Limit theorems for random fields

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Let $\Omega, \mathcal{A}, \mu)$ be a probability space, $T_1, \dots, T_d$ commuting measure preserving transformations.
For $\underline{i} = (i_1 \dots, i_d)$ we denote $T_{\underline{i}} = T_1^{i_1} \circ \cdots \circ T_d^{i_d}$.
$(T_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$ is thus a $\mathbb{Z}^d$ action on $\Omega, \mathcal{A}, \mu)$.
In the spaces L^p we denote

$$U_{\underline{i}} f = f \circ T_{\underline{i}}.$$

For f measurable,

$$f \circ T_{\underline{i}},\ \underline{i} \in \mathbb{Z}^d$$

is thus a (strictly) stationary random field and any (strictly) stationary random field can be represented in this way.

Suppose that there is a measurable function e on Ω such that $e \circ T_{\underline{i}}$ are independent (hence iid) and generate $\mathcal{A}$. Then we say that the action is Bernoulli and $f \circ T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, is a Bernoulli random field.
Notice that for $d = 1$ we get $f = g(\dots, e_{-1}, e, e_1, \dots)$ where e_i are iid and $g : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is measurable. M.Rosenblatt posed a question whether each strictly stationary process can be represented in this way; the answer is no.

Let $\mathcal{F}_{\underline{i}} \subset \mathcal{A}$, $\underline{i} \in \mathbb{Z}^d$, be σ-algebras such that

- $\mathcal{F}_{\underline{i}} \subset \mathcal{F}_{\underline{j}}$ if for all $1 \leq q \leq d$, $i_q \leq j_q$,
- $\mathcal{F}_{\underline{i}} \cap \mathcal{F}_{\underline{j}} = \mathcal{F}_{\underline{i} \wedge \underline{j}}$,
- $E\Big(E(f \mid \mathcal{F}_{\underline{i}}) \mid \mathcal{F}_{\underline{j}}\Big) = E(f \mid \mathcal{F}_{\underline{i} \wedge \underline{j}})$ for any integrable f.

Then we say that the filtration $\mathcal{F}_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, is completely commuting.

Observation.
If the random field is Bernoulli
and for every $\underline{j}$, $\mathcal{F}_{\underline{j}}$ is generated by $e \circ T_{\underline{i}}$ with $i_q \leq j_q$, $1 \leq q \leq d$,
then $\mathcal{F}_{\underline{j}}$, $\underline{j} \in \mathbb{Z}^d$, is a completely commuting filtration.

For random fields, unfortunately, martingale differences can be defined in different ways (and we get different notions).

For a completely commuting filtration $(\mathcal{F}_{\underline{j}})$, a natural way is the following.

By $e_i \in \mathbb{Z}^d$ we denote the vector $(0, \ldots, 1, \ldots, 0)$ having 1 on the i-th place and 0 at all other places, $1 \le i \le d$.

Then we say that $f \circ T_{\underline{i}}$ are martingale differences if f is integrable and $\mathcal{F}_{\underline{0}}$-measurable, $E(f \mid \mathcal{F}_{-\underline{e_i}}) = 0$ for all $1 \le i \le d$.

A substantial problem is that (unlike in the one dimensional case) for a field of L^2 martingale differences the CLT might not take place.

As noticed in a paper by Wang and Woodroofe, for a 2-dimensional random field

$Z_{i,j} = X_i Y_j$ where X_i and Y_j, $i, j \in \mathbb{Z}$, are mutually independent $\mathcal{N}(0,1)$ random variables,

we get a convergence towards a non normal law.

The random field of $Z_{i,j}$ can be represented by an ergodic action of $\mathbb{Z}^2$.

In the case of a Bernoulli random field the CLT is true, however.

An idea of a proof:

Suppose that $f \circ T_{\underline{i}}$ are martingale differences.

For $m \geq 1$, define $\mathcal{G}_m = \sigma\{e \circ T_{\underline{i}} : -m \leq i_q \leq 0,\ q = 1, \ldots, d\}$ and $f_m = E(f \,|\, \mathcal{G}_m)$.

Then $\|f - f_m\|_2 \to 0$ and $f_m \circ T_{\underline{i}}$ are martingale differences.

Then f_m are m-dependent and their partial sums are close to the partial sums of f.

We can show a stronger result.

Theorem

Let $f \in L^2$*, be such that* $(f \circ T_{\underline{i}})_{\underline{i}}$ *is a field of martingale differences for a completely commuting filtration* $\mathcal{F}_{\underline{i}}$*. If at least one of the transformations* T_{e_i}*,* $1 \leq i \leq d$*, is ergodic then the central limit theorem holds, i.e. for* $n_1, \ldots, n_d \to \infty$ *the distributions of*

$$\frac{1}{\sqrt{n_1 \ldots n_d}} \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} f \circ T_{(i_1,\ldots,i_d)}$$

weakly converge to $\mathcal{N}(0, \sigma^2)$ *where* $\sigma^2 = \|f\|_2^2$*.*

This result allows us to think of martingale approximations.

An idea of a proof:

For simplicity, suppose that $d = 2$.

Denote

$$F_{u,v} = \frac{1}{\sqrt{v}} \sum_{j=1}^{v} U_{u,j} f, \quad X_{u,n,v} = \frac{1}{\sqrt{n}} F_{u,v}.$$

Let us fix a positive integer m and for constants $a_1, \dots, a_m$ consider the sums

$$\sum_{i=1}^{m} a_i \sum_{j=1}^{v} U_{i,j} f, \quad v \to \infty.$$

Then $(\sum_{i=1}^{m} a_i U_{i,j} f)_j$, $j = 1, 2, \dots,$ are martingale differences for the filtration $(\mathcal{F}_j^{(2)})$, $\mathcal{F}_j^{(2)} = \sigma(\cup\{\mathcal{F}_{i_1,i_2},\, :\, i_2 \leq j\})$, and by the central limit theorem of Billingsley and Ibragimov (we can also use the McLeish's theorem)

$$\frac{1}{\sqrt{v}} \sum_{j=1}^{v} \big(\sum_{i=1}^{m} a_i U_{i,j} f \big)$$

converge in law to $\mathcal{N}(0, \sum_{i=1}^{m} a_i^2)$.

Notice that here we use the assumption of ergodicity of $T_{0,1}$ and that with respect to the σ-algebra $(\mathcal{F}_j^{(2)})_j$ the convergence is quenched.
From this it follows that the random vectors $(F_{1,v}, \ldots, F_{m,v})$ converge in law to a vector $(W_1, \ldots, W_m)$ of m mutually independent and $\mathcal{N}(0,1)$ distributed random variables.

For a given $\epsilon > 0$, if $m = m(\epsilon)$ is sufficiently big then we have

$$\left\|1 - (1/m) \sum_{u=1}^{m} W_u^2\right\|_1 < \epsilon/2.$$

Using a truncation argument we can from the convergence in law of $(F_{u,v}, \ldots, F_{m,v})$ towards $(W_1, \ldots, W_m)$ deduce that for $m = m(\epsilon)$ sufficiently big and v bigger than some $v(m, \epsilon)$,

$$\left\|1 - \frac{1}{m} \sum_{u=1}^{m} F_{u,v}^2\right\|_1 < \epsilon,$$

$F_{u,v} = \frac{1}{\sqrt{v}} \sum_{j=1}^{v} U_{u,j} f.$

From this we deduce that for v and N sufficiently large

$$\Big\|1 - \frac{1}{N}\sum_{i=1}^{N} F_{i,v}^2\Big\|_1 = \Big\|1 - \frac{1}{Nv}\sum_{i=1}^{N}\big(\sum_{j=1}^{v} U_{i,j}f\big)^2\Big\|_1 < 2\epsilon.$$

We use McLeish's theorem again for

$$\big((1/\sqrt{v})\textstyle\sum_{j=1}^{v} U_{i,j}f\big)_i$$

and get the CLT.

The CLT is quenched with respect to $\mathcal{F}_{0,0}$ hence we get a convergence to a Brownian in finitely dimensional distributions.
Another proof of the finitely dimensional convergence has been recently published by Cuny, Dedecker, and V. Tightness was proved by V. and Wang.

To get the conditions for martingale approximation let us define orthonormal projections:
Denote by $\mathcal{F}_l^{(q)}$ the σ-algebra generated by all $\mathcal{F}_{i_1,i_2,\dots,i_d}$ with $i_q \leq l$ ($i_j \in \mathbb{Z}$ for $1 \leq j \leq d$, $j \neq q$), $1 \leq q \leq d$.

Similarly as in the one dimensional case we can define orthogonal projection operators

$$P_{j_1,j_2,\dots,j_d} \text{ onto } \bigcap_{1\leq q\leq d} L^2(\mathcal{F}_{j_1}^{(q)}) \ominus L^2(\mathcal{F}_{j_1-1}^{(q)}).$$

First efforts to get an approximation are due to Gordin in 2009 who, however, did not have a CLT for martingale differences.

In dimension $d = 1$, useful approximation is given by the martingale-coboundary decomposition

$$f = m + g - g \circ T$$

where $m, g \in L^2$ and $(m \circ T^i)_i$ is a martingale difference sequence. This was first used by Gordin in 1969.
For simplicity we'll deal with the causal case only. Then a NSC for the martingale-coboundary decomposition is the convergence of

$$\sum_{j=0}^{\infty} E(U^j f | \mathcal{F}_0)$$

(in L^p; both martingale differences and the cobounding function are then in L^p.)

Gordin used a sufficient condition

$$\sum_{j=0}^{\infty} \|E(U^j f | \mathcal{F}_0)\|_2 < \infty.$$

This condition was generalised to random fileds with completely commuting filtration by El Machkouri and Giraudo.

For $d \geq 2$ the decomposition has the following form

$$f = \sum_{S \subset \{1,\ldots,d\}} \prod_{q \in S^c} (I - U_{\epsilon_q}) g_S$$

where $g_S \in \in_q S \to \bigcap L^2(\mathcal{F}_0^{(q)}) \ominus L^2(\mathcal{F}_{-1}^{(q)})$,
$S \subset \{1, \ldots, d\}$,
$\prod_{q \in \emptyset} (I - U_{\epsilon_q})$ is defined as I.

For $d = 2$ we get

$$f = m + [g_1 - U_{1,0} g_1] + [g_2 - U_{0,1} g_2] + [g - U_{1,0} g - U_{0,1}(g - U_{1,0} g)]$$

where $m, g \in L^2$,
$P_{0,0} m = m$,
$g_1 \in L^2(\mathcal{F}_0^{(2)}) \ominus L^2(\mathcal{F}_{-1}^{(2)})$,
$g_2 \in L^2(\mathcal{F}_0^{(1)}) \ominus L^2(\mathcal{F}_{-1}^{(1)})$.

Theorem

The martingale-coboundary decomposition takes place if and only if

$$\sum_{j_1=0}^{\infty}\cdots\sum_{j_d=0}^{\infty}\|\sum_{i_1=j_1}^{\infty}\cdots\sum_{i_d=j_d}^{\infty}P_{0,\dots,0}U_{i_1,\dots,i_d}f\|_2^2<\infty.$$

For $d=1$ the condition was found (as sufficient) by C.C.Heyde.

The decomposition can take place in L^p spaces, $p\geq 2$. There we have

Theorem

Let $p\geq 2$. If

$$\sum_{i_1}\cdots\sum_{i_d}i_1^2 i_2^2\dots i_d^2\|P_{i_1,i_2,\dots,i_d}f\|_p^2<\infty$$

then the martingale-coboundary decomposition in L^p takes place.

As shown by Lesigne and Volný, for martingale differences one can prove estimates of large deviations $\mu(S_n > n)$ of ordre $n^{-p/2}$, if S_n is a partial sum of uniformy L^p bounded martingale difference (no stationarity is needed).

Remark that the martingale-coboundary decomposition can be proved for non stationary processes, too:

Let (X_i), $i \in \mathbb{Z}$, be a sequence of random variables, $X_i \in L^p$, and $(\mathcal{F}_i)$ a filtration. Suppose that all X_i are $\mathcal{F}_\infty$-measurable and $E(X_i|\mathcal{F}_{-\infty}) = 0$.

Then it is equivalent:

(i) There exist a sequence of martingale differences $Y_i \in L^p$ with the filtration $(\mathcal{F}_i)$ and random variables $U_i \in L^p$ such that for all $i \in \mathbb{Z}$

$$(1) \quad X_i = Y_i + U_i - U_{i+1}$$

and for all $k \in \mathbb{Z}$, $i \to \infty$,

$$(2) \quad E(U_{k+i}|\mathcal{F}_k) \to 0, \quad U_{k-i} - E(U_{k-i}|\mathcal{F}_k) \to 0$$

in L^1.

(ii) For every $k \in \mathbb{Z}$ the sums

$$(3) \quad V_k = \sum_{i=0}^{\infty} E(X_{k+i}|\mathcal{F}_{k-1}), \quad W_k = \sum_{i=1}^{\infty} [X_{k-i} - E(X_{k-i}|\mathcal{F}_{k-1})]$$

converge in L^1 and the limits belong to L^p.

Moreover, if (1) and (3) take place then

$$U_k = V_k - W_k,$$

$$Y_k = X_k - (V_k - W_k) + (V_{k+1} - W_{k+1}) = \sum_{i \in \mathbb{Z}} P_k X_{k+i}$$

where for an integrable function X,

$$P_i X = E(X|\mathcal{F}_i) - E(X|\mathcal{F}_{i-1}), \quad i \in \mathbb{Z}.$$

A classical condition implying CLT and weak invariance principle is the Hannan's condition. In dimension 1 it reads as

$$\sum_i \|P_i f\|_2 < \infty.$$

If the process is regular, it implies both CLT and WIP.

For random fields with completely commuting filtration we get

Theorem

Let $(f \circ T_{\underline{i}})$ be a random field with completely commuting filtration, $f \in L^2$ and $E(f \mid \mathcal{F}^{(q)}_{-\infty}) = 0$, $q = 1, \dots, d$. If

$$\sum_{\underline{i} \in \mathbb{Z}^d} \|P_{\underline{i}} f\|_2 < \infty$$

then the WIP takes place.

To prove the theorem, we show that f can be approximated by

$$f_m = \sum_{\underline{i} \in \{-m,\dots,m\}^d} P_{\underline{i}} f$$

and to f_m the martingale-coboundary decomposition can be applied.

The limit theorems can be proved for a more general summation. Nevertheless, some assumptions on the $\mathbb{Z}^d$ action are needed (everything goes well if it is Bernoulli).

For the CLT, we get e.g. the convergence for $S_{\Gamma_n}/|\Gamma_n|$ (Klicnarová, Wang, V.).

The invariance principle can be defined for a summation over more general sets as well.
If $\mathcal{A}$ is a collection of Borel subsets of $[0,1]^d$, define the smoothed partial sum process $\{S_n(A)\,;\,A \in \mathcal{A}\}$ by

$$S_n(A) = \sum_{\underline{i} \in \{1,\ldots,n\}^d} \lambda(nA \cap R_i) X_i$$

where $R_{\underline{i}} =]i_1 - 1, i_1] \times \ldots \times]i_d - 1, i_d]$ is the unit cube with upper corner at i, λ is the Lebesgue measure on $\mathbb{R}^d$.
We equip the collection $\mathcal{A}$ with the pseudo-metric ρ defined for any A, B in $\mathcal{A}$ by

$$\rho(A,B) = \sqrt{\lambda(A \Delta B)}.$$

To measure the size of $\mathcal{A}$ one considers the metric entropy: denote by $H(\mathcal{A},\rho,\varepsilon)$ the logarithm of the smallest number $N(\mathcal{A},\rho,\varepsilon)$ of open balls of radius ε with respect to ρ which form a covering of $\mathcal{A}$. The function $H(\mathcal{A},\rho,.)$ is the entropy of the class $\mathcal{A}$.

Let $C(\mathcal{A})$ be the space of continuous real functions on $\mathcal{A}$, equipped with the norm $\|.\|_{\mathcal{A}}$ defined by

$$\|f\|_{\mathcal{A}} = \sup_{A \in \mathcal{A}} |f(A)|.$$

A standard Brownian motion indexed by $\mathcal{A}$ is a mean zero Gaussian process W with sample paths in

$$C(\mathcal{A}) \text{ and } \mathrm{Cov}(W(A), W(B)) = \lambda(A \cap B).$$

Such a process exists if

$$\int_0^1 \sqrt{H(\mathcal{A}, \rho, \varepsilon)}\, d\varepsilon < +\infty.$$

We say that the invariance principle or functional central limit theorem (FCLT) holds if the sequence

$$\{n^{-d/2} S_n(A)\,;\, A \in \mathcal{A}\}$$

converges in distribution to an $\mathcal{A}$-indexed Brownian motion in the space $C(\mathcal{A})$.

A key took in proving the theorems is the following inequality:

Denote $\Delta_p = \sum_{\underline{i}\in\mathbb{Z}^d} \|P_{\underline{i}} f\|_p$, $p \geq 2$. Then for any set of $a_{\underline{i}} \in \mathbb{R}$

$$\Big\| \sum_{\underline{i}\in\mathbb{Z}^d} a_{\underline{i}} U_{\underline{i}} f \Big\|_p \leq (p-1)^{d/2} \Big(\sum_{\underline{i}\in\mathbb{Z}^d} a_{\underline{i}}^2 \Big)^{1/2} \Delta_p.$$

Another approach was used by El Machkouri, V. and Wu.
In 2005 Wei Biao Wu introduced "physical dependence measure" for one dimensional stationary processes in Bernoulli dynamical systems.
This means that he considered processes of

$$X_i = g(\dots, e_{i-1}, e_i, e_{i+1}, \dots)$$

where e_i are iid and defined

$$X_i^* = g(\dots, e_{i-1}, e_i^*, e_{i+1}, \dots)$$

where e_i^* is an independent copy of e_i.
We define

$$\delta_{i,p} = \|X_i - X_i^*\|_p \text{ and } \Delta_p = \sum_i \delta_{i,p}.$$

The notion was generalized to $\mathbb{Z}^d$ Bernoulli random fields.
Then $\Delta_2 < \infty$ implies WIP.

Theorem

Let $(X_i)_{i\in\mathbb{Z}^d}$ be the stationary centered random field and let $\mathcal{A}$ be a collection of regular Borel subsets of $[0,1]^d$. Assume that one of the following condition holds:

(*i*) *The collection $\mathcal{A}$ is a Vapnik-Chervonenkis class with index V and there exists $p > 2(V-1)$ such that X_0 belongs to $\mathbb{L}^p$ and $\Delta_p := \sum_{i\in\mathbb{Z}^d} \delta_{i,p} < \infty$.*

(*ii*) *There exists $\theta > 0$ and $0 < q < 2$ such that $E[\exp(\theta|X_0|^{\beta(q)})] < \infty$ where $\beta(q) = 2q/(2-q)$ and $\Delta_{\psi_{\beta(q)}} := \sum_{i\in\mathbb{Z}^d} \delta_{i,\psi_{\beta(q)}} < \infty$ and such that the class $\mathcal{A}$ satisfies the condition*

$$\int_0^1 (H(\mathcal{A},\rho,\varepsilon))^{1/q}\, d\varepsilon < +\infty.$$

Then the sequence of processes $\{n^{-d/2}S_n(A)\,;\, A\in\mathcal{A}\}$ converges in distribution in $\mathcal{C}(\mathcal{A})$ to σW where W is a standard Brownian motion indexed by $\mathcal{A}$ and $\sigma^2 = \sum_{k\in\mathbb{Z}^d} E(X_0X_k)$.

$\Delta_p < \infty$ implies Hannan's condition hence the V.-Wang's result is stronger in L^p spaces.
On the other hand, for variables with exponential moments and more general summation the E-V-W method gives more.

For proving the invariance principles, a key inequality is

$$\Big\| \sum_{\underline{i} \in \Gamma} U_{\underline{i}} f \Big\|_p \leq \Big(2p \sum_{\underline{i} \in \Gamma} a_{\underline{i}}^2 \Big)^{1/2} \Delta_p$$

where $a_{\underline{i}} \in \mathbb{R}$ and $\Gamma \subset \mathbb{Z}^d$ finite.

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