# Discrete Schur-Constant Models in Insurance

**Claude Lefèvre**

ISFA and ULB

Jointly with **A. Castañer, M.M. Claramunt, S. Loisel**

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# Outline

# 1. Preliminaries: the continuous case

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# 1.1. Archimedean survival copula

A **survival copula** is defined as

$$C(u_1, \ldots, u_n) = P(U_1 > 1 - u_1, \ldots, U_n > 1 - u_n),$$

where $U_1, \ldots, U_n$ are $n$ (dependent) uniforms on $(0, 1)$.

Such a copula is **Archimedean** if

$$C(u_1, \ldots, u_n) = \psi(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_n))$$

for some univariate survival function $\psi$ (called Archimedean generator).

A survival function $\psi$ may be an Archimedean generator iff $\psi$ is a ***n*-monotone** function.

# 1.2. Schur-constant model

For **continuous** positive random variables (e.g. lifetimes in reliability, claim interarrival times in insurance): see

R.E. Barlow, M.B. Mendel (1993), paper in a book,
L. Caramellino, F. Spizzichino (1996), JMA 56, 153-163,
R.B. Nelsen (2005), BJPS 19, 179-190,
Y. Chi, J. Yang, Y. Qi (2009), IME 44, 398-408, ...

**Definition.**

$(X_1, \ldots, X_n)$ forms a **Schur-constant model** if

$$P(X_1 > x_1, \ldots, X_n > x_n) = S(x_1 + \ldots + x_n),$$

for some univariate survival function $S$ (called Schur-constant).

The model is a special case of exchangeable vector.

All the lower dimensional subvectors of $(X_1, \dots, X_n)$ are also Schur-constant.

The model translates a property of *indifference relative to aging* -or no-aging-:

$$X_i - x_i | (\mathbf{X} > \mathbf{x}) =_d X_j - x_j | (\mathbf{X} > \mathbf{x}).$$

The function $S$ is both Schur-convex and Schur-concave, hence the appellation of Schur-constant.

## Characterization 1.

A survival function $S$ may be a Schur-constant generator iff $S$ is a $n$-monotone function.

$S$ can then be written as

$$S(x) = E\left(1 - \frac{x}{Z}\right)_+^{n-1},$$

where $Z =_d X_1 + \ldots + X_n$. And reciprocally.

Thus, a Schur-constant model is such that

$$P(X_1 > x_1, \ldots, X_n > x_n) = E\left(1 - \frac{x_1 + \ldots + x_n}{Z}\right)_+^{n-1}.$$

## Characterization 2.

A Schur-constant model has the radial representation

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} =_d Z \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}$$

where

* $Z$ is independent of the $U_i$'s, and
* $(U_1, \ldots, U_n)$ is a Schur-constant vector of sum 1 with

$$P(U_1 > u_1, \ldots, U_n > u_n) = [1 - (u_1 + \ldots + u_n)]_+^{n-1} .$$

# 1.3. Schur-constant versus Archimedean

Schur-constant models and Archimedean copulas are closely related. Indeed:

(i) The copula of a Schur-constant model of generator $S$ is Archimedean with the same generator $S$.

(ii) If $(U_1, \ldots, U_n)$ form a Archimedean copula of generator $\psi$, then $[\psi^{-1}(1 - U_1), \ldots, \psi^{-1}(1 - U_n)]$ forms a Schur-constant model with the same generator $\psi$.

# 2. Discrete Schur-constancy

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# 2.1. Discrete Schur-constant models

For **discrete** random variables valued in $N_0 = \{0, 1, 2 \ldots\}$:

A. Castañer, M.M. Claramunt, C. Lefèvre, S. Loisel (2015), JMA 140, 343-362.

**Definition.**

$(X_1, \ldots, X_n)$ forms a **Schur-constant model** if

$$P(X_1 \geq x_1, \ldots, X_n \geq x_n) = S(x_1 + \ldots + x_n),$$

for some univariate survival function $S : N_0 \to [0, 1]$ (called Schur-constant generator).

Probabilities on subvectors are directly obtained.
For instance:

**Property.**

For $(x_1, \ldots, x_j) \in N_0^j$,

$$P(x_1 \leq X_1 < x_1 + h_1, \ldots, x_j \leq X_j < x_j + h_j)$$
$$= (-1)^j \, \Delta_{1,h_1} \ldots \Delta_{j,h_j} S(x_1 + \ldots + x_j),$$
$$P(X_1 = x_1, \ldots, X_j = x_j) = (-1)^j \, \Delta^j S(x_1 + \ldots + x_j),$$

Let $T_j = X_1 + \ldots + X_j$ (partial sums). For $0 \leq t_{j-k+1} \leq \ldots \leq t_j$,

$$P(T_{j-k+1} = t_{j-k+1}, \ldots, T_j = t_j) = (-1)^j \, \Delta^j S(t_j) \binom{t_{j-k+1} + j - k}{j - k}.$$

A function $f(x) : N_0 \to R$ is said to be $n$-monotone if

$$(-1)^j \Delta^j f(x) \geq 0, \quad j = 0, \ldots, n,$$

**Characterization 1.**

A survival function $S$ may be a Schur-constant generator iff $S$ is a $n$-monotone function on $N_0$.

Equivalently, the associated p.m.f. $p$ is a $(n-1)$-monotone function on $N_0$.

$S$ can then be written as

$$S(x) = E\left[\binom{Z-x+n-1}{n-1}\Big/\binom{Z+n-1}{n-1}\right],$$

where $Z =_d X_1 + \ldots + X_n$ with a p.m.f.

$$P(Z = z) = (-1)^n \binom{Z+n-1}{n-1} \Delta^n S(z).$$

Thus, a Schur-constant model is such that

$$P(X_1 \geq x_1, \ldots, X_n \geq x_n) = \\ E\left[\binom{Z-(x_1+\ldots+x_n)+n-1}{n-1}\Big/\binom{Z+n-1}{n-1}\right].$$

**Characterization 2.**

A Schur-constant model has the 'discrete radial' representation that is of doubly mixed multinomial form, namely

$$(X_1, \ldots, X_n) =_d \mathcal{M}M(Z; U_1, \ldots, U_n),$$

where

* $Z$ is independent of the $U_i$'s, and

* $(U_1, \ldots, U_n)$ is a continuous Schur-constant vector of sum 1 with

$$P(U_1 \geq u_1, \ldots, U_n \geq u_n) = [1 - (u_1 + \ldots + u_n)]_+^{n-1}.$$

# 2.2. The special geometric case

The Schur-constancy generalizes the lack of memory of geometrics.

**Property 1.**

In a Schur-constant model, the components $X_i$, $1 \leq i \leq n$, are independent if and only if they are geometrically distributed.

**Property 2.**

An infinite sequence of random variables $\{X_i,\ i \geq 1\}$ with finite mean is Schur-constant iff for all $j \geq 1$, $(X_1, \ldots, X_j)$ has a mixed geometric distribution, namely

$$P(X_1 \geq x_1, \ldots, X_j \geq x_j) = E\left[\left(\frac{\Theta}{\Theta + 1}\right)^{x_1 + \ldots + x_j}\right],$$

where $\Theta = \lim_{n \to \infty} T_n/n$ a.s.

# 3. Monotone survival functions

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# 3.1. Representations

Discrete monotone distributions: see
Lefèvre and Loisel (2013), JAP 50, 827-847.

The previous results allow us to characterize $n$-monotone survival function $S$.

(1) Such a function admits a general representation

$$S(x) = E\left\{ \binom{Z - x + n - 1}{n - 1} \Big/ \binom{Z + n - 1}{n - 1} \right\},$$

for some random variable $Z$ valued in $N$ whose p.m.f. is

$$P(Z = z) = (-1)^n \binom{z + n - 1}{n - 1} \Delta^n S(z).$$

(2) $S$ is the survival function of a r.v. $X$ whose distribution is of doubly mixed binomial, namely

$$X =_d \mathcal{M}B(Z, 1 - U^{1/(n-1)}),$$

where $U$ is uniform on $(0, 1)$, and $Z$ is independent of $U$.

# 3.2. Examples

*Bernoulli model*

$X$ is a Bernoulli random variable of survival function

$$S(0) = 1, \quad S(1) = p, \quad S(x) = 0, \; x \geq 2.$$

$\rightarrow S(x)$ is $n$-monotone iff $p \leq 1/n$.

## *Stop-loss model*

$X$ has a survival function of stop-loss type

$$S(x) = \frac{(k-x)_+^t}{k^t}, \quad x \in N,$$

where $k$ and $t$ are positive integers.

$\rightarrow S(x)$ is $(t+1)$-monotone.

Proof. Based on the expansion

$$\frac{(k-x)_+^t}{t!} = \sum_{i=0}^{t-1} \alpha_i(t) \binom{k-x+i}{t},$$

where $\{\alpha_i(t), 0 \leq i \leq t-1\}$ is a symmetric p.m.f.

*Other models*

* Power-type: for $k$ positive integer and $t$ positive real,

$$S(x) = [1 - (x/k)^t]_+, \quad x \in N.$$

$\to S(x)$ is 2-monotone iff $t \leq 1$.

* Gompertz-type: for $\theta$ positive real,

$$S(x) = exp[\theta(1 - e^x)], \quad x \in N.$$

$\to S(x)$ is $n$-monotone iff $\theta \geq \theta_n = \ldots$

* Logarithmic, Benford, Pareto . . . . . . . . . . . .

# 4. Schur-constant interarrival models

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# 4.1. Claim counting process

We introduce an associated counting process as

$$N(t) = \sum_{i=1}^{n} I(T_i \leq t), \quad t \in N,$$

where $T_i = X_1 + \ldots + X_i$ and $\{X_1, \ldots, X_n\}$ is Schur-constant.

In insurance, suppose that a maximum number of $n$ claims can arise in a portfolio. Let $T_i$ denote the claim arrival time of the $i$-th claim. Then, $N(t)$ represents the total number of claims that occur until time $t$.

**Property 1.**

(1) For $t \geq 0$,

$$P[N(t) = n] = P(T_n \leq t), \text{ with } T_n =_d Z,$$
$$P[N(t) = k] = (-1)^k \Delta^k S(t+1) \binom{t+k}{k}, \; 0 \leq k \leq n-1.$$

(2) For $0 \leq t_1 \leq \ldots \leq t_k \leq t$,

$$P[T_1 = t_1, \ldots, T_k = t_k | N(t) = k] = 1/\binom{t+k}{k}, \; 1 \leq k \leq n-1.$$

(Given $N(t) = k$, the arrival times are obtained by throwing $k$ undistinguishible balls in $t+1$ urns (the instants $0, \ldots, t$)).

**Property 2.**

In an infinite Schur-model, $N(t)$ has a mixed negative binomial distribution

$$N(t) =_d \mathcal{M}NB[t+1, 1/(\Theta+1)],$$

where $\Theta$ is the limit defined as before. Explicitly,

$$P[N(t)=k] = \binom{t+k}{k} E\left[\left(\frac{1}{\Theta+1}\right)^k \left(\frac{\Theta}{\Theta+1}\right)^{t+1}\right], \quad k \geq 0.$$

# 4.2. Random payment process

A compound Schur-constant sum of discounted claims:

$$R(t) = \sum_{i=1}^{N(t)} C_i \prod_{j=1}^{T_i} v_j = \sum_{i=1}^{n} I(T_i \leq t)\, C_i \prod_{j=1}^{T_i} v_j, \quad t \in N,$$

where $T_i = X_1 + \ldots + X_i$ is the $i$-th payment time, $C_i$ is the claim amount at $T_i$, independent of the payment times, and $v_j$ is a deterministic discount factor for the period $(j-1, j)$.

Our purpose is to determine the Laplace transform of $R(t)$, in terms of the Laplace transform of $C_i$.

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# 4.3. Insurance risk process

A discrete-time risk model in which claims occur according to a Schur-constant counting process $N(t)$:

$$U(t) = h(t) - \sum_{i=1}^{N(t)} C_i, \quad t \in N,$$

where the claim amounts $C_i$ are independent of the claim arrival process (but may be dependent), and the cumulated premiums until time $t$ are given by an increasing function $h(t)$.

Ruin occurs when the reserves $U(t)$ become negative. Our purpose is to derive a formula for $\phi(t)$, the probability of non-ruin until time $t$.

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