Detecting a changed segment in a sample

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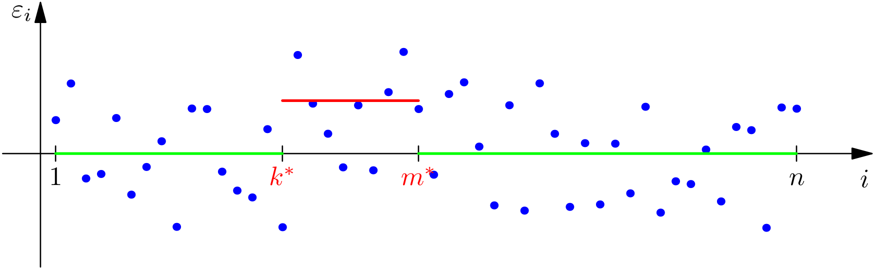
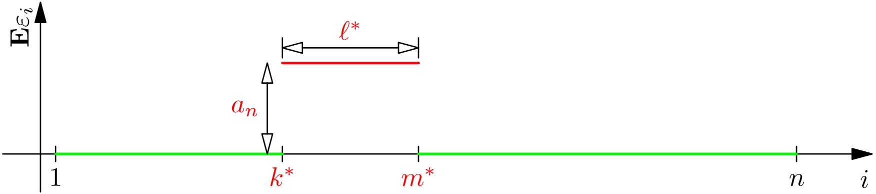
Content

Epidemic change in the mean of $\varepsilon_1, \dots, \varepsilon_n$



$\mathbf{E}\,\varepsilon_i = a_n \neq 0$ for $k^* < i \leq m^*$, $\mathbf{E}\,\varepsilon_i = 0$ outside $(k^*, m^*]$.



k^*, m^*, $\ell^* \to \infty$ as $n \to \infty$. All these parameters are unknown.

Suppose we have observations (ε_k) which are i.i.d. with zero mean, except, maybe, for a short interval, where $\mathbf{E}\,\varepsilon_k$ is a nonzero constant. Such model can be interpreted as an epidemic one. The length of the interval describes the duration of the epidemic.

Problem

How to decide whether such an interval is present ?

Pioneering works:

- LEVIN, B., KLINE, J. (1985). The CUSUM test of homogeneity with an application in spontaneous abortion epidemiology. *Statistics in Medicine*, 4, 469–488.
- COMMENGES, D., SEAL, J., PINATEL, F. (1986). Inference about a change point in experimental neurophysiology. *Math. Biosc.*, 80, 81–108.

Put $\mathbb{I}_n := (k^*, m^*] \cap \mathbb{N}$ and $\mathbb{I}_n^c := \{1, \dots, n\} \setminus \mathbb{I}_n$. Define

$$S(0) := 0,\ S(J) := \sum_{i \in J} \varepsilon_i,\ S(t) := S((0, t]),\ S_j = S(j).$$

Assume for a moment that k^* and m^* are known, that ε_i's $(1 \le i \le n)$ are independent with the same variance and test H_0 against H_1 where

$H_0)$ ε_i, $1 \le i \le n$, have the same distribution with $\mathbf{E}\,\varepsilon_1 = 0$,

$H_1)$ ε_i's are distributed as ε_1 for $i \in \mathbb{I}_n^c$ and as ε_{k^*+1} for $i \in \mathbb{I}_n$, with $\mathbf{E}\,\varepsilon_{k^*+1} = a_n$.

Then we have a two-samples problem for which it is classical to use the statistics $|Q|$ where

$$Q = \sqrt{\ell^*}\Big(\frac{S(\mathbb{I}_n)}{\ell^*} - \frac{S_n}{n}\Big) - \sqrt{n - \ell^*}\Big(\frac{S(\mathbb{I}_n^c)}{n - \ell^*} - \frac{S_n}{n}\Big).$$

which can be recast with $t_k = k/n$ and elementary computations as

$$Q = n^{-1/2} \frac{S(\mathbb{I}_n) - t_{\ell^*} S(n)}{\big(t_{\ell^*}(1 - t_{\ell^*})\big)^{1/2}} \underbrace{\Big(\sqrt{1 - t_{\ell^*}} + \sqrt{t_{\ell^*}}\Big)}_{1 \le \quad \le 2}.$$

After dropping the blue bounded factor and accounting that k^*, m^* are unknown, we replace $|Q|$ by $\mathrm{UI}(n, 1/2)$ where

$$\mathrm{UI}(n, \alpha) := \max_{1 \le i \le j \le n} \frac{|S_j - S_i - S_n(t_j - t_i)|}{\big|(t_j - t_i)(1 - (t_j - t_i))\big|^{\alpha}}, \quad 0 \le \alpha \le \frac{1}{2}.$$

Unfortunately, nothing is known about the limiting distribution of $\mathrm{UI}(n, 1/2)$ in general. So we propose to use instead $\mathrm{UI}(n, \alpha)$ with $0 < \alpha < 1/2$ and more generally, with the weight functions on $[0, 1]$,

$$\rho(h) := h^{\alpha} L(1/h), \quad \tilde{\rho}(h) = \rho(h(1-h)), \quad L \text{ slowly varying at } \infty,$$

the test statistics:

$$\mathrm{UI}(n, \rho) := \max_{1 \le i \le j \le n} \frac{|S_j - S_i - S_n(t_j - t_i)|}{\tilde{\rho}(t_j - t_i)}$$

$$\mathrm{DI}(n, \rho) := \max_{1 \le j \le \log n} \frac{1}{\rho(2^{-j})} \max_{r \in \mathbb{D}_j} \left| S(nr) - \frac{S(nr^-) + S(nr^+)}{2} \right|,$$

where $\mathbb{D}_j := \{(2k-1)2^{-j};\ 1 \le k \le 2^{j-1}\}$ and $r^{\pm} = r \pm 2^{-j}$.

A first look at consistency

From now on, we replace H_1 by the more general alternative:

H_A) $(\varepsilon_i)_{i\geq 1}$, are independent with $\sup_{i\geq 1} \operatorname{Var} \varepsilon_i < \infty$ and $\mathbf{E}\, \varepsilon_i = a_n \mathbf{1}_{\mathbb{I}_n}(i)$.

Theorem (Račkauskas and S., 2004)

Under H_A, if

$$\lim_{n\to\infty} n^{1/2} \frac{a_n h_n}{\rho(h_n)} = \infty, \quad \textit{where } h_n := \frac{\ell^*}{n}\Big(1 - \frac{\ell^*}{n}\Big)$$

then

$$n^{-1/2}\mathrm{UI}(n,\rho) \xrightarrow[n\to\infty]{\mathrm{Pr}} \infty, \quad n^{-1/2}\mathrm{DI}(n,\rho) \xrightarrow[n\to\infty]{\mathrm{Pr}} \infty.$$

If a_n is constant, then consistency is obtained

- for $\rho(h) = h^\alpha$, $0 \leq \alpha < 1/2$ when $\ell^* \gg n^\gamma$ where $\gamma = \frac{1-2\alpha}{2-2\alpha}$,
- for $\rho(h) = h^{1/2} |\ln h|^\beta$, $\beta > 1/2$, when $\ell^* \gg \ln^{2\beta} n$.

Asymptotic distributions under H_0

When they exists, the limits in law of our statistics under H_0 are

$$\mathrm{UI}(\rho) := \sup_{0\le s<t\le 1} \frac{|B(t)-B(s)|}{\tilde\rho(t-s)}, \quad B \text{ Brownian bridge},$$

$$\mathrm{DI}(\rho) := \sup_{j\ge 1} \frac{1}{\rho(2^{-j})} \max_{r\in\mathbb{D}_j} \left| W(r) - \frac{W(r^+)+W(r^-)}{2} \right|, \quad W \text{ Brownian.}$$

No explicit formula is known for the distribution of $\mathrm{UI}(\rho)$.

Proposition

With $\rho(h) = h^\alpha L(1/h)$, let $c := \limsup_{j\to\infty} j^{1/2} 2^{j(\alpha-1/2)} L(2^j)^{-1}$.

- *If $c=\infty$, then $\mathrm{DI}(\rho)=\infty$ a.s.,*
- *if $0\le c<\infty$, then the d.f. of $\mathrm{DI}(\rho)$ is continuous with support $[c\sqrt{\log 2}, \infty)$ and recalling $\operatorname{erf} x = \frac{2}{\pi^{1/2}}\int_0^x \exp(-s^2)\,\mathrm{d}s$,*

$$\mathbf{P}(\mathrm{DI}(\rho)\le x) = \prod_{j=1}^{\infty} \left\{ \operatorname{erf}\left(2^{j(1/2-\alpha)} L(2^j) x\right) \right\}^{2^{j-1}}.$$

Theorem (Račkauskas and S., 2004)

Under H_0, if for every positive constant A,

$$\lim_{t\to\infty} P\big(|\varepsilon_1| > At^{1/2-\alpha}L(t)\big) = 0, \qquad (\star)$$

then with $\sigma^2 := \operatorname{Var}\varepsilon_1$,

$$\frac{1}{\sigma\sqrt{n}}\mathrm{UI}(n,\rho) \xrightarrow[n\to\infty]{\text{law}} \mathrm{UI}(\rho), \quad \frac{1}{\sigma\sqrt{n}}\mathrm{DI}(n,\rho) \xrightarrow[n\to\infty]{\text{law}} \mathrm{DI}(\rho).$$

- If $\alpha < 1/2$, we may skip A in $(\star)$.
- If $\rho(h) = h^\alpha$, $\alpha < 1/2$, defining $p(\alpha) := (1/2-\alpha)^{-1}$,

$$(\star) \Leftrightarrow \lim_{t\to\infty} t^{p(\alpha)} P(|\varepsilon_1| > t) = 0.$$

- If $\rho(h) = h^{1/2}\ln^\beta 1/h$, $\beta > 1/2$,

$$(\star) \Leftrightarrow \forall c > 0,\ \mathbf{E}\exp\left(c\,|\varepsilon_1|^{1/\beta}\right) < \infty.$$

The convergence $\mathrm{UI}(n,\rho) \to \mathrm{UI}(\rho)$ follows from the Hölderian FCLT through the representation

$$\mathrm{UI}(n,\rho) = g_n(\xi_n) = g(\xi_n) + o_P(1), \quad g(W) = \mathrm{UI}(\rho),$$

where ξ_n is the random polygonal line with vertices $(k/n, S(k))$, g_n and g are defined through the functional

$$J(f,s,t) := \frac{|f(t) - f(s) - (t-s)f(1)|}{\tilde{\rho}(t-s)}, \quad 0 \leq s < t \leq 1,$$

by

$$g_n(f) := \max_{1\leq i<j\leq n} J\Big(f, \frac{i}{n}, \frac{j}{n}\Big), \quad g(f) := \sup_{0\leq s<t\leq 1} J(f,s,t).$$

Theorem (Hölderian FCLT, Račkauskas and S., 2004)

For i.i.d. centered ε_i's and $\rho(h) = h^\alpha L(1/h)$,

$$\frac{1}{\sigma\sqrt{n}}\xi_n \xrightarrow[\mathrm{H}^o_\rho]{\text{law}} W \iff \forall A > 0,\ \lim_{t\to\infty} P\big(|\varepsilon_1| > At^{1/2-\alpha}L(t)\big) = 0.$$

Hölder spaces

Let $(\mathbb{B}, \| \ \|)$ be a separable Banach space.

Definition

$$\mathrm{H}^o_\rho(\mathbb{B}) := \left\{ x : [0,1] \to \mathbb{B};\ \omega_\rho(x,\delta) \xrightarrow[\delta\to 0]{} 0 \right\},$$

where

$$\omega_\rho(x,\delta) := \sup_{\substack{s,t\in[0,1],\\ 0<t-s<\delta}} \frac{\|x(t)-x(s)\|}{\rho(t-s)}.$$

Endowed with $\|x\|_\rho := \|x(0)\| + \omega_\rho(x,1)$, $\mathrm{H}^o_\rho(\mathbb{B})$ is a separable Banach space.

- $\mathrm{H}^o_\rho := \mathrm{H}^o_\rho(\mathbb{R})$.
- With $\rho(h) = h^\alpha$, $\mathrm{H}^o_\alpha(\mathbb{B}) := \mathrm{H}^o_\rho(\mathbb{B})$, $\mathrm{H}^o_\alpha := \mathrm{H}^o_\alpha(\mathbb{R})$.

How to extend to dependent ε_i's?

To extend the previous results to dependent variables it suffices to have

1. an Hölderian FCLT for the partial sums of the observations (under H_0). Available results include linear processes (R+S,), α-mixing (Hamadouche 1997), τ dependence, ρ-mixing, α-mixing, martingale increments, (Giraudo 2015), ...
2. an estimate of variance : $\mathrm{Var}\left(\sum_{i \in K} \varepsilon_i\right) = O(\mathrm{Card} K)$, for K finite.

CLT in Banach space

Let ε be a random element in the Banach space $\mathbb{B}$, with independent copies $\varepsilon_1, \dots, \varepsilon_n, \dots$ and put $S_n := \varepsilon_1 + \dots + \varepsilon_n$.

Definition

$$\varepsilon \in \mathrm{CLT}(\mathbb{B}) \quad \textit{if} \quad n^{-1/2} S_n \quad \textit{converges in distribution in } \mathbb{B}.$$

If $\varepsilon \in \mathrm{CLT}(\mathbb{B})$,

- $\mathbf{E}\,\varepsilon = 0$;
- ε is pregaussian;
- the limit of $n^{-1/2} S_n$ is Gaussian;
- $\lim_{t\to\infty} t^2 \sup_{n\geq 1} \mathbf{P}\left(\|S_n\| \geq t\sqrt{n}\right) = 0$.

It is well known that if $\dim(\mathbb{B}) < \infty$,

$$\varepsilon \in \mathrm{CLT}(\mathbb{B}) \quad \Leftrightarrow \quad \mathbf{E}\,\varepsilon = 0 \text{ and } \mathbf{E}\,\|\varepsilon\|^2 < \infty.$$

If $\dim(\mathbb{B}) = \infty$, the problem is more intricate and involves geometry of $\mathbb{B}$

- Hilbertian case is O.K. (Kwapien theorem)
- Worst case : c_0, $\mathrm{C}[0,1]$, ...

Definition ($\mathbb{B}$-valued Brownian motion)

Let Z be a Gaussian random element in $\mathbb{B}$, with distribution μ. A $\mathbb{B}$-valued Brownian motion W with parameter Q is a Gaussian process $W : [0,1] \to \mathbb{B}$ with independent increments such that

$$W(t) - W(s) = |t-s|^{1/2} Z \quad \textit{(in distribution)},$$

where Z is a centered Gaussian random element in $\mathbb{B}$ with covariance operator Q.

FCLT in $\mathrm{C}(\mathbb{B})$ and $\mathrm{H}^o_\rho(\mathbb{B})$

Consider the partial sums process built on the ε_i's, that is the random $\mathbb{B}$ valued polygonal line defined by

$$\xi_n(t) := S_{[nt]} + (nt - [nt])\varepsilon_{[nt]+1}, \quad t \in [0,1].$$

ξ_n is a random element in the spaces $\mathrm{C}(\mathbb{B})$ and $\mathrm{H}^o_\rho(\mathbb{B})$.

Theorem (Kuelbs, 1973)

$$n^{-1/2}\xi_n \xrightarrow[\mathrm{C}(\mathbb{B})]{\text{law}} W \quad \Leftrightarrow \quad \varepsilon_1 \in \mathrm{CLT}(\mathbb{B}).$$

Theorem (Račkauskas and S., 2004)

$n^{-1/2}\xi_n$ converges in law in $\mathrm{H}^o_\rho(\mathbb{B})$ to W if and only if

a) $\varepsilon_1 \in \mathrm{CLT}(\mathbb{B})$;

b) $\forall A > 0, \quad \lim_{t\to\infty} t\,\mathbf{P}\big(\|\varepsilon_1\| > t^{1/2-\alpha}L(t)\big) = 0.$

Back to epidemic detection

Null hypothesis H_0'

The ε_i's are i.i.d. pregaussian random elements in $\mathbb{B}$ with covariance Q.

Let Y, Y_r (r dyadic) be i.i.d. Gaussian centered random elements in $\mathbb{B}$ with covariance Q. Set

$$\mathrm{DI}(\rho, Q) := \frac{1}{\sqrt{2}} \sup_{j \geq 1} \frac{1}{2^{j(1/2-\alpha)} L(2^j)} \max_{r \in \mathbb{D}_j} \| Y_r \|.$$

With $G_Q(x) := P(\| Y \| \leq x)$, we obtain

$$\forall x > 0, \quad P\left(\mathrm{DI}(\rho, Q) \leq x\right) = \prod_{j=1}^{\infty} \left(G_Q\left(2^{j(1/2-\alpha)+1/2} L(2^j) x\right)\right)^{2^{j-1}}.$$

Theorem (Račkauskas and S., 2006)

Under H_0', if $\varepsilon_1 \in \mathrm{CLT}(\mathbb{B})$ and $\lim_{t\to\infty} t\,\mathbf{P}\left(\|\varepsilon_1\| > At^{1/2-\alpha} L(t)\right) = 0$ for every $A > 0$,

$$n^{-1/2}\mathrm{DI}(n, \rho) \xrightarrow[n\to\infty]{\text{law}} \mathrm{DI}(\rho, Q).$$

Consistency

Alternative hypothesis H'_A

For $k = 1, \dots, n$, $\varepsilon_k = \varepsilon'_k + a_n \mathbf{1}_{\mathbb{I}_n}(k)$ where $\alpha_n \in \mathbb{B}$ is deterministic and $(\varepsilon'_k)_{1 \le k \le n}$ satisfies H'_0.

Theorem (Račkauskas, S., 2006)

Under H'_A, if

$$\lim_{n\to\infty} n^{1/2} \frac{a_n h_n}{\rho(h_n)} = \infty, \quad \text{where } h_n := \frac{\ell^*}{n}\Big(1 - \frac{\ell^*}{n}\Big)$$

then

$$n^{-1/2} \mathrm{DI}(n, \rho) \xrightarrow[n\to\infty]{\Pr} \infty.$$

Application to epidemic change in distribution function

Let $Z_1, \dots, Z_n$ be independent real valued with continuous distribution functions $F_1, \dots, F_n$ respectively. We test H_0' vs H_A' where

H_0': $F_k = F$, for $k = 1, \dots, n$

H_A': $F_k = F + (G - F)\mathbf{1}_{\mathbb{I}_n}(k)$, for $k = 1, \dots, n$, where G is another d.f.

Put $U_k := F(Z_k)$ and define $\varepsilon_k(t) := \mathbf{1}_{U_k \leq t} - t$, $t \in [0,1]$. Note that under H_0' the Bochner (or Pettis) integral of ε_k is 0, since U_k is uniformly distributed on $[0,1]$. Define

$$\kappa_n(s,t) := \sum_{i \leq ns} \left(\mathbf{1}_{\{U_i \leq t\}} - t\right), \quad s,t \in [0,1].$$

For $r \in \mathbb{D}_j$, $j \geq 1$, define random functions $\lambda_r(\kappa_n)$ by

$$\lambda_r(\kappa_n)(t) := \kappa_n(r,t) - \frac{1}{2}\left(\kappa_n(r^-,t) + \kappa_n(r^+,t)\right), \quad t \in [0,1].$$

Cramér-von Mises type DI statistics ($\mathbb{B} = L^2[0,1]$)

$$\mathrm{CMDI}(n,\rho) = \max_{1\le j\le \log n} \frac{1}{\rho^2(2^{-j})} \max_{r\in\mathbb{D}_j} \|\lambda_r(\kappa_n)\|_2^2.$$

Note that

$$\|\lambda_r(\kappa_n)\|_2^2 = \frac{1}{4}\int_0^1 \left| \sum_{nr^- < k \le nr} \varepsilon_k(t) - \sum_{nr<k\le nr^+} \varepsilon_k(t) \right|^2 \mathrm{d}t.$$

Let B be a standard Brownian bridge on $[0,1]$. Put

$$N_2(u) = \mathbf{P}\left(\int_0^1 B^2(t)\,\mathrm{d}t \le u \right), \quad u \ge 0.$$

Theorem (Račkauskas and S., 2006)

Under H_0' and for any admissible ρ and every $u \ge 0$,

$$\lim_{n\to\infty} \mathbf{P}(n^{-1}\mathrm{CMDI}(n,\rho) \le u) = \prod_{j=1}^{\infty} \Big[N_2(2^{j(1-2\alpha)+1} L(2^j)^2 u) \Big]^{2^{j-1}}.$$

Kolmogorov-Smirnov type DI statistics

The Kolmogorov-Smirnov type dyadic increments statistics are defined by

$$\mathrm{KSDI}(n,\rho) = \max_{1\le j\le \log n} \frac{1}{\rho(2^{-j})} \max_{r\in\mathbb{D}_j} \|\lambda_r(\kappa_n)\|_\infty .$$

Let $N_\infty(u)$ be the limit distribution of the classical Kolmogorov-Smirnov statistic,

$$N_\infty(u) = \mathbf{P}\left(\max_{0\le t\le 1} |B(t)| \le u\right), \quad u \ge 0.$$

Theorem (Račkauskas and S., 2006)

Under H_0' and for any admissible ρ,

$$\lim_{n\to\infty} \mathbf{P}\{n^{-1/2}\mathrm{KSDI}(n,\rho) \le u\} = L_{\infty,\rho}(u),$$

for each $u > 0$, where

$$L_{\infty,\rho}(u) = \prod_{j=1}^{\infty} \left[L_\infty\left(2^{j(1/2-\alpha)+1/2} L(2^j) u\right)\right]^{2^{j-1}} .$$

Testing change of characteristic function

Let $Z_1, \dots, Z_n$ be independent real valued with characteristic functions $\varphi_1, \dots, \varphi_n$ respectively. We test H'_0 vs H'_A where

H'_0: $\varphi_k = \varphi$, for $k = 1, \dots, n$
H'_A: $\varphi_k = \varphi + (\psi - \varphi)\mathbf{1}_{\mathbb{I}_n}(k)$, for $k = 1, \dots, n$, where ψ is another ch.f.

Define

$$c_n(s,t) := \sum_{1 \le k \le ns} \big(\exp(itZ_k) - \mathfrak{c}(t) \big), \quad s \in [0,1], \quad t \in \mathbb{R}$$

and define for $r \in \mathbb{D}_j$, $j \ge 1$,

$$\lambda_r(c_n)(t) := c_n(r,t) - \frac{1}{2}\big(c_n(r^-,t) + c_n(r^+,t)\big), \quad t \in [0,1].$$

With any probability measure μ on $\mathbb{R}$ set

$$C^2_\mu(n,r) := \int_{-\infty}^{\infty} \left|\lambda_r(c_n)(t)\right|^2 \mu(\mathrm{d}t)$$

and define the test statistics

$$\mathrm{CDI}(n,\rho) := \max_{1\le j\le \log n} \frac{1}{\rho^2(2^{-j})} \max_{r\in\mathbb{D}_j} C_\mu^2(n,r).$$

The limiting distribution function of this statistics depends on the d.f.

$$G_2(u) = \mathbf{P}\left\{\int_{\mathbb{R}} \left|Y_\varphi(t)\right|^2 \mu(\,\mathrm{d}t) \le u\right\}, \quad u \ge 0,$$

where Y_φ is a complex Gaussian process with zero mean and covariance

$$\mathbf{E}\, Y_\varphi(t)\overline{Y_\varphi(s)} = \varphi(t-s) - \varphi(t)\varphi(-s), \quad s,t\in\mathbb{R}.$$

With $\mathbb{B} = L^2(\mathbb{R}\to\mathbb{C},\mu)$, we obtain

Theorem

Under H_0' and for any admissible ρ, every $u > 0$,

$$\lim_{n\to\infty} \mathbf{P}\{n^{-1}\mathrm{CDI}(n,\rho)\le u\} = G_{2,\rho}(u), \quad \textit{where}$$

$$G_{2,\rho}(u) = \prod_{j=1}^{\infty}\Big[G_2\big(2^{j(1-2\alpha)+1}L(2^j)^2 u\big)\Big]^{2^{j-1}}.$$

Problem

Detection of an epidemic interval in the innovations of a n.n.s. $AR(1)$

The first order *nearly nonstationary* autoregressive process (n.n.s. AR(1)) is generated by the triangular array scheme

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k, \quad n \geq 0, \quad 0 \leq k \leq n,$$

where for simplicity, we choose $y_{n,0} = 0$ and

- $\phi_n \to 1$, as $n \to \infty$,
- (ε_k) is a sequence of innovations with $\mathbf{E}\,\varepsilon_k = 0$ and finite variance σ^2,

Parametrization

First type model

$\phi_n = \mathrm{e}^{\gamma/n}$
with constant $\gamma < 0$.
Parametrization suggested by Phillips (1987).

Second type model

$\phi_n = 1 - \frac{\gamma_n}{n}$,
$\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$
Parametrization suggested by Giraitis-Phillips (2004).

Null hypothesis H_0

$(y_{n,k})_{n\geq 1, 0\leq k\leq n}$ is a n.n.s. AR(1).

Alternative hypothesis H_A

$y_{n,k} = z_{n,k} + a_n \mathbf{1}_{\mathbb{I}_n}(k)$, where $(z_{n,k})_{n\geq 1, 0\leq k\leq n}$ is a n.n.s. AR(1).

Set

$$T_{\alpha,n}(x_1,\dots,x_n) = \max_{1\leq \ell\leq n} \ell^{-\alpha} \max_{1\leq k\leq n-\ell} \left| \sum_{j=k+1}^{k+\ell} x_j - \frac{\ell}{n}\sum_{j=1}^{n} x_j \right|.$$

Test statistics

$$\widetilde{T}_{\alpha,n} := T_{\alpha,n}(y_{n,1},\dots,y_{n,n}) =: T_{\alpha,n}(y),\ \widehat{T}_{\alpha,n} := T_{\alpha,n}(\widehat{\varepsilon}_1,\dots,\widehat{\varepsilon}_n) =: T_{\alpha,n}(\widehat{\varepsilon}),$$

where $(\widehat{\varepsilon}_k)$ are residuals of the model defined by

$$\widehat{\varepsilon}_k = y_{n,k} - \widehat{\phi}_n y_{n,k-1}, \quad k = 1,\dots,n,$$

and $\widehat{\phi}_n$ is the classical least square estimator of ϕ_n:

$$\widehat{\phi}_n = \frac{\sum_{k=1}^n y_{n,k} y_{n,k-1}}{\sum_{k=1}^n y_{n,k-1}^2}.$$

Theorem (Markevičiūtė,Račkauskas, S., 2013)

Suppose $\phi_n = \mathrm{e}^{\gamma/n}$ with $\gamma < 0$ constant and that the ε_i are i.i.d. centered and in $\mathcal{L}^o_{p,\infty}$ for some $p > 2$. Then under H_0 for any $\alpha \in (0, \alpha_p)$

$$n^{-3/2+\alpha} \sigma^{-1} \widetilde{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_{\alpha,\infty}(J) = \|L(J)\|_\alpha,$$

where $\sigma^2 = \mathbb{E}\varepsilon_1^2$ and J is an integrated Ornstein-Uhlenbeck process $J(t) = \int_0^t \int_0^s \mathrm{e}^{\gamma(s-r)} \,\mathrm{d}W(r) \,\mathrm{d}s$. Under the same assumptions,

$$n^{-1/2+\alpha} \sigma^{-1} \widehat{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_{\alpha,\infty}(W - A^{-1}BJ)$$

where $A = \int_0^1 \left(\int_0^s \mathrm{e}^{\gamma(s-r)} \,\mathrm{d}W(r) \right)^2 \mathrm{d}s$, $B = \int_0^1 \int_0^s \mathrm{e}^{\gamma(s-r)} \,\mathrm{d}W(r) \,\mathrm{d}W(s)$.

Theorem (Markevičiūtė, Račkauskas, S., 2015)

Suppose that $\phi_n = 1 - \gamma_n/n$, *where* (γ_n) *is a sequence of nonnegative numbers such that* $\gamma_n/n \to 0$. *Assume that* (ε_i)*'s are i.i.d. centered and in* $\mathcal{L}^o_{p,\infty}$ *for some* $p > 2$. *Then for* $\alpha \in (0, \alpha_p]$, *under* H_0, *if*

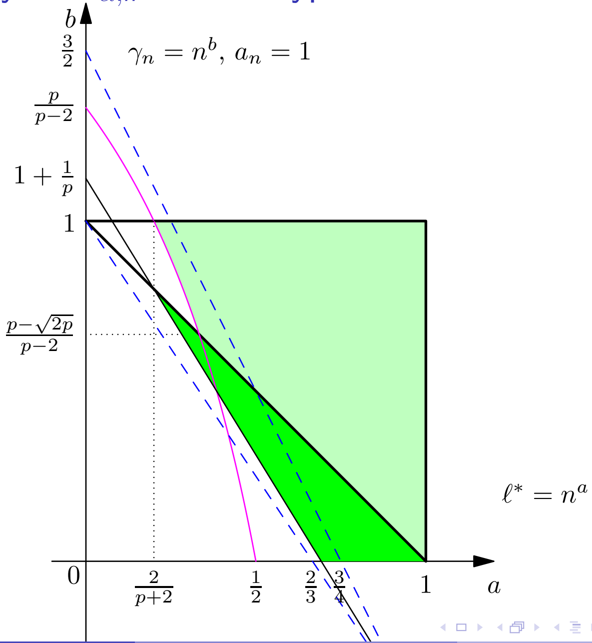
$$\liminf_{n\to\infty} n^{1-\delta}(1 - \phi_n) > 0, \text{ for some } \delta > 0,$$

then

$$n^{-1/2+\alpha}(1 - \phi_n)\sigma^{-1}\widetilde{T}_{\alpha,n} \xrightarrow[n\to\infty]{\text{law}} T_{\alpha,\infty}(W) = \|B\|_\alpha,$$

where B *is a standard Brownian bridge on* $[0, 1]$.

Consistency of $\widetilde{T}_{\alpha,n}$, second type model



Consistency of $\widehat{T}_{\alpha,n}$, second type model

Conditions on the changed segment and the amplitude of the jump.

- $k^* \geq \lambda n$ with some fixed $0 < \lambda < 1$ and $\ell^* = \ell^*(n) \to \infty$.
- $\ell^* = o(n)$.
- $a_n = O(1)$.

Condition depending on the integrability of innovations.

$$n^{-1/2+\alpha} \left| a_n \right| \ell^{*(1-\alpha)} \to \infty.$$

Condition on the dependence structure of the model

$$\frac{n(1-\phi_n)}{a_n^2 \ell^*} \to \infty.$$

Theorem ([Markevičiūtė,Račkauskas, S., 2015)

Suppose that in the second type model, the ε_i's are in $\mathcal{L}_2$ (then $p = 2$) or in $\mathcal{L}^o_{p,\infty}$ for some $p > 2$. Suppose moreover that $\phi_n \in (0,1)$, $\phi_n \to 1$ and that $n(1-\phi_n)$ is non decreasing or regularly varying. Under H_A, assume that the above conditions are satisfied. Then $n^{-1/2} T_{\alpha,n}(\widehat{\varepsilon})$ tends in probability to infinity.