An Approximation Algorithm for the p-Hub Median Problem

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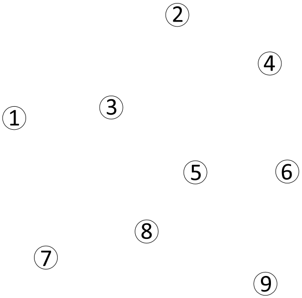
p-Hub Median Problem (pHM)

- Metric pHM

Instance

- a set of clients $V = \{v_1, v_2, \ldots, v_n\}$
- a set os demands $D \subseteq V \times V$
- a cost function $\rho : V \times V \longrightarrow \mathbb{R}^+$
- an integer $p > 0$



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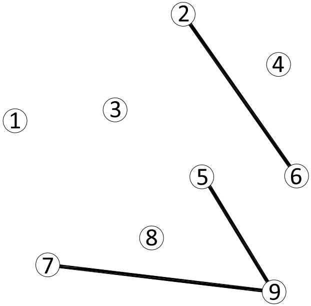

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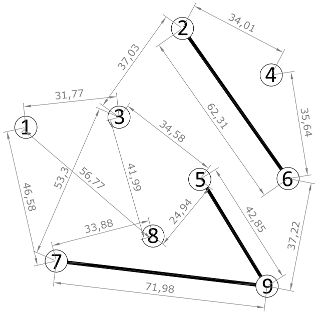

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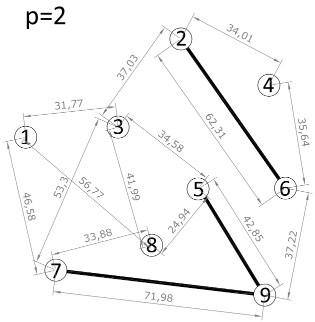

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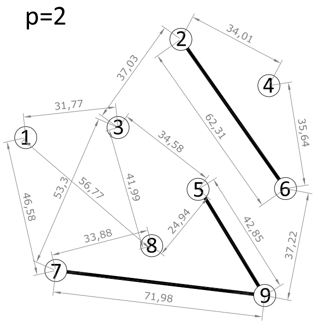
Objective

Select $T \subseteq V$ of terminals, where $|T| \leq p$, and assign each demand to a terminal, in order to minimize the total cost between demands and terminals.



p-Hub Median Problem (pHM)

Theorem

K-Median is a particular case of pHM.

Theorem [Jain et. al., 2002]

K-Median has a approximation factor $\geq 1 + \frac{2}{e}$ if NP $\not\subset$ DTIME($n^{O(\log \log n)}$).

Corollary

pHM has a approximation factor $\geq 1 + \frac{2}{e}$ if NP $\not\subset$ DTIME($n^{O(\log \log n)}$).

p-Hub Median Problem (pHM)

Theorem

A (4α)-approximation algorithm that opens at most $\left(\frac{2\alpha}{2\alpha-1}\right)p$ terminals, where $\alpha > 1$ is a trade off parameter.

Linear Program

- First, we preprocess the input, defining a new cost function $\hat{\rho}$ such that $\hat{\rho}(d, i) = \rho(u, i) + \rho(i, v), \forall d = (u, v) \in D, i \in V$.
- The integer program (IP) formulation for pHM:

$$
\textbf{(IP)} \quad
\begin{array}{lll}
\text{minimize} & \sum_{d \in D} \sum_{i \in V} x_{di} \, \hat{\rho}(d, i) & \\
\text{subject to} & \sum_{i \in V} y_i \leq p & \\
 & \sum_{i \in V} x_{di} = 1, & \forall d \in D \\
 & x_{di} \leq y_i, & \forall d \in D, i \in V \\
 & x_{di} \in \{0, 1\}, & \forall d \in D, i \in V \\
 & y_i \in \{0, 1\}, & \forall i \in V
\end{array}
$$

- Our algorithm uses the LP relaxation of IP 1, where $x_{di} \geq 0$ and $y_i \geq 0, \forall d \in D, i \in V$.

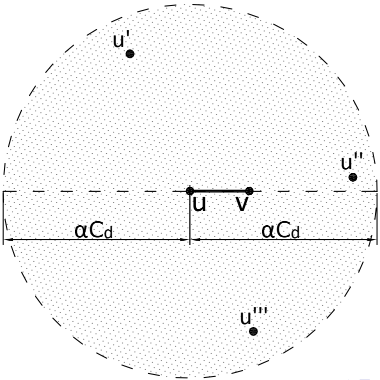
Rounding Algorithm

To each demand $d{=}(u,v) \in D$:

- *"Mean Distance"*: $C_d = \sum_{i \in V} x_{di}\ \hat{\rho}(d,i)$

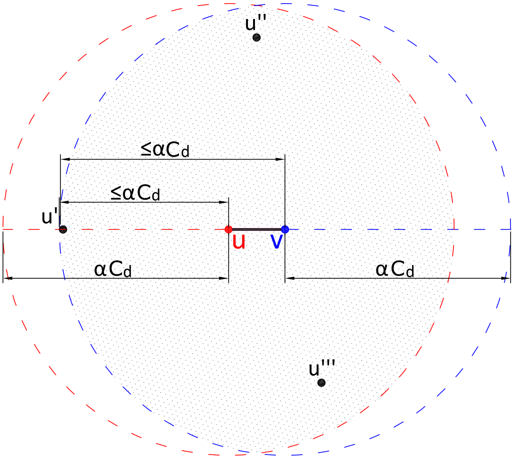
For each $u \in V$, let:

- $\mathrm{B}(u, \alpha C_d)$



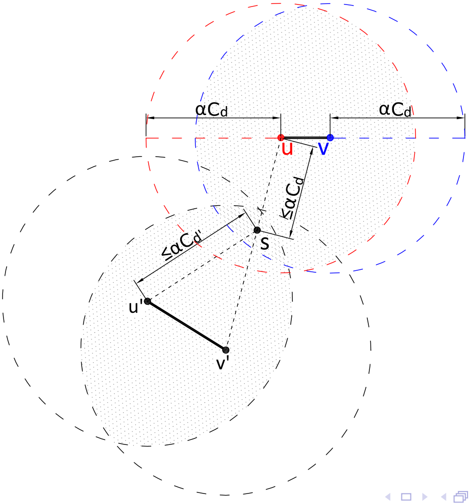
For each $d = (u, v) \in D$, let:

- *Neighborhood*: $I_d = \{u' \in B(u, \alpha C_d) \cap B(v, \alpha C_d)\}$



For each $d \in D$, let:

- *Extended neighborhood*:
  $\overline{V_d} = \{(u', v') \in V^2 : (u', v') \in D \text{ and } I_d \cap I_{(u',v')} \neq \emptyset\}$



Rounding Algorithm

Algorithm 1: Rounding Algorithm.

Solve the LP and use it to compute the C_d values
$T := \{\}$
$\overline{D} := D$
while $\overline{D} \neq \emptyset$ **do**
    Choose $d = (u, v) \in \overline{D}$ with the lowest value of C_d
    $T := T \cup \{u\}$
    for $(u', v') \in \overline{D}$ **do**
        if $(u' \in \overline{V_d})$ ***and*** $(v' \in \overline{V_d})$ **then**
            $\overline{D} := \overline{D} \setminus (u', v')$
    $\overline{D} := \overline{D} \setminus (u, v)$
return T

Conclusions and Future Works

- We got a (4α)-approximation algorithm, with $\alpha > 1$, that opens at most $\left(\frac{2\alpha}{2\alpha-1}\right) p$ terminals.
- We used the rounding technique of linear programs.
- As future works: opening the right number of terminals and improve the approximation factor.

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