Time-frequency analysis of locally stationary Hawkes processes

CIRM, Luminy

Rainer von Sachs

Institut de statistique, biostatistique et sciences actuarielles, UCL

joint work with
F. Roueff *(Télécom ParisTech)* and L. Sansonnet *(Agro ParisTech)*.

February 18, 2016

Contents

Introduction

- Diverse application fields of Hawkes processes (Hawkes, 1971) = self-exciting point processes:
  seismology (Ogata, 1988), genomics (Reynaud-Bouret and Schbath, 2010), neuroscience (Reynaud-Bouret *et al.*, 2013), finance: microstructure dynamics (Bacry *et al.*, 2012), order arrival rate modelling and high-frequency data (Bowsher, 2007), , ...
- Stationary linear Hawkes process N:
  - its conditional intensity:

$$\lambda(t) = \lambda_c + \int_{-\infty}^{t^-} p(t-s)\, N(\mathrm{d}s) = \lambda_c + \sum_{T_i < t} p(t - T_i)$$

  - alternatively: description by clusters of point processes: immigrants (follow a Poisson process and define the centers of clusters) and offsprings.

Aim of this work: Model and capture *time-varying* dynamics, i.e. define a model of point process that can be locally interpreted as a stationary Hawkes process (such as for "classical" time series by R. Dahlhaus).

Introduction - cont'd

Existing approaches of time-varying dynamics for Hawkes processes

Essentially only $\lambda_c = \lambda_c(t)$ (e.g. Chen and Hall, 2013) and no "locally stationary" approach via rescaling in time (taking $u = t/T$)

Some notation for this talk

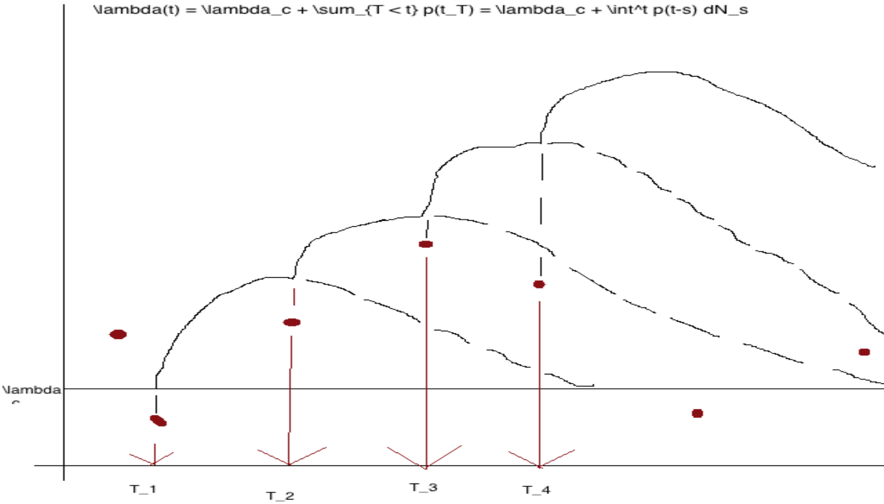
- We look at (time-continuous) processes with values in $\mathbb{R}^\ell$ (but for today mostly $\ell = 1$)
- A point process is identified with a random measure with discrete support: $N = \sum_k \delta_{T_k}$, where δ_t is the Dirac measure at point t and $\{T_k\}$ is the corresponding (countable) random set of points.
- For a test function g, $N(g) = \sum_k g(T_k)$.
- (Some) Functional norms:
  - L^q-norms $|\cdot|_q$ for $q \in [1; \infty]$, and
  - β-norm: $|h|_{(\beta)} := \left|h \times |\cdot|^\beta\right|_1 = \int |h(s)||s|^\beta \, \mathrm{d}s$, for a given $\beta > 0$.

Hawkes processes as cluster processes

- Cluster processes = point processes constructed via conditioning on the realization of a so-called *center process*, usually a PPP in the sequel (Daley and Vere-Jones, 2003).
- Starting point: N_c a PPP with intensity measure μ_c and density λ_c that represents the *immigrants* which appear spontaneously.
- At each *center point* t of N_c, a (independent branching) PPP $N(\cdot|t)$ with finite intensity $\mu(\cdot|t)$, generating the *offsprings* at t.
- Cluster process N = all the immigrants + all the offspring of each immigrant.
- Standard Hawkes processes are made stationary by assuming that N_c is a homogeneous PPP on the whole space $\mathbb{R}^\ell$ and $t \mapsto \mu(\cdot|t)$ is shift invariant: $\mu(\cdot|t) = \mu \circ S^t$, where μ is fixed (i.e. $\mu = \mu(\cdot|0)$).

Dynamics of Hawkes processes

$$\lambda(t) = \lambda_c + \int_{-\infty}^{t^-} p(t-s)\, N(\mathrm{d}s) = \lambda_c + \sum_{T_i<t} p(t-T_i)$$



Our work on non- and locally stationary Hawkes processes

Roadmap of our approach on *approximation* (SPA 2015):

- First, we show existence of *non-stationary* Hawkes processes
- Then, we define *locally stationary* Hawkes processes using a series of regularity conditions on immigrant intensity $\lambda_c(t/T)$ and fertility function $p(\cdot; t/T)$
- The key tool for existence of an approximating stationary process: local approximation of the log Laplace functional
- This allows for approximation of the first and second order moment structure of the non-stationary by the (locally) stationary Hawkes process
- In particular: existence of a local (i.e. time-varying) Bartlett spectrum (time-frequency analysis)

Existence of *non-stationary* Hawkes process

Give up shift invariance of $t \mapsto \mu_c(t)$ and $t \mapsto \mu(\cdot|t)$, i.e. their densities $\lambda_c(t)$ (immigrant intensity) and $p(\cdot; t)$ (fertility function) depend on t.

We can define a *non stationary* Hawkes process via these t-v densities if

$$\zeta_1 := \sup_{t\in\mathbb{R}} \mu\left(\mathbb{R}^\ell \middle| t\right) = \sup_{t\in\mathbb{R}^\ell} \int p(s;t)\,\mathrm{d}s < 1 \quad \text{and} \quad |\lambda_c|_\infty < \infty \,.$$

Theorem 1: The component process $N(\cdot|t)$ has finite moment measure $\mathbb{E}[N(\mathbb{R}^\ell|t)] \leqslant \frac{1}{1-\zeta_1}$ and its density is uniformly bounded by $\frac{|\lambda_c|_\infty}{1-\zeta_1}$.

Proof via the cluster construction: each component $N(\cdot|t)$ can be constructed as the superposition of point processes defined iteratively.

Non-stationary Hawkes processes (under the density assumption) with λ_c and $p(\cdot;\cdot)$ still quite general: can still evolve quite arbitrarily in the space.

Locally stationary Hawkes processes

- Aim: Define a locally stationary approximation - similarly to autoregressive processes. In the following, to simplify notation, consider univariate case only, i.e. $\ell = 1$ w.l.o.g.

Definition of a locally stationary Hawkes process

A locally stationary Hawkes process with *local immigrant intensity* $\lambda_c^{<\text{LS}>}$ and *local fertility function* $p^{<\text{LS}>}(\cdot;\cdot)$ is

- a collection $(N_T)_{T>0}$ of non-stationary Hawkes processes with
- immigrant intensity $\lambda_{cT}(t) = \lambda_c^{<\text{LS}>}(t/T)$ and
- fertility function $p_T(\cdot;t) = p^{<\text{LS}>}(\cdot;t/T)$.

For a given *real location* t, the scaled location t/T is typically called an *absolute location* in $[0;1]$ and denoted by u.

Analogy with locally stationary AR-processes

Note the similarities/differences with an AR(p)-process ($p = 1$ w.l.o.g.):

Dahlhaus (SPA 1996) shows that ($t = 1, \ldots, T$, $T \geqslant 1$)

$$X_{t,T} = a(t/T)\ X_{t-1,T} + \varepsilon_t \ , \quad \varepsilon_t \sim (0,1)\ ,$$

has a representation

$$X_{t,T} = \int_{-\pi}^{\pi} A^o_{t,T}(\omega) \exp(i\omega t)\ \mathrm{d}\xi(\omega)$$

with the existence of an $A(u,\omega) = \left(1 - a(u) \exp(-i\omega)\right)^{-1}$
such that $\sup_{t,\omega} |A^o_{t,T}(\omega) - A(t/T,\omega)| = O(T^{-1})$.

Here, $A^o_{t,T}(\omega) = \sum_{\ell \geqslant 0} (-1)^\ell \prod_{j=0}^{\ell-1} a(\frac{t-j}{T})\ \exp(-i\omega\ell)$.

Unfortunately, this is not possible for the more elaborated recursive dynamics of a Hawkes process.

Under assumption (LS-1):

$$\zeta_1^{<\mathrm{LS}>} := \sup_{u\in[0,1]} \int p^{<\mathrm{LS}>}(t;u)\,\mathrm{d}t < 1 \quad \text{and} \quad \left|\lambda_c^{<\mathrm{LS}>}\right|_\infty < \infty\,,$$

for all $T > 0$, the non-stationary Hawkes process N_T admits a uniformly bounded intensity function.

- Under (LS-1), moreover, for each $u \in \mathbb{R}^\ell$, the function $t \mapsto p^{<\mathrm{LS}>}(t;u)$ satisfies the required condition for the fertility function of a *stationary* Hawkes process.
- Notation: $N(\cdot;u) =$ a "stationary" Hawkes process with immigrant intensity $\lambda_c^{<\mathrm{LS}>}(u)$ and fertility function $t \mapsto p^{<\mathrm{LS}>}(t;u)$.
- More regularity assumptions:
  - (LS-2): smoothness conditions on ${\lambda_c}^{<\mathrm{LS}>}(u)$
  - (LS-3): smoothness conditions on $p^{<\mathrm{LS}>}(t,u)$ w.r.t. its second argument u
  - (LS-4): some uniform decay condition on $p^{<\mathrm{LS}>}(\cdot,\cdot)$ w.r.t. its first argument

Local approximation of the log Laplace functional

The process $N_T \circ S^{-Tu}$, i.e. N_T shifted at the *real location* Tu, follows approximately the distribution of a stationary Hawkes process $N(\cdot; u)$.

Proof: via Convergence of the log-Laplace function of N_T (i.e. log of $\mathcal{L}_T(g) = \mathbb{E}[\exp N_T(g)]$):

Theorem 2

Under Ass. (LS-1:4), for $\beta \in (0; 1]$ and an appropriate test-function g

- the log-Laplace transform $\log \mathcal{L}_T(S^{-Tu}g)$ of $N_T \circ S^{-Tu}$, converges to the log-Laplace $\log \mathcal{L}(g; u)$ of $N(\cdot; u)$,
- with rate $T^{-\beta}$ and with explicit bounds for the constants depending only on the ("functional") $\beta-$ and ℓ_1-norms of the test function g.

Corollary 3

As application, for any $u \in [0, 1]$, as $T \to \infty$, the point process $N_T \circ S^{-Tu}$ converges in distribution to $N(\cdot; u)$.

Local approximation of the cumulants

Apply Theorem 2 to $\mathcal{L}_T(S^{-Tu}g) : z \mapsto \mathbb{E}\left[\exp N_T(g(\cdot - Tu, z))\right]$.

Key result for treatment of "all" moments (used in estimation theory) via

$$\mathrm{Cum}\left(N(g_1), \dots, N(g_m)\right) = \partial^{1_m}|_{z=0_m} \log \mathcal{L}(z_1 g_1 + \cdots + z_m g_m) ,$$

Theorem 4

Let $\beta \in (0,1]$. Assume (LS-1:4).
Let for any $m \geqslant 1$, $g_1, \dots, g_m$ be real valued bounded integrable.

Then for any T and any $u \in [0,1]$, we have

$$\left|\mathrm{Cum}\left(N_T(S^{-Tu}g_1), \dots, N_T(S^{-Tu}g_m)\right) - \mathrm{Cum}\left(N(g_1; u), \dots, N(g_m; u)\right)\right|$$
$$\leqslant \frac{2^{m-1} C_1\, T^{-\beta}}{(-\log \zeta_1^{<\mathbf{LS}>})^{m-1}} \left\{\sum_{j=1,\dots,m} \left(|g_j|_{(\beta)} + C_2 |g_j|_1\right)\right\} \left\{\sum_{j=1,\dots,m} \left(|g_j|_\infty + C_3 |g_j|_1\right)\right\}^{m-1} .$$

First Application: Convergence of local mean density

Theorem 4 with $m = 1$ implies:

- For any T, N_T admits a uniformly bounded density function $m_{1T}(t)$.
- The intensity measure of the (approximating) stationary Hawkes process admits a time-constant mean density:

$$m_1^{<\mathbf{LS}>}(u) = \frac{\lambda_c^{<\mathbf{LS}>}(u)}{1 - \int p^{<\mathbf{LS}>}(\cdot; u)}$$

  $\rightarrow$ *local mean density* at absolute location u.
- Convergence of $m_{1T}(t)$ to $m_1^{<\mathbf{LS}>}(\cdot; u)$ with rate $T^{-\beta}$ in a local neighbourhood of Tu
  (and explicit bounds on the constants depending only on the $\beta-$ and ℓ_1-norms of the test function g).

Second application: Local Bartlett spectrum

- Locally stationary time series provide a nonparametric statistical framework for time-frequency analysis of time series (Dahlhaus, 2009).
- Such ideas can now be applied to locally stationary Hawkes processes.
- Stationary case
  - The Bartlett spectrum Γ of a second order point process N on $\mathbb{R}$ is defined as the (unique) nonnegative measure on $\mathbb{R}$ s.t., for any bounded and compactly supported function f on $\mathbb{R}$,

$$\mathrm{Var}\big(N(f)\big) = \Gamma(|\hat{f}|^2) = \int \left|\hat{f}(\omega)\right|^2 \gamma(\omega) \, \mathrm{d}\omega \, ,$$

    where $\hat{f}(\omega) = \int f(t) \, \mathrm{e}^{-\mathrm{i}t\omega} \, \mathrm{d}t$.
  - For Hawkes processes, the density of the Bartlett spectrum is given by

$$\gamma(\omega) = \frac{\lambda_c}{2\pi(1 - \int p)} \, |1 - \hat{p}(\omega)|^{-2} \, .$$

Under (LS-1), applying this result to the stationary Hawkes process $N(\cdot; u)$, we have, for any bounded and compactly supported function f,

$$\mathrm{Var}\big(N(f;u)\big) = \Gamma^{<\mathrm{LS}>}\left(|\hat{f}|^2; u\right) = \int \left|\hat{f}(\omega)\right|^2 \gamma^{<\mathrm{LS}>}(\omega; u)\,\mathrm{d}\omega\ ,$$

where

- $\gamma^{<\mathrm{LS}>}(\omega; u) \;=\; \dfrac{m_1^{<\mathrm{LS}>}(u)}{2\pi} \left|1 - \hat{p}^{<\mathrm{LS}>}(\omega; u)\right|^{-2}$, with
- $\hat{p}^{<\mathrm{LS}>}(\omega; u) = \displaystyle\int p^{<\mathrm{LS}>}(t; u)\, \mathrm{e}^{-\mathrm{i}t\omega}\,\mathrm{d}t.$
- $\Gamma^{<\mathrm{LS}>}(\cdot; u)$ and $\gamma^{<\mathrm{LS}>}(\omega; u)$ are called the *local Bartlett spectrum* and the *local spectral density*, respectively, at absolute location u.
- Corollary 6: Under (LS-1:4), for $\beta \in (0, 1]$ and bounded functions f supported inside $[-b, b]$, for some $b > 0$:

  Convergence of $\mathrm{Var}\left(N_T(S^{-Tu} f)\right)$ to $\Gamma^{<\mathrm{LS}>}(|\hat{f}|^2; u)$ with rate of order $T^{-\beta}$ (and again, explicit bounds on constants ..)

Kernel estimation of the local Bartlett spectrum

- f a test function and m a moment function (such as $m(x) = x$, $m(x) = x^2, \dots$).
- b_1 a given time bandwidth and u_0 a fixed time in $[0; 1]$ (namely, $u_0 = t_0/T$ with $t_0 \in [0; T]$).
- We build an estimator of $\mathbb{E}[m(N(f; u_0))]$ based on the empirical observations of N_T and defined by

$$\widehat{E}[m \circ N_T(S^{-Tu_0} f); W_{b_1}] := \frac{1}{T} \int m \circ N_T(f(\cdot - t - Tu_0)) W_{b_1}(t/T) \, \mathrm{d}t,$$

  for some fixed kernel function W.
- In practice, f should be compactly supported, so that this integral can be computed from a finite set of observations in $[0, T]$.

- Localisation (smoothing) in frequency by a real valued kernel K, compactly supported, with Fourier transform $\hat{K}$ such that $\int |\hat{K}(\omega)|^2 \, \mathrm{d}\omega = 1$.
- b_2 a given frequency bandwidth and ω_0 a fixed frequency.
- We wish to estimate the quantity

$$\gamma_{b_2}(\omega_0; u_0) := \int \frac{1}{b_2} |\hat{K}((\omega - \omega_0)/b_2)|^2 \, \Gamma^{<\mathsf{LS}>}(\mathrm{d}\omega; u_0),$$

  which in turn is an approximation of the local spectral density $\gamma^{<\mathsf{LS}>}(\omega_0; u_0)$ of $\Gamma^{<\mathsf{LS}>}(\cdot; u_0)$ at ω_0.
- $f = K_{b_2,\omega_0}$ is the kernel having Fourier transform $\omega \mapsto b_2^{-1/2} \hat{K}((\omega - \omega_0)/b_2)$.
- Consequently, we get that $K_{b_2,\omega_0}(t) = b_2^{1/2} \mathrm{e}^{\mathrm{i}\omega_0 t} K(b_2 t)$.
- Finally, take $m(x) = x^2$ and $m(x) = x$ to define: $\widehat{\gamma}_{b_2,b_1}(\omega_0; u_0) =$

$$= \widehat{E}\left(|N_T(S^{-Tu_0} K_{b_2,\omega_0})|^2; W_{b_1}\right) - \left|\widehat{E}\left(N_T(S^{-Tu_0} K_{b_2,\omega_0}); W_{b_1}\right)\right|^2.$$

Numerical experiments

Specific class of Gamma-shaped local fertility functions $p^{<\mathbf{LS}>}(\cdot; u)$ and time-constant immigrant intensity $\lambda_c^{<\mathbf{LS}>}(u) = 1/2$.

- Example 1 [Exponential without delay]:
  $$p^{<\mathbf{LS}>}(s; u) = \zeta(u)\, \frac{\theta(u)\mathrm{e}^{-\theta(u)s}}{\Gamma(1)} \mathbb{1}_{s>0}\ , \quad \zeta(u),\ \theta(u) \text{ of cosine form.}$$
- Example 2 [Gamma with varying delay $\delta(u)$ and constant $\zeta(u)$]:
  $$p^{<\mathbf{LS}>}(s; u) = \frac{1}{2}(s - \delta(u))\frac{\mathrm{e}^{-(s-\delta(u))}}{\Gamma(2)} \mathbb{1}_{s>\delta(u)}$$
  with $\delta(u) = (6 - 10u) \times \mathbb{1}_{[0;1/2]}(u) + (10u - 4) \times \mathbb{1}_{(1/2;1]}(u)$
  inducing a periodic phenomenon in the self-excitation.

Here, $\zeta(u), \theta(u)$ and $\delta(u)$ are Lipschitz $\beta = 1$, and one gets explicit formulas for the local mean density

$$m_1^{<\mathbf{LS}>}(u) = m_1^{<\mathbf{LS}>}(\zeta(u)) = \frac{\lambda_c}{1 - \zeta(u)}.$$

and local Bartlett spectrum $\Gamma^{<\mathbf{LS}>}(\mathrm{d}\omega; u) = \Gamma^{<\mathbf{LS}>}(\mathrm{d}\omega; \delta(u), \zeta(u), \theta(u))$.

Simulation of locally stationary Hawkes processes

Use time-varying conditional intensity

$$\lambda_T(t) := \lambda_c^{<\mathsf{LS}>}(t/T) + \sum_{T_i<t} p^{<\mathsf{LS}>}(t - T_i; t/T) \ ,$$

where $(T_i)_{i\in\mathbb{Z}}$ denote the points of N_T.

Use Ogata's modified thinning algorithm (Ogata, 1981) to simulate the non-stationary Hawkes process N_T on the interval $[0, T]$.

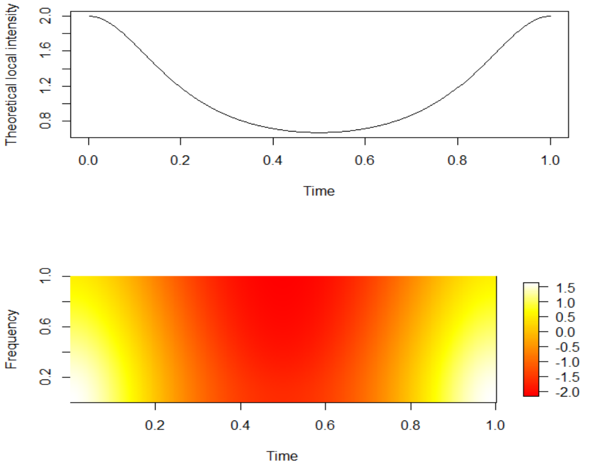


Figure 1 : *Theoretical local mean density (top) and Bartlett spectrum (bottom) for Example 1.*

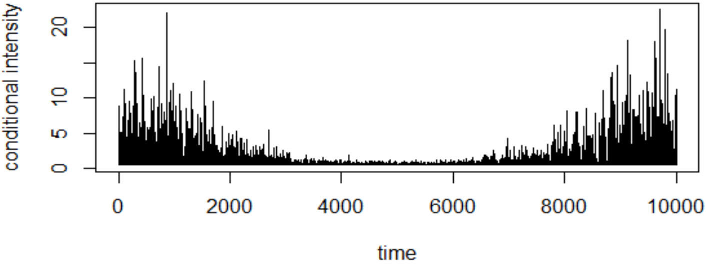


Figure 2 : *Conditional intensity function of a simulated Hawkes process following Example 1, with* $T = 10000$.

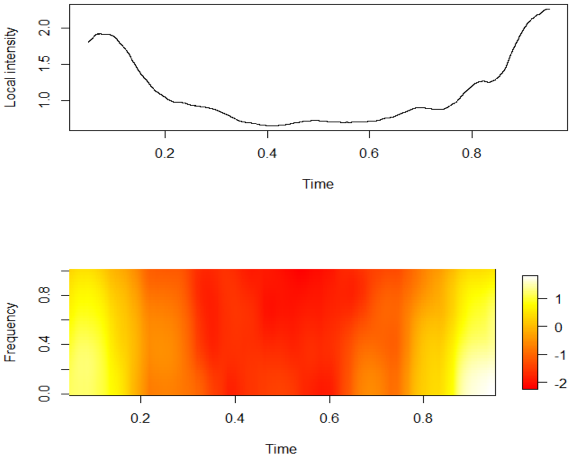


Figure 3 : *Estimation of the local mean density (top) and of the local Bartlett spectrum (bottom) for Example 1.*

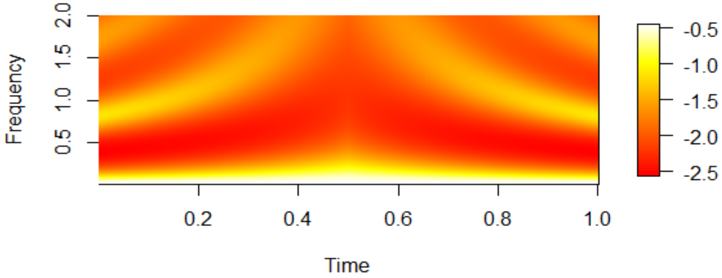


Figure 4 : *Theoretical local Bartlett spectrum for Example 2 (local mean density being constant over time).*

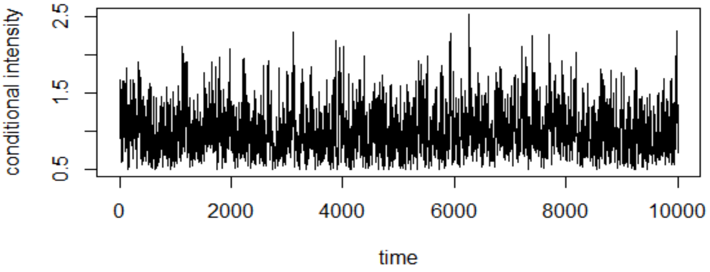


Figure 5 : *Conditional intensity function of a simulated Hawkes process following Example 2, with* $T = 10000$.

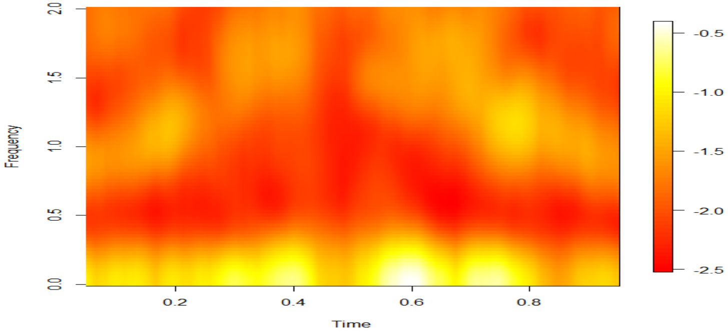


Figure 6 : *Estimation of the local Bartlett spectrum for Example 2.*

Some asymptotic estimation theory (work in progress)

Estimation of the local mean intensity

A straightforward estimator of

$$m_1^{<\mathbf{LS}>}(u_0) = \frac{\lambda_c^{<\mathbf{LS}>}(u_0)}{1 - \int p^{<\mathbf{LS}>}(\cdot; u_0)} .$$

is

$$\widehat{m_1}(u_0) \ := \ N_T(S^{-Tu_0} W_{b_1 T}) \ = \int W_{b_1 T}(s - Tu_0) \ N_T(ds) \ ,$$

with time-kernel $W_{b_1 T}(\cdot) = \frac{1}{b_1 T} W(\frac{\cdot}{b_1 T})$.

Under suitable conditions (of Theorem 4, integrability of W), with $\beta \in (0, 1]$, we have

- Bias: $O\left(b_1^\beta + T^{-\beta}\right)$
- Variance: $O\left((Tb_1)^{-1} + b_1^\beta + T^{-\beta}\right)$

This gives an MSE-rate of convergence of order $T^{-\frac{\beta}{\beta+1}}$.

Principle of proof, using Theorem 4 with $g = W_{b_1 T}$

- Bias (Thm 4 with $m = 1$):

$$\mathbb{E}\widehat{m_1}(u_0) - m_1(u_0) \lesssim T^{-\beta} \left(|W_{\mathbf{b_1} T}|_1 + |W_{\mathbf{b_1} T}|_{(\beta)}\right)$$

$$\lesssim T^{-\beta} \left(1 + (b_1 T)^{\beta}\right) \lesssim b_1^{\beta} + T^{-\beta} \,.$$

- Variance (Thm 4 with $m = 2$ and $g_1 = g_2 = g$):

$$Var(\widehat{m_1}(u_0)) \lesssim Var((N(W_{\mathbf{b_1} T,\mathbf{u_0}})) \; +$$

$$+ \; T^{-\beta}(|W_{\mathbf{b_1} T}|_1 + |W_{\mathbf{b_1} T}|_{(\beta)}) \times (|W_{\mathbf{b_1} T}|_{\infty} + |W_{\mathbf{b_1} T}|_1) \; \lesssim$$

$$\lesssim (Tb_1)^{-1} + T^{-\beta} \left(1 + (b_1 T)^{\beta}\right) \times \left((b_1 T)^{-1} + 1\right) \lesssim \; (Tb_1)^{-1} + b_1^{\beta} + T^{-\beta}.$$

Asymptotic estimation theory for local Bartlett spectrum

Employed technique:

- Control of the expectation and the variance of

$$\widehat{E}[N_T(f); w] := \frac{1}{T} \int N_T(f(\cdot - t)) w(t/T) \, \mathrm{d}t$$

  estimator of first moment of "stationary" Hawkes process $N(f; u_0)$.
- Control of the expectation and the variance of

$$\widehat{E}[N_T^2(f); w] := \frac{1}{T} \int N_T^2(f(\cdot - t)) w(t/T) \, \mathrm{d}t$$

  estimator of the second moment of $N(f; u_0)$.

Here again with the time-kernel $w = W_{b_1, u_0}$ and now with a "test-function" $f = K_{b_2, \omega_0}$, i.e. the kernel in frequency direction.

Again use Theorem 4, similarly to the proof of consistency of the local mean density estimator (but MUCH more involved).

Possibility of a CLT (future work)....

Bias of the estimator $\widehat{\gamma}_{b_2,b_1}(\omega_0; u_0)$

Recall: Estimation of $\gamma_{\mathbf{b_2}}(\omega_0; u_0)$ via

$$\widehat{\gamma}_{\mathbf{b_2},\mathbf{b_1}}(\omega_0; u_0) = \widehat{E}\left(|N_T(S^{-Tu_\mathbf{0}} K_{\mathbf{b_2},\omega_\mathbf{0}})|^2; W_{\mathbf{b_1}}\right) - \left|\widehat{E}\left(N_T(S^{-Tu_\mathbf{0}} K_{\mathbf{b_2},\omega_\mathbf{0}}); W_{\mathbf{b_1}}\right)\right|^2 .$$

Under suitable conditions (of Theorem 4, integrability of K and W), with $\beta \in (0, 1]$, and for a Bartlett spectrum sufficiently regular in frequency, we have for the *Bias* a rate of order

$$\frac{1}{Tb_1 b_2} \; + \; b_2^{-1}\left((Tb_2)^{-\beta} \; + b_1^{\beta}\right) \; + \; b_2^2$$

(The behaviour of the *Variance* has still to be worked out.)

Convergence of the bias under conditions for the two bandwidths:

$$Tb_1 b_2 \to \infty, \; Tb_2^{1+\frac{1}{\beta}} \to \infty \text{ and } \frac{b_1^{\beta}}{b_2} \to 0 \; ,$$

plus a traditional bias-condition $b_2 \to 0$ for the convergence of $\gamma_{\mathbf{b_2}}(\omega_0; u_0)$ to the "point" spectral density $\gamma^{<\mathbf{LS}>}(\cdot; u_0)$ in ω_0 (for a given u_0).

Conclusion

- Self-exciting point processes ("Hawkes" processes) show a far more evolved dynamics over time than, e.g., autoregressive processes. Consequently, locally stationary approximations are more difficult to derive.
- Unfortunately, unlike "classical" time series (under usual conditions) no spectral representation exists that would render the approach more "direct".
- The key to success: approximation theory for local Laplace transforms and its derivatives (control of cumulants):
- Existence of (local) mean density (no "explosive" behaviour) and local Bartlett spectrum via
  control of convergence of first and second moment structure
- Work in progress: Asymptotic estimation theory (control of convergence of first and second empirical moments);
  real data analysis