Applications of the multivariate tail process for extremal inference

Anja Janßen

University of Copenhagen

(joint project with Holger Drees, University of Hamburg)

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Multivariate regularly varying time series

- We will deal with a stationary multivariate regularly varying time series $(\mathbf{X}_t)_{t\in\mathbb{Z}}$, $\mathbf{X}_t \in \mathbb{R}^d$.
- The multivariate regular variation is equivalent to existence of a so-called "spectral tail process" $(\Theta_t)_{t\in\mathbb{Z}}$, such that

$$\mathcal{L}\left(\frac{\mathbf{X}_{-n}}{x},\ldots,\frac{\mathbf{X}_m}{x}\,\middle|\,\|\mathbf{X}_0\|>x\right)\overset{w}{\Rightarrow}\mathcal{L}(Y\cdot\Theta_{-n},\ldots,Y\cdot\Theta_m), x\to\infty,$$

for a random variable Y which is Par(α)-distributed and independent of $(\Theta_t)_{t\in\mathbb{Z}}$ (cf. Basrak & Segers (2009)). This, in turn, is equivalent to $\|\mathbf{X}_0\|$ being regularly varying with index α and

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Example and aim

- Think for example of Random Difference Equations with

$$\mathbf{X}_t = \mathbf{A}_t \mathbf{X}_{t-1} + \mathbf{B}_t, \quad t \in \mathbb{Z},$$

for random i.i.d. $(\mathbf{A}_t, \mathbf{B}_t), t \in \mathbb{Z}$, with $\mathbf{A}_t \in \mathbb{R}^{d \times d}, \mathbf{B}_t \in \mathbb{R}^d$.

$\Rightarrow$ Under assumptions of Kesten (1973) the stationary solution is a multivariate regularly varying time series.

- Our aim: Estimation of the distribution of Θ_t, in particular for $t = 1$.
- The distribution of Θ_1 is of particular importance if $(\mathbf{X}_t)_{t \in \mathbb{Z}}$ is Markovian, because in this case the joint distribution of (Θ_0, Θ_1) (together with α) determines the whole structure of $(\Theta_t)_{t \in \mathbb{Z}}$ (cf. J. & Segers (2014))!

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- Our aim: Estimation of the distribution of $\mathbf{\Theta}_t$, in particular for $t = 1$.
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The straightforward thing to do...

For the estimation of the law of Θ_1, use for $A \in \mathbb{B}^d$

$$P\left(\frac{\mathbf{X}_1}{\|\mathbf{X}_0\|} \in A \,\middle|\, \|\mathbf{X}_0\| > x\right) = \frac{P\left(\frac{\mathbf{X}_1}{\|\mathbf{X}_0\|} \in A, \|\mathbf{X}_0\| > x\right)}{P(\|\mathbf{X}_0\| > x)} \overset{w}{\Rightarrow} P(\mathbf{\Theta}_1 \in A),$$

as $x \to \infty$ (if $P(\mathbf{\Theta}_1 \in \partial A) = 0$) to motivate the estimator

Forward estimator

$$\hat{P}_{n,f}(A) := \frac{\sum_{i=1}^{n-1} \mathbb{1}_{\{\|\mathbf{X}_i\| > u_n\}} \mathbb{1}_{\left\{\frac{\mathbf{X}_{i+1}}{\|\mathbf{X}_i\|} \in A\right\}}}{\sum_{i=1}^{n-1} \mathbb{1}_{\{\|\mathbf{X}_i\| > u_n\}}} \quad \text{for} \quad P(\mathbf{\Theta}_1 \in A),$$

based on the observations $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ with suitable threshold u_n.

... but we know more about $(\boldsymbol{\Theta}_t)_{t\in\mathbb{Z}}$!

The stationarity assumption about $(\mathbf{X}_t)_{t\in\mathbb{Z}}$ implies some properties of the spectral tail process $(\boldsymbol{\Theta}_t)_{t\in\mathbb{Z}}$.

"Time change formula" (Basrak & Segers (2009))

Let $(\boldsymbol{\Theta}_t)_{t\in\mathbb{Z}}$ be a spectral tail process of a stationary time series. For all $i, s, t \in \mathbb{Z}$ with $s \le 0 \le t$ and for all bounded and measurable functions $f : (\mathbb{R}^d)^{t-s+1} \to \mathbb{R}$ with $f(y_s, \ldots, y_t) = 0$ if $y_0 = 0$:

$$E\left(f(\Theta_{s+i}, \ldots, \Theta_{t+i})\right) = E\left(f\left(\frac{\Theta_s}{\|\Theta_{-i}\|}, \ldots, \frac{\Theta_t}{\|\Theta_{-i}\|}\right)\|\Theta_{-i}\|^{\alpha}\right).$$

(Remember that α is the index of regular variation of the underlying time series $(\mathbf{X}_t)_{t\in\mathbb{Z}}$.)

Application to the estimation of $P(\mathbf{\Theta}_1 \in A)$

From the last slide:

$$E\left(f(\Theta_{s+i},\ldots,\Theta_{t+i})\right) = E\left(f\left(\frac{\Theta_s}{\|\Theta_{-i}\|},\ldots,\frac{\Theta_t}{\|\Theta_{-i}\|}\right)\|\Theta_{-i}\|^{\alpha}\right).$$

For $A \in \mathbb{B}^d$ with $\mathbf{0} \notin A$ set $f(y_0) = \mathbb{1}_A(y_0)$. Then

$$\begin{aligned} P(\mathbf{\Theta}_1 \in A) &= E(f(\mathbf{\Theta}_{0+1})) \\ &= E\left(f\left(\frac{\mathbf{\Theta}_0}{\|\mathbf{\Theta}_{-1}\|}\right)\|\mathbf{\Theta}_{-1}\|^{\alpha}\right) \\ &= E\left(\mathbb{1}_A\left(\frac{\mathbf{\Theta}_0}{\|\mathbf{\Theta}_{-1}\|}\right)\|\mathbf{\Theta}_{-1}\|^{\alpha}\right) \end{aligned}$$

The not so straightforward thing to do...

For the estimation of the law of Θ_1, use

$$E\left(\mathbb{1}_A\left(\frac{\mathbf{X}_0}{\|\mathbf{X}_{-1}\|}\right)\left(\frac{\|\mathbf{X}_{-1}\|}{\|\mathbf{X}_0\|}\right)^\alpha \;\middle|\; \|\mathbf{X}_0\| > x\right)$$

$$= \frac{E\left(\mathbb{1}_A\left(\frac{\mathbf{X}_0}{\|\mathbf{X}_{-1}\|}\right)\left(\frac{\|\mathbf{X}_{-1}\|}{\|\mathbf{X}_0\|}\right)^\alpha \mathbb{1}_{(x,\infty)}(\|\mathbf{X}_0\|)\right)}{P(\|\mathbf{X}_0\| > x)} \overset{w}{\Rightarrow} P(\mathbf{\Theta}_1 \in A)$$

as $x \to \infty$, to motivate the

Backward estimator

$$\hat{P}_{n,b}(A) := \frac{\sum_{i=2}^n \mathbb{1}_{\{\|\mathbf{X}_i\|>u_n\}} \mathbb{1}_{\left\{\frac{\mathbf{X}_i}{\|\mathbf{X}_{i-1}\|}\in A\right\}} \left(\frac{\|\mathbf{X}_{i-1}\|}{\|\mathbf{X}_i\|}\right)^\alpha}{\sum_{i=2}^n \mathbb{1}_{\{\|\mathbf{X}_i\|>u_n\}}}$$

based on the observations $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ with suitable threshold u_n.
Cf. Drees, Segers and Warchoł (2015) for univariate setting!

Why this?

In the following, let $d = 2$ and concentrate on sets

$$A_{y,t} = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| > y, \varphi(\mathbf{x}) \le t\},$$

where $\varphi(\mathbf{x}) \in [0, 2\pi]$ denotes the angle between $\mathbf{x}$ and the positive x-axis. Compare

$$\hat{P}_{n,f}(A_{y,t}) = \frac{\sum_{i=1}^{n-1} \mathbb{1}_{\{\|\mathbf{X}_i\| > u_n\}} \mathbb{1}_{\left\{\frac{\|\mathbf{X}_{i+1}\|}{\|\mathbf{X}_i\|} > y, \varphi(\mathbf{X}_{i+1}) \le t\right\}}}{\sum_{i=1}^{n-1} \mathbb{1}_{\{\|\mathbf{X}_i\| > u_n\}}}$$

with

$$\hat{P}_{n,b}(A_{y,t}) = \frac{\sum_{i=2}^{n} \mathbb{1}_{\{\|\mathbf{X}_i\| > u_n\}} \mathbb{1}_{\left\{\frac{\|\mathbf{X}_i\|}{\|\mathbf{X}_{i-1}\|} > y, \varphi(\mathbf{X}_i) \le t\right\}} \left(\frac{\|\mathbf{X}_{-1}\|}{\|\mathbf{X}_0\|}\right)^{\alpha}}{\sum_{i=2}^{n} \mathbb{1}_{\{\|\mathbf{X}_i\| > u_n\}}}$$

Variance comparison

Let y be large. For an extreme value of $\|\mathbf{X}_i\|$ it is rare that $\|\mathbf{X}_{i+1}\| > y\|\mathbf{X}_i\|$ and more likely that $\|\mathbf{X}_i\| > y\|\mathbf{X}_{i-1}\|$, which heuristically suggests a smaller variance of latter estimator for higher values of y.

Theorem

Let $(\mathbf{X}_t)_{t\in\mathbb{Z}}$ be a stationary multivariate regularly varying time series. Under suitable conditions (including β-mixing, continuity of $P^{\mathbf{\Theta}_1}$, assumptions about duration and moments of extremal clusters), for $y_0 \geq 0, \tilde{y}_0 > 0$, the process

$$\sqrt{nv_n}\begin{pmatrix} \left(\hat{P}_{n,f}(A_{y,t}) - P\left(\frac{\|\mathbf{X}_1\|}{\|\mathbf{X}_0\|} > y, \varphi(\mathbf{X}_1) \leq t \,\middle|\, \|\mathbf{X}_0\| > u_n\right)\right)_{y\geq y_0, t\in[0,2\pi]} \\ \left(\hat{P}_{n,b}(A_{\tilde{y},t}) - E\left(\left(\frac{\|\mathbf{X}_{-1}\|}{\|\mathbf{X}_0\|}\right)^{\alpha} \mathbb{1}_{\{\frac{\|\mathbf{X}_0\|}{\|\mathbf{X}_{-1}\|} > \tilde{y}, \varphi(\mathbf{X}_0) \leq t\}} \,\middle|\, \|\mathbf{X}_0\| > u_n\right)\right)_{\tilde{y}\geq\tilde{y}_0, t\in[0,2\pi]} \end{pmatrix}$$

with $v_n = P(\|\mathbf{X}_0\| > u_n)$ converges weakly in ℓ^∞ to a centered Gaussian process.

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Generalized tail array sums

For the theory: Observe $(\mathbf{X}_0, \dots, \mathbf{X}_{n+1})$ and write estimators as

$$\hat{P}_{n,f}(A_{y,t}) = \frac{\sum_{i=1}^{n} \Phi_{y,t}(\mathbf{X}_{n,i})}{\sum_{i=1}^{n} \chi(\mathbf{X}_{n,i})} \quad \text{and} \quad \hat{P}_{n,b}(A_{y,t}) = \frac{\sum_{i=1}^{n} \Psi_{y,t}(\mathbf{X}_{n,i})}{\sum_{i=1}^{n} \chi(\mathbf{X}_{n,i})}$$

for functions

$$\chi(\mathbf{X}_{n,i}) = \mathbb{1}_{\{\|\mathbf{X}_i\|>1\}}, \Phi_{y,t}(\mathbf{X}_{n,i}) = \mathbb{1}_{\left\{\|\mathbf{X}_i\|>1, \frac{\|\mathbf{X}_{i+1}\|}{\|\mathbf{X}_i\|}>y, \varphi(\mathbf{X}_{i+1})\leq t\right\}},$$

$$\Psi_{y,t}(\mathbf{X}_{n,i}) = \mathbb{1}_{\left\{\|\mathbf{X}_i\|>1, \frac{\|\mathbf{X}_i\|}{\|\mathbf{X}_{i-1}\|}>y, \varphi(\mathbf{X}_i)\leq t\right\}} \left(\frac{\|\mathbf{X}_{i-1}\|}{\|\mathbf{X}_i\|}\right)^{\alpha}$$

of

$$\mathbf{X}_{n,i} := (\mathbf{X}_{i-1}, \mathbf{X}_i, \mathbf{X}_{i+1})/u_n \cdot \mathbb{1}_{\{\|\mathbf{X}_i\|>u_n\}} \in \mathbb{R}^6.$$

$\Rightarrow$ Use theory of Drees and Rootzén (2010) (cf. also Drees and Rootzén (2016)) to show convergence of empirical processes.

Simulations

- The variance of the limiting expressions are in general quite complicated which makes comparisons between the two estimators difficult.
- In dimension $d = 1$ it possible to show that the backward estimator has lower variance if $y > 1$ (cf. Drees, Segers & Warchoł (2015))

Example: The following simulations where done to estimate

$$P(\|\Theta_1\| > y, \varphi(\Theta_1) \leq 1.5\pi)$$

for a random difference equation of the form

$$\mathbf{X}_t = \mathbf{A}_t \mathbf{X}_{t-1} + \mathbf{B}_t, \quad t \in \mathbb{Z},$$

where the distribution of $\varphi(\Theta_1)$ is uniform on $[0, 2\pi]$, $\|\Theta_1\|$ ist distributed like the absolute value of a standard normal r.v. and both are independent (cf. Buraczewski et al. (2009)). The value of α is equal to 2.

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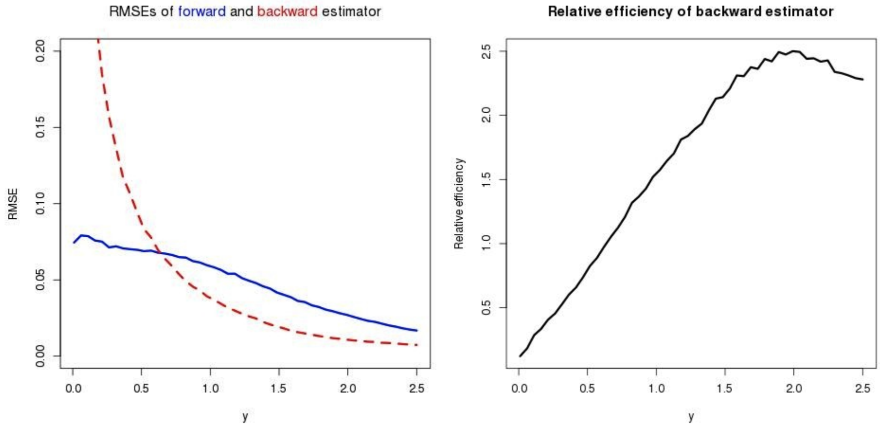

Left: RMSEs for forward (blue,solid) and backward (red, dashed) estimator of $P(\|\mathbf{\Theta}_1\| > y, \varphi(\mathbf{\Theta}_1) \leq 1.5\pi)$ for different values of y. Based on 5.000 simulations of observations of length $n = 1.000$, setting u_n as the 95%-quantile of the observations of $\|\mathbf{X}_t\|$. Right: Ratio of RMSEs of forward and backward estimator.

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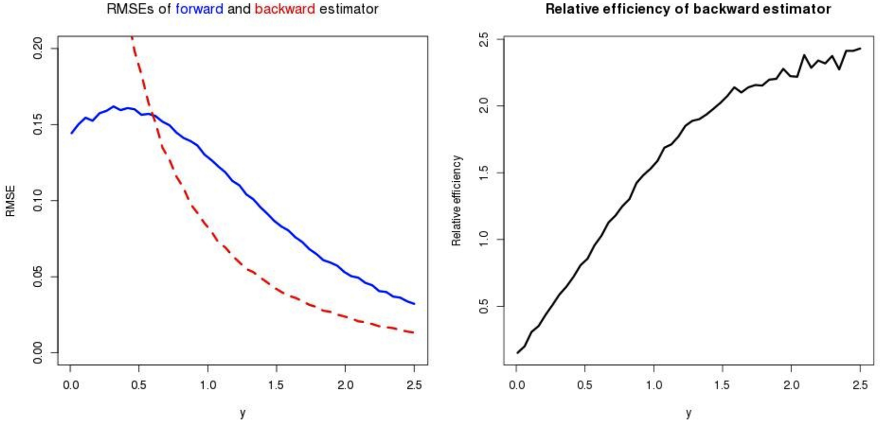

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- That the backward estimator needs knowledge of α is of course a drawback in applications.
- Solution: Plug-in an estimator for α, e.g. Hill-estimator. Under slightly stronger assumptions we can show that asymptotic normality of the estimator still holds.
- The next slide shows the estimators for the same model as before, but α is now estimated by the Hill-estimator based on the exceedances of $\|\mathbf{X}\|$ over u_n.

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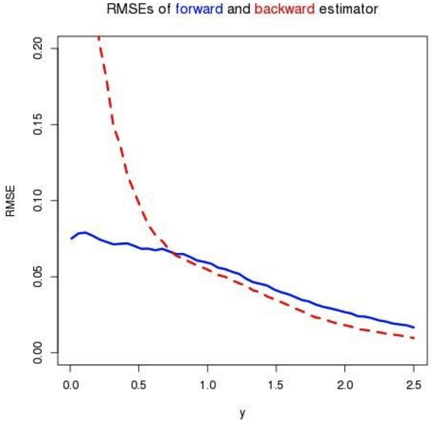
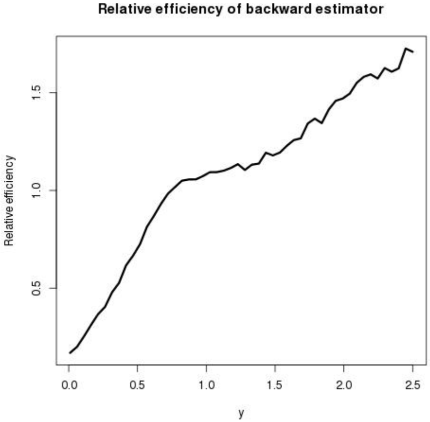
Simulations with estimated α





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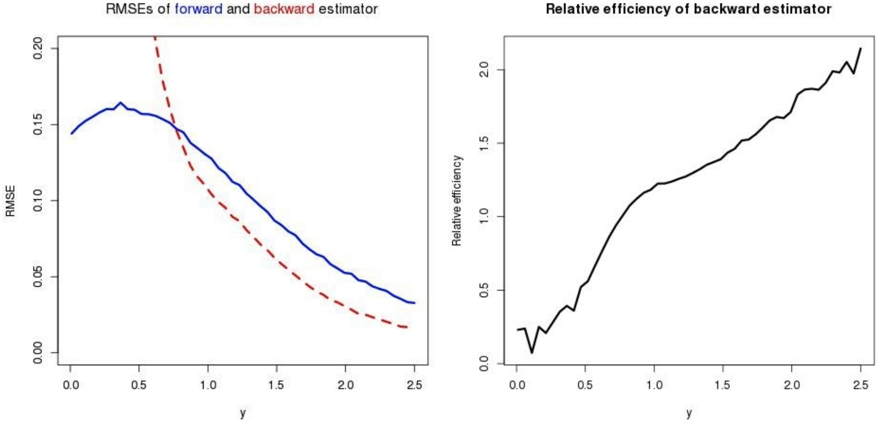

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Summary / Outlook

- The "time change formula" can also in the multivariate setting be useful to improve estimation.
- However, expressions of estimator variances become much more tedious and more specific assumptions are needed to allow for comparisions

$\Rightarrow$ Look at special models like RDEs for concrete statements.

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Thank you for your attention!

Some references

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Conditions to ensure asymptotic normality I

1. $P(\mathbf{\Theta}_1 \in \partial\{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| > y, \varphi(\mathbf{x}) \leq t\}) = 0$, $\forall\, y \geq \min(y_0, \tilde{y}_0), t \in [0, 2\pi]$
2. The exist sequences $l_n \to \infty$, $r_n = o((nv_n)^{1/2})$ such that $l_n = o(r_n), r_n v_n \to 0$ and the mixing coefficients

$$\beta_{n,k} := \sup_{1 \leq l \leq n-k-1} E\left(\sup_{B \in \mathcal{B}^n_{n,l+k+1}} \left| P(B \mid \mathcal{B}^l_{n,1}) - P(B) \right| \right)$$

with

$$\mathcal{B}^j_{n,i} := \sigma((\mathbf{X}_{n,l})_{i \leq l \leq j})$$

satisfy $\beta_{n,l_n} \frac{n}{r_n} \to 0$.

Conditions to ensure asymptotic normality II

3. For all $k \in \{0, \ldots, r_n\}$ there exists

$$s_n(k) \geq P(\|\mathbf{X}_k\| > u_n \mid \|\mathbf{X}_0\| > u_n)$$

such that $\lim_{n\to\infty} s_n(k) = s(k) \in \mathbb{R}$ exists and

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} s_n(k) = \sum_{k=1}^{\infty} s(k) < \infty.$$

4.
$$E\left(\left(\sum_{i=1}^{r_n} \mathbb{1}_{\{\|\mathbf{X}_i\|>u_n\}}\right)^{2+\delta}\right) = O(r_n v_n)$$

for some $\delta > 0$.

back to process convergence