Weak convergence of the empirical copula process with respect to weighted metrics

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Outline

- The empirical copula process and its weak convergence
  Known results and applications
- Convergence with respect to weighted metrics
  Standard empirical processes and empirical copula processes
- Application
  Estimation of Pickands dependence function

The empirical copula process

The empirical copula

- Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ with $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{id})'$ be identically distributed with unknown copula C and unknown continuous marginal cdfs $F_1, \ldots, F_d$. In particular: $C(\boldsymbol{u}) = \mathbb{P}(\boldsymbol{U}_i \leq u)$ where $\boldsymbol{U}_i = (F_1(X_{i1}), \ldots, F_d(X_{id}))'$.

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where $\hat{\boldsymbol{U}}_i = \frac{n}{n+1}(\hat{F}_{n1}(X_{i1}), \ldots, \hat{F}_{nd}(X_{id}))'$ are (observable) pseudo-observations from C.

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- Properties of $\hat{C}_n$: a cdf, not a copula, jumps of size of at most d/n (if there are no ties), ...

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- **(Minimum-distance) estimators of parametric copulas** (Tsukahara, 2005). $\{C_\theta \mid \theta \in \Theta\}$ class of parametric models. Estimator:

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- **Goodness-of fit tests, Asymptotics of estimators for Pickands dependence function, ...**

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- Asymptotics: consider the standardized version

The empirical copula process:

$$\boldsymbol{u} \mapsto \mathbb{C}_n(\boldsymbol{u}) = \sqrt{n}\{\hat{C}_n(\boldsymbol{u}) - C(\boldsymbol{u})\}$$

and investigate its functional weak convergence

A toy example on the usage of the empirical copula process

Suppose we knew that $\mathbb{C}_n \rightsquigarrow \mathbb{C}_C$. Consider **Spearman's rho**:

- population version $\rho = \mathrm{Cor}(F_1(U_1), F_2(U_2)) = 12 \int_{[0,1]^2} C(\boldsymbol{u})\mathrm{d}\boldsymbol{u} - 3$
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By the continuous mapping theorem:

$$\sqrt{n}(\rho_n - \rho) = 12 \int_{[0,1]^2} \mathbb{C}_n(\boldsymbol{u}) \mathrm{d}\boldsymbol{u} + o_P(1) \rightsquigarrow 12 \int_{[0,1]^2} \mathbb{C}_C(\boldsymbol{u}) \mathrm{d}\boldsymbol{u}$$

The general workhorses when working with empirical (copula) processes

Continuous mapping theorem:

- Provided $\Psi : (\mathbb{D}_1, d_1) \to (\mathbb{D}_2, d_2)$ is continuous:

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General observation: The stronger the metric on $\mathbb{D}_1$, the more functions are continuous (and differentiable), the more useful a weak convergence result.

Functional weak convergence of the empirical copula process

Pointwise consideration suggests a Gaussian limit:

$$\mathbb{C}_n(\boldsymbol{u}) \rightsquigarrow \mathbb{C}_C(\boldsymbol{u}) = \mathbb{B}_C(\boldsymbol{u}) - \sum_{j=1}^{d} C^{[j]}(\boldsymbol{u})\mathbb{B}_C(\boldsymbol{u}^{(j)})$$

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- **B., Segers, Volgushev, 2014:** If $C^{[j]}$ exists and is continuous almost everywhere, then

$$\mathbb{C}_n \rightsquigarrow \mathbb{C}_C \quad \text{in} \quad (L_p([0,1]^d, \|\cdot\|_p)$$

(in fact even with respect to d_{hypi})

Implications and possible Limitations (?)

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However, there are **limitations:**

Anderson-Darling-type goodness-of-fit statistic for $H_0 : C = C_0$:

$$T_n = \int_{(0,1)^2} \frac{n\{\hat{C}_n(\boldsymbol{u}) - C_0(\boldsymbol{u})\}^2}{\min(u_1, u_2)} \mathrm{d}\boldsymbol{u}$$

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Question: Is $\Psi : F \mapsto \int_{(0,1)^2} \frac{F(\boldsymbol{u})^2}{\min(u_1,u_2)} \mathrm{d}\boldsymbol{u}$ continuous?

The Anderson-Darling map is not continuous (wrp to $\|\cdot\|_\infty$)

A simplified (univariate) version of the problem:

Considered as map on the set of functions such that $\frac{F(u)}{u}$ is integrable,

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Example: $F_n(u) = \frac{1}{n}\mathbb{1}(u \geq e^{-n})$. Then $F_n \to 0$ uniformly, but

$$\int_0^1 \frac{F_n(u)}{u}\, du = \frac{1}{n}\int_{e^{-n}}^1 \frac{1}{u}\, du = \frac{1}{n}\{-\log(e^{-n})\} = 1 \not\to 0.$$

Way out? Convergence with respect to weighted metrics

Considered as map on the set of functions such that $\frac{F(u)}{u}$ is integrable, equipped with the metric

$$d(F,G) = \sup_{u \in (0,1]} \frac{|F(u) - G(u)|}{u^\omega}$$

with some $\omega > 0$,

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Proof: $$\left| \int_0^1 \frac{F_n(u)}{u}\, du - \int_0^1 \frac{F(u)}{u}\, du \right| \leq \int_0^1 \frac{1}{u^{1-\omega}}\, du \cdot d(F_n, F).$$

Weighted weak convergence

Weighted supremum distances

- Let f_1, f_2 be functions, continuous in u_0 with $f_j(u_0) = 0$. (keep in mind the standard empirical process and the standard Brownian bridge):

$$\mathbb{B}_n(u) = n^{-1/2} \sum_{i=1}^{n} \{\mathbb{1}(U_i \leq u) - u\} \rightsquigarrow \mathbb{B}(u), \qquad \boldsymbol{u} \in [0,1]^d$$

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Weighted supremum distance:

$$d(f_1, f_2) = \left\| \frac{f_1 - f_2}{g} \right\|_\infty = \sup_u \frac{|f_1(u) - f_2(u)|}{g(u)},$$

g some positive weight function that is approaching 0 at u_0.

Weighted convergence of empirical processes

Univariate empirical process:

- Chibisov, 1964, …(i.i.d.); Shao, Yu, 1996 (weakly dependent):

$$g_\omega(u) = \min\{u, 1-u\}^\omega, \qquad \omega \in [0, b)$$

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- Genest, Segers, 2009 (i.i.d.), relying on Van der Vaart and Wellner, 1996:

$$g_\omega(u, v) = \min\{u, v, 1-\min\{u, v\}\}^\omega, \qquad \omega \in [0, 1/2)$$

Recall that $\mathbb{B}_C(u, v) = 0$ iff $\{u = 0\}$ or $\{v = 0\}$ or $\{(u, v) = (1, 1)\}$.

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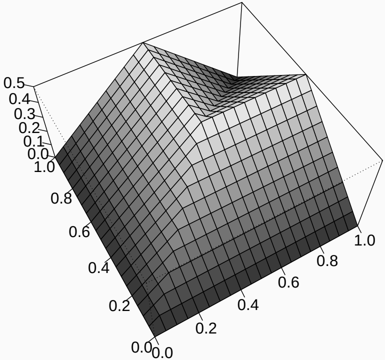
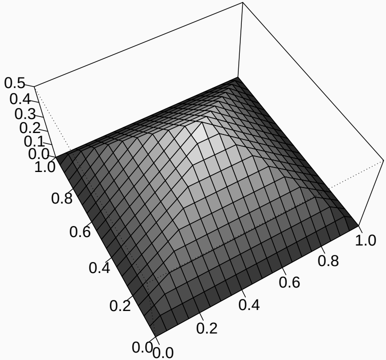
- Suggested weight function:

$$d = 2: \qquad g_\omega(u,v) = \min\{u, v, 1-u, 1-v\}^\omega$$

$$d > 2: \qquad g_\omega(\boldsymbol{u}) = \min\{\wedge_{j=1}^d u_j, \wedge_{j=1}^d (1 - \min_{j' \neq j} u_{j'})\}^\omega$$

Graphs of weight functions





- Left: graph of $\bar{g}_1(u, v) = \min\{u, v, 1 - \min(u, v)\}$
- Right: graph of $g_1(u, v) = \min\{u, v, (1 - u), (1 - v)\}$.

Stronger conditions on the copula

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- Moreover, there exists a constant $K > 0$ such that

$$|C^{[j_1,j_2]}(\boldsymbol{u})| \leq K \min\left\{\frac{1}{u_{j_1}(1-u_{j_1})}, \frac{1}{u_{j_2}(1-u_{j_2})}\right\}, \qquad \forall\, \boldsymbol{u} \in V_{j_1} \cap V_{j_2}.$$

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Can be shown to be satisfied for many common copula families.

Also sufficient for an almost sure error bound on the Stute representation of $\mathbb{C}_n$ (Segers, 2012) and for weak convergence of kernel based estimators (Omelka, Gijbels, Veraverbeke, 2009)

Main result

Theorem (Berghaus, B., Volgushev, 2016+, Bernoulli):

Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be stationary and geometric alpha-mixing. If the marginals of the stationary distribution are continuous and if the copula C satisfies the above Condition, then, for any $c \in (0,1)$ and any $\omega \in (0, 1/2)$,

$$\sup_{\boldsymbol{u} \in [\frac{c}{n}, 1-\frac{c}{n}]^d} \left| \frac{\mathbb{C}_n(\boldsymbol{u})}{g_\omega(\boldsymbol{u})} - \frac{\bar{\mathbb{C}}_n(\boldsymbol{u})}{g_\omega(\boldsymbol{u})} \right| = o_P(1).$$

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Moreover,

$$\bar{\mathbb{C}}_n \rightsquigarrow \mathbb{C}_C \qquad \text{in } (\ell^\infty([0,1]^d), d_{g_\omega}).$$

Applications

Estimation of tail dependence via block maxima

Object of interest: **tail dependence** (finance, actuarial science, hydrology, ...)

Two general methods to assess tail dependence:

- **POT-method:** only consider observations that are larger than some threshold
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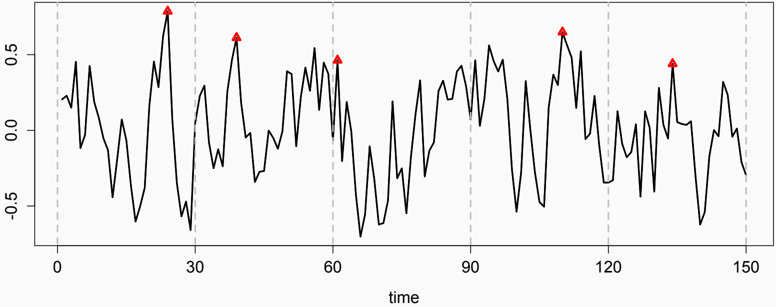
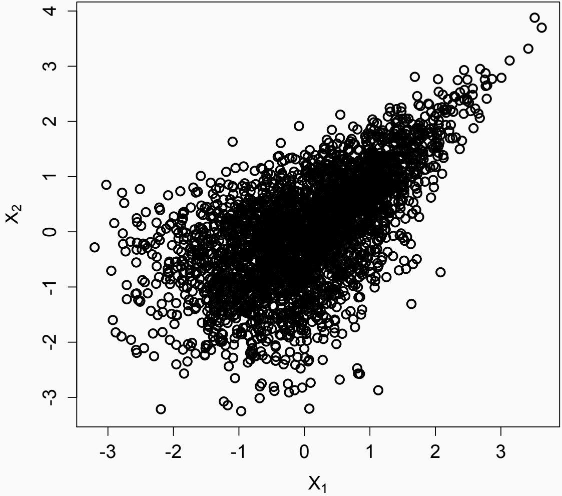
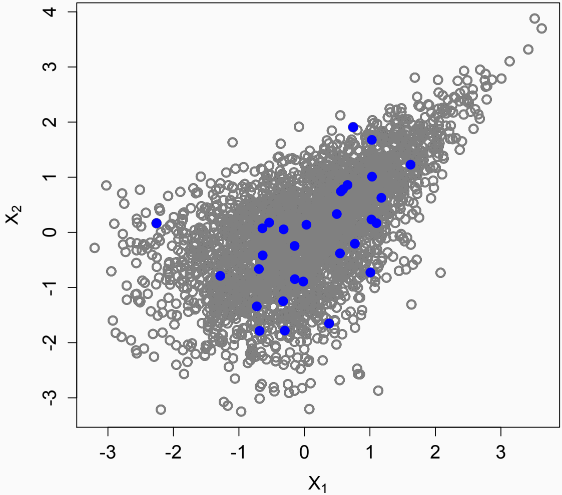


The multivariate case: componentwise block maxima



Original Time Series

The multivariate case: componentwise block maxima



Observations
in the
first block

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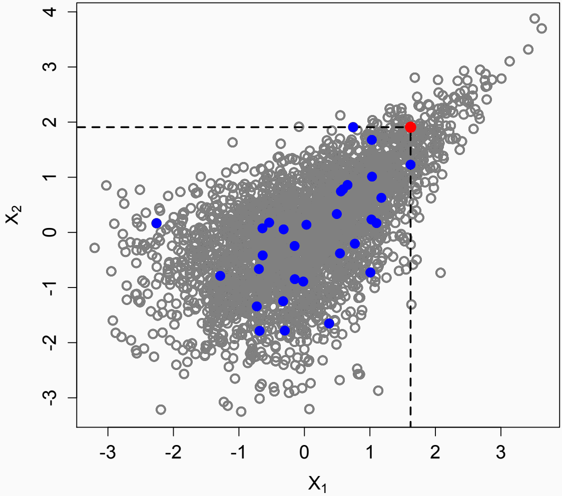

Componentwise
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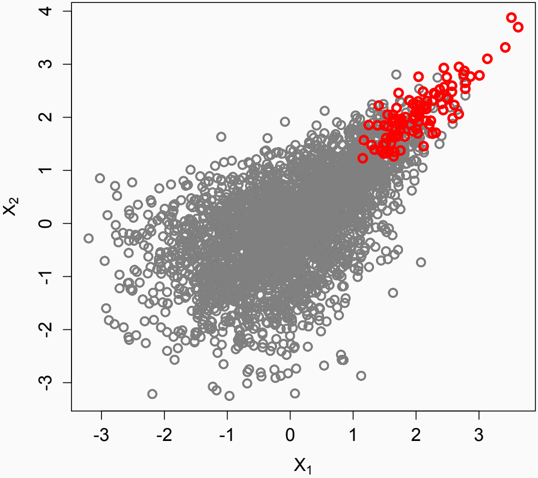
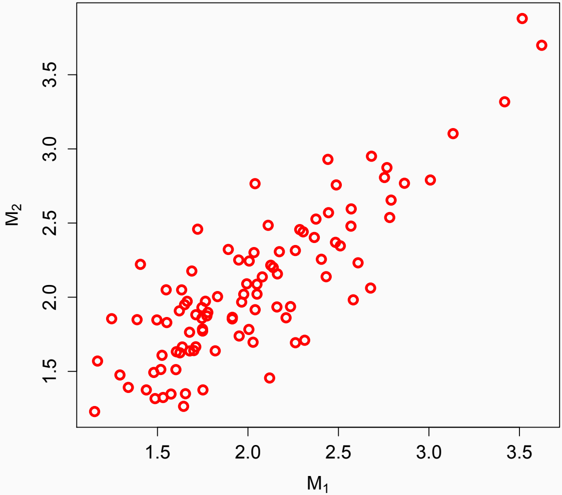


All block maxima

The multivariate case: componentwise block maxima



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Extreme value copulas and Pickands dependence functions

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for a function $A : \Delta_{d-1} \to [1/d, 1]$ (**Pickands dependence function**).

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  for a function $A : \Delta_{d-1} \to [1/d, 1]$ (**Pickands dependence function**).
- If the data generating process is some block maxima scheme, then model by extreme value copulas (annual maximal water levels, ...)

Estimation of Pickands dependence function

(Nonparametric) estimation for i.i.d. observations:

- **Known marginals:** Pickands, 1981, Deheuvels, 1991, Capéràa, Fougères and Genest, 1997, Hall and Tajvidi, 2000, Jiménez, Villa-Diharce and Floers, 2001, Zhang, Wells and Peng, 2008, ...
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- Example: the **Pickands-estimator** (with estimated marginals)

$$\hat{A}_n^P(\boldsymbol{w}) = \left[\frac{1}{n}\sum_{i=1}^{n}\min\left\{\frac{-\log(\hat{U}_{i1})}{w_1},\ldots,\frac{-\log(\hat{U}_{id})}{w_d}\right\}\right]^{-1}.$$

Asymptotics: the empirical copula process comes into play

Genest and Segers, 2009, AoS:

$$\sqrt{n}(\hat{A}_n^P - A) = -A^2 \mathbb{K}_n / (1 + n^{-1/2} A \mathbb{K}_n)$$

where

$$\mathbb{K}_n(\boldsymbol{w}) = \int_0^1 \mathbb{C}_n(u^{w_1}, \dots, u^{w_d}) \, \frac{\mathrm{d}u}{u}.$$

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Asymptotics in Genest and Segers, 2009: use Stute's representation for the empirical copula process based on i.i.d. observations. Weighted convergence of the standard empirical process becomes available. Careful case-by-case study of appearing integrals necessary.

A greatly simplified approach to the asymptotics based on weighted convergence

- For any $\omega > 0$,

$$\Psi : f \mapsto \left\{ \boldsymbol{w} \mapsto \int_0^1 f(u^{w_1}, \ldots, u^{w_d}) \, \frac{\mathrm{d}u}{u} \right\}$$

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- Note: This approach is not restricted to the i.i.d. case.

Summary

Wrapping up

The **empirical copula process**:

- if first order partial derivatives exist almost everywhere, then convergence wrt. $\|\cdot\|_p$ (or d_{hypi})
- if first order partial derivatives exist on the entire interior, then convergence wrt. $\|\cdot\|_\infty$
- if second order partial derivatives exist and do not explode too heavily, then convergence wrt. weighted supremum distances

The stronger the metric, the more applications through the continuous mapping theorem and the functional delta method.

Thank you!

B. Berghaus, A. Bücher, S. Volgushev (2016+): Weak convergence of the empirical copula process with respect to weighted metrics. To appear in *Bernoulli*. Arxiv:1411.5888.

Appendix

A: Some comments on the main result

- The restriction to $[\frac{c}{n}, 1-\frac{c}{n}]^d$ is necessary: Consider $d=2$, then for any (u_1, u_2) such that $u_2 > n/(n+1)$

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- Also holds under more general high level conditions on the serial dependence