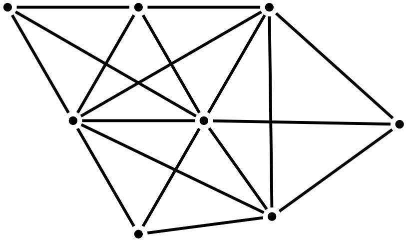

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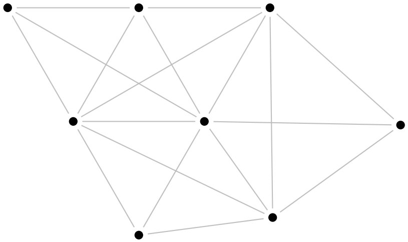

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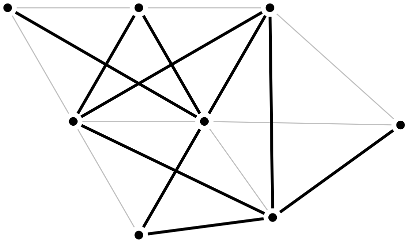

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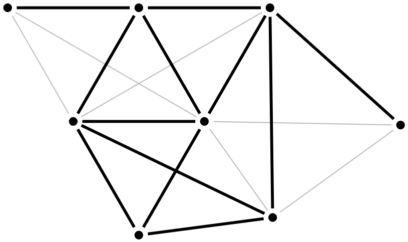

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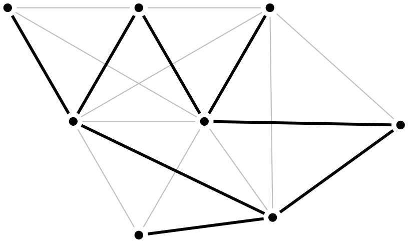

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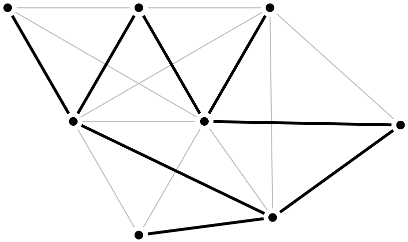

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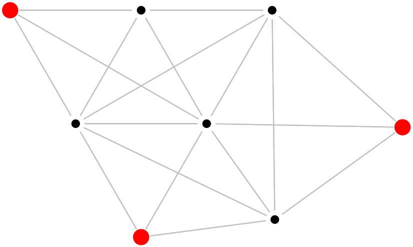

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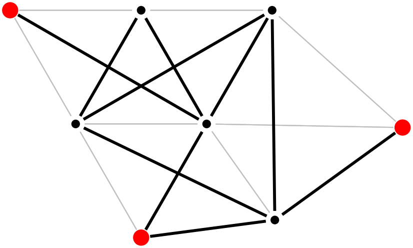

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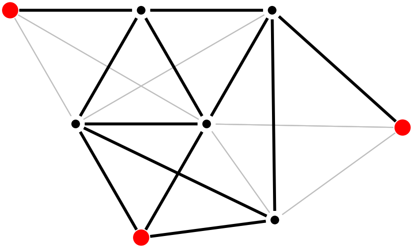

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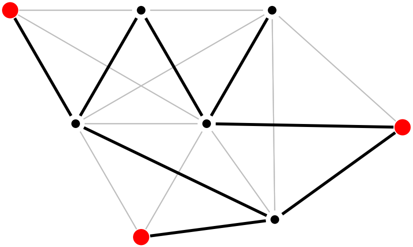

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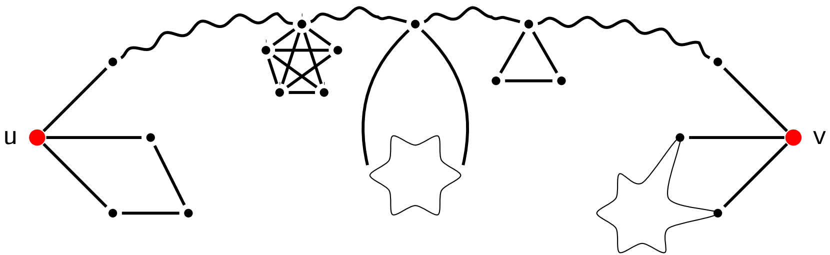

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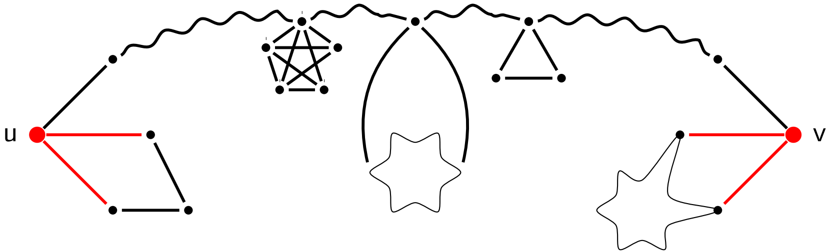

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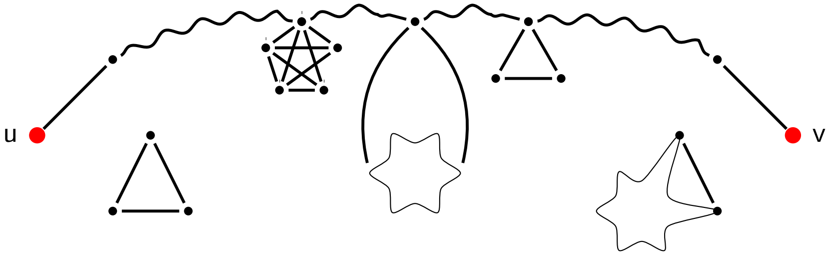

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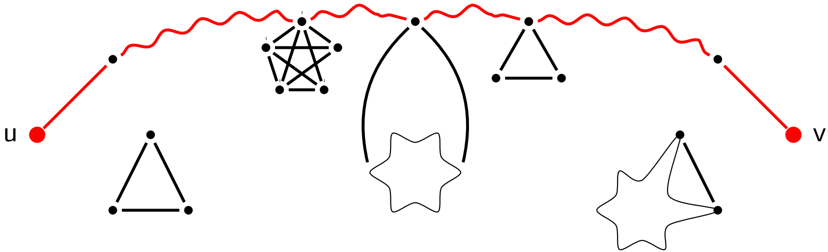

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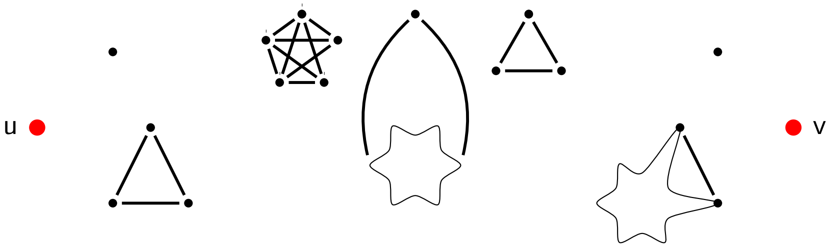

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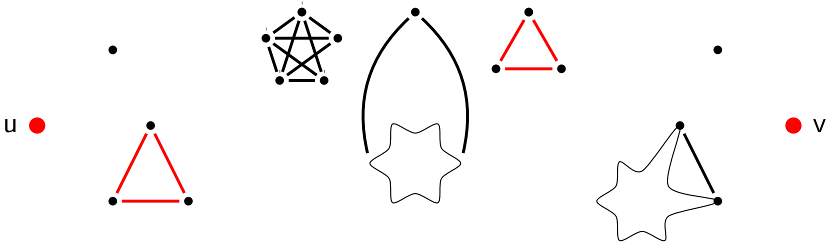

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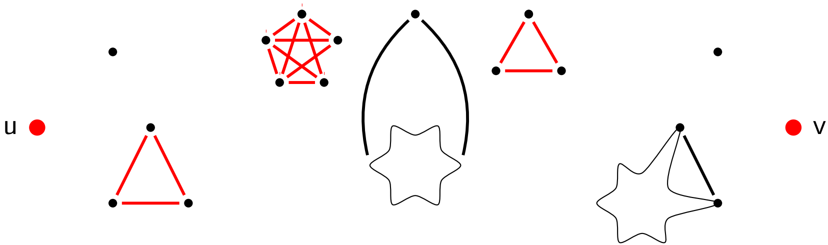

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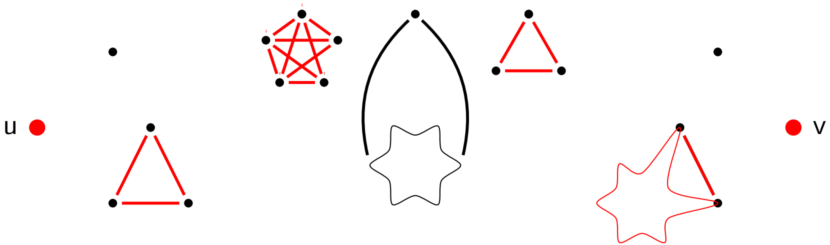

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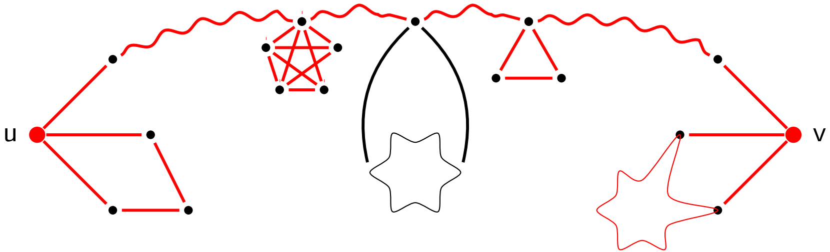

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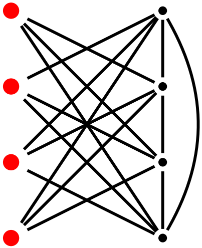

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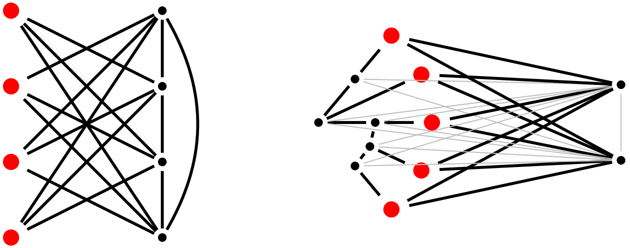

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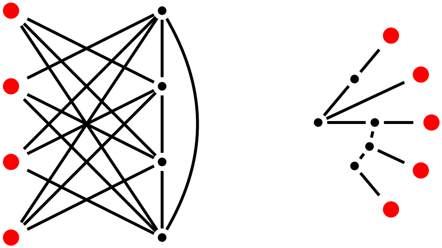

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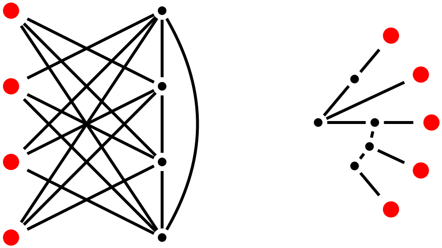

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- If G has maximum degree at most 4, then G is a Gallai graph or $G \in \{K_3, K_5, K_5^-\}$;
- Develop more techniques to obtain reducing subgraphs.

Gallai's Conjecture for graphs with treewidth 3.

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