Periodogram Based Tests of Stationarity

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- A smoothed L_2-type Test

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with innovations $\{\varepsilon_t, t \in \mathbb{Z}\} \sim I.I.D.(0,1)$, $\sum_j |a(j)| < \infty$. Its covariance structure is fully described by the spectral density $f(\cdot)$, which, for $\gamma(h) = Cov(X_t, X_{t+h})$, $h \in \mathbb{Z}$, is defined by

$$f(\lambda) := \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-ih\lambda}$$
$$= \frac{1}{2\pi} |A(e^{-i\lambda})|^2, \qquad \lambda \in [-\pi, \pi],$$

where $A(z) = \sum_{j\in\mathbb{Z}} a(j) z^j$, $z \in \mathbb{C}$.

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To make such a class rich enough and mathematically tractable, it is commonly assume that smooth functions $\alpha(\cdot, j) : (0, 1] \to \mathbb{R}$ and a non-negative sequence $\{l(j), j \in \mathbb{Z}\}$ exist such that

$$\sup_u |\alpha(u, j)| \leq \frac{K}{l(j)}, \qquad \sum_{j\in\mathbb{Z}} |j| \frac{1}{l(j)} < \infty \qquad \text{and}$$

$$\sup_{1\leq t\leq n} |a_{t,n}(j) - \alpha(\frac{t}{n}, j)| \leq \frac{K}{nl(j)}.$$

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- $f(u,\lambda)$ is called the local spectral density and is defined by

$$f(u,\lambda) = \frac{1}{2\pi}|A(u,e^{-i\lambda})|^2, \qquad u \in [0,1], \quad \lambda \in (-\pi,\pi],$$

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  the linear stationary class $\mathcal{F}_S$ where $a_{t,n}(j)$ are time invariant, that is $a_{t,n}(j) = a(j)$ for all t, n and j, is a subclass of $\mathcal{F}_{LS}$.

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$$H_0: \; f(u, \cdot) = g(\cdot) \qquad \text{for almost all } u \in [0, 1]$$
$$H_1: \; f(u, \cdot) \neq g(\cdot) \text{ for } u \in A \subseteq [0, 1] \text{ with } \lambda(A) > 0.$$

Basic Statistics and Properties

- Consider the periodogram based on the entire time series, i.e., $I_n(\lambda) = (2\pi n)^{-1} | \sum_{t=1}^{n} X_{t,n} e^{-i\lambda t}|^2, \quad \lambda \in [0, \pi],$

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$$\widehat{g}(\lambda) = n^{-1} \sum_{j} K_h(\lambda - \lambda_j) I_n(\lambda_j),$$

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  where $\lambda_j = 2\pi j/n$ are the Fourier frequencies, $K_h(\cdot) = h^{-1} K(\cdot/h)$ a smoothing kernel and $0 < h = h(n)$ a smoothing bandwidth.

- It yields that if $\{\mathbf{X}_n\}_{n \in \mathbb{N}}$ is locally stationary with time varying spectral density $f(u, \lambda)$ and $h \to 0$, $nh^2 \to \infty$, as $n \to \infty$, then

$$\sup_{\lambda \in [0,\pi]} \left| \widehat{g}(\lambda) - \int_0^1 f(u, \lambda) du \right| \xrightarrow{P} 0.$$

- Consider next periodograms based on segments of the time series. In particular, for $u \in (0,1)$ define the local periodogram

$$I_N(u,\lambda) = \frac{1}{2\pi N}\Big|\sum_{t=1}^{N} X_{t+[un]-N/2-1,n} e^{-i\lambda t}\Big|^2,$$

where $0 < N \ll n$ is a time window width. $I_N(u,\lambda)$ is the periodogram calculated over a window of N observations around the time point $[un]$.

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- Then,

$$E\Big(\frac{I_N(u,\lambda)}{g(\lambda)}\Big) = \frac{f(u,\lambda)}{g(\lambda)} + O(N^{-1})$$

$$\xrightarrow{N\to\infty} \begin{cases} 1 & \text{if } H_0 \text{ true} \\ f(u,\lambda)\big/g(\lambda) & \text{if } H_1 \text{ true}, \end{cases}$$

recall $g(\lambda) = \int_0^1 f(u,\lambda)du$.

- Thus the mean function $m : [0,1] \times [-\pi, \pi] \to \mathbb{R}$ defined by

$$m(u,\lambda) = E\left(\frac{I_N(u,\lambda)}{g(\lambda)}\right) - 1,$$

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$$\widehat{m}(u,\lambda) = \frac{1}{N}\sum_k K_b(\lambda - \lambda_k)\Big(\frac{I_N(u,\lambda_k)}{\widehat{g}(\lambda_k)} - 1\Big),$$

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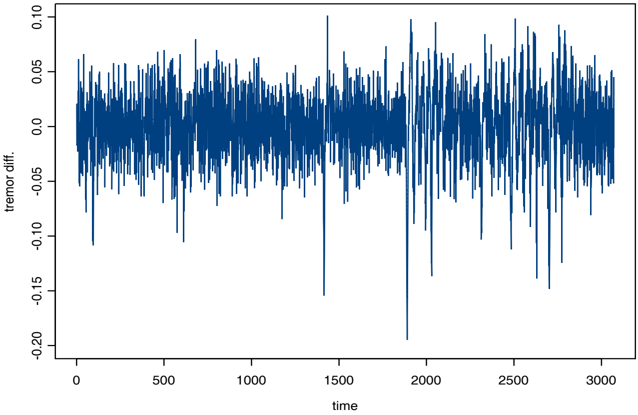
$$Q_n(u) = \int_{-\pi}^{\pi} \big[\widehat{m}(u,\lambda)\big]^2 d\lambda.$$

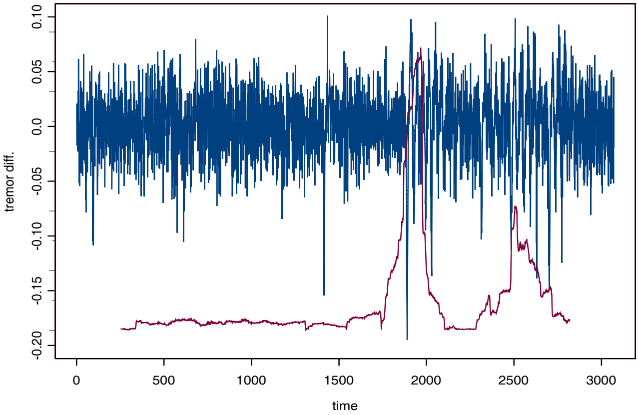
$Q_n(u), u \in [0,1]$, can be interpreted as an estimated measure of second order stationarity of a time series; P. (2009).

Data Example: Consider the series of n=3072 observations of a set of tremor data (first differences) recorded in the Cognitive Neuroscience Laboratory, Univ. of Quebec, Montreal. Compare different regions of tremor activity coming from a subject with Parkinson's disease (Data has been analyzed by von Sachs and Neumann (2000)).

Data example 1 (con.): Tremor series.



Data example 1 (con.): Tremor series and statistic $Q_n(t/n)$, $t = [N/2] + 1, [N/2] + 2, \ldots, n - [N/2]$, (red line). Time window width $N = 256$, Bandwidths h, b chosen by CV.



Data example 2 (Egg price series) : n=1201 observations of weekly egg prices (first differences) at a German agriculture market between April 1967 and May 1990 (Fan and Yao (2003)).

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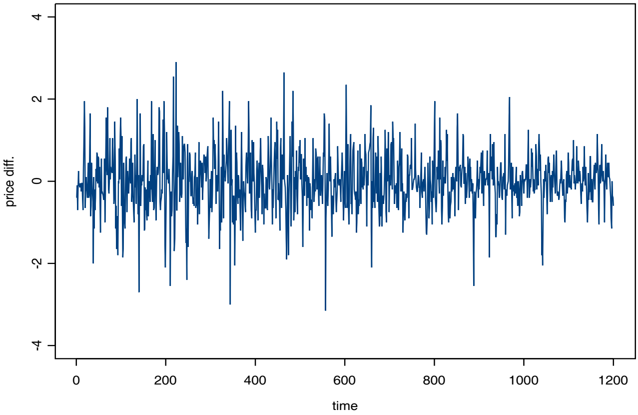
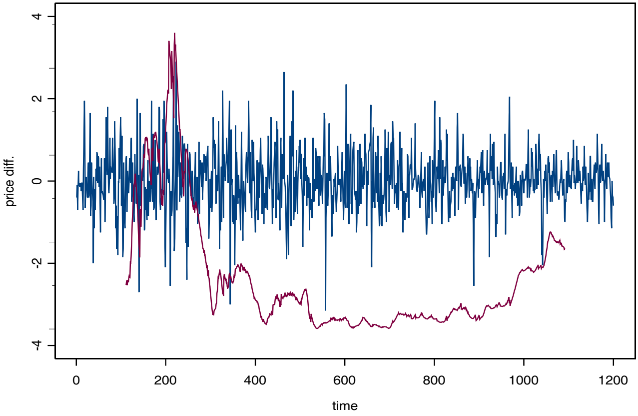


Data example 2 (con.): Egg price series and statistic $Q_n(t/n)$, $t = [N/2]+1, [N/2]+2, \ldots, n-[N/2]$, (red line). Time window width $N = 128$.



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$$N\sqrt{b}\Big(Q_n(u_1) - \mu_n, \ldots, Q_n(u_M) - \mu_n\Big)' \Rightarrow N_M(0, \Sigma_Q),$$

where

$$\mu_n = b^{-1/2} \int_{-\pi}^{\pi} K^2(x)dx$$

and

$$\Sigma_Q = \sigma_Q^2 \mathbf{I}_M, \quad \sigma_Q^2 = \frac{1}{2\pi^2} \int_{-2\pi}^{2\pi} \Big(\int_{\pi}^{\pi} K(x)K(x+y)dx \Big)^2 dy.$$

Limiting distribution of $Q_n(u)$ does not dependent on characteristics or parameters of the underlying process $\mathbf{X}_n$.

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- However, for a given sample size n and for $|u_1 - u_2| < N/n$, there is obviously nonnegligible dependence between $Q_n(u_1)$ and $Q_n(u_2)$ due to the overlap of the segments of random variables used to calculate the corresponding local periodograms $I_N(u_1, \lambda)$ and $I_N(u_2, \lambda)$.

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- To appropriately describe this local dependence structure, we consider the random variables

$$Q_n(x; u_0) = N\sqrt{b} \int_{-\pi}^{\pi} \left(\widehat{m}(u_0 + x\delta_n, \lambda)\right)^2 d\lambda, \quad \text{for } x \in [-1/2, 1/2],$$

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  where $u_0 \in (0,1)$ and $\delta_n = N/n$.
- Notice: For $x_1 \neq x_2 \in [-1/2, 1/2]$ the time distance between the random variables $Q_n(x_1; u_0)$ and $Q_n(x_2; u_0)$ is $|x_2 - x_1|\delta_n$. Thus we allow the time distance between $Q_n(x_1; u_0)$ and $Q_n(x_2; u_0)$ to shrink to zero at the rate $\delta_n = N/n$ as $n \to \infty$.

- We then have the following result for the process $\{Q_n(x; u_0), x \in [-1/2, 1/2]\}$, P. (2010):

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Theorem: Under H_0 and as $n \to \infty$,

$$\{Q_n(x;u_0) - b^{-1/2} \int_{-\pi}^{\pi} K^2(y)dy\}_{x\in[-1/2,1/2]} \Rightarrow \{G(x)\}_{x\in[-1/2,1/2]},$$

where G is a zero mean Gaussian process on $[-1/2, 1/2]$ with

$$Cov(G(x_1), G(x_2)) = \frac{1}{\pi}\Big(1 - |x_1 - x_2|\Big)^4 \int (K * K)^2(y)dy.$$

A L_2-type Test

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- This can be done as follows:
  - Let $0 < u_1 < u_2 < \cdots < u_M < 1$ be a set of $M = M(n) \in \mathbb{N}$ distinct and equidistant time points in the interval $(0, 1)$ given by

    $$u_j = \frac{t_j}{n}, \quad \text{where} \quad t_j = S(j-1) + N/2,$$

    $j = 1, 2, \ldots, M$.
    The positive integer $S = S(n)$ denotes the shift from time point to time point and $n = S(M-1) + N$.
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    The positive integer $S = S(n)$ denotes the shift from time point to time point and $n = S(M-1) + N$.
    (Simple case: The shift S is equal the time window width N. Then $M = n/N$).
  - The proposed test statistic, P. (2009), is then given by

$$T_n = \frac{1}{M} \sum_{s=1}^{M} Q_n(u_s).$$

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Theorem: If $nh^2 \to \infty$, $Nb^2 \to \infty$, $Nb/(nh^2) \to 0$ and $Nhb \to \infty$, then, as $n \to \infty$,

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$$\mu_n = \left[\sqrt{\frac{M}{b}} \int_{-\pi}^{\pi} K^2(x)dx + \sqrt{M\,b}\left(\frac{1}{4\pi}\int_{-2\pi}^{2\pi}(K*K)(y)dy + 2\pi\kappa_4\right)\right],$$

$$\tau_0^2 = \frac{2}{\pi^2}\int_{-2\pi}^{2\pi}(K*K)^2(y)dy,$$

  $\kappa_4 = E(\varepsilon_1^4)/\sigma_\varepsilon^4 - 3$ and $K * K(\cdot)$ denotes convolution of the kernel K.

- Limiting distribution under the null

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$$N\sqrt{Mb}\ T_n \ - \ \mu_n \ \xrightarrow{D} \ N(0, \tau_0^2),$$

  where

$$\mu_n = \Big[\sqrt{\frac{M}{b}} \int_{-\pi}^{\pi} K^2(x)dx + \sqrt{M\,b}\,\Big(\frac{1}{4\pi}\int_{-2\pi}^{2\pi}(K*K)(y)dy + 2\pi\kappa_4\Big)\Big],$$

$$\tau_0^2 = \frac{2}{\pi^2}\int_{-2\pi}^{2\pi}(K*K)^2(y)dy,$$

  $\kappa_4 = E(\varepsilon_1^4)/\sigma_\varepsilon^4 - 3$ and $K * K(\cdot)$ denotes convolution of the kernel K.

- The centering sequence μ_n depends on the rescaled fourth order cumulant of the innovation process $\kappa_4 = E(\varepsilon_1^4)/\sigma_\varepsilon^4 - 3$.

- A nonparametric, and consistent estimator of κ_4 can be constructed; see Grenander and Rosenblatt (1956), Janas and Dahlhaus (1994) and Kreiss and P. (2012). An improved estimator has been proposed by Frangeskou and P. (2015).

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- The test T_n rejects the null hypothesis H_0 if

$$(N\sqrt{Mb}T_n - \widehat{\mu}_n)/\tau_0 > z_\alpha,$$

where $\widehat{\mu}_n$ is obtained by replacing κ_4 in μ_n by a consistent estimator $\widehat{\kappa}_4$, and, z_α is the upper α-percentage point of the standard Gaussian distribution.

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where $\widehat{\mu}_n$ is obtained by replacing κ_4 in μ_n by a consistent estimator $\widehat{\kappa}_4$, and, z_α is the upper α-percentage point of the standard Gaussian distribution.
- Convergence against the limiting Gaussian distribution is slow (common for L_2-tests). Bootstrapping the distribution of the test statistic T_n (under the null) is possible using an AR-sieve bootstrap with wild bootstrapped i.i.d. pseudo-innovations; Frangeskou and P. (2016).

- Consistency of the test can be established if $\{X_{t,n}, t = 1, 2, \ldots, n\}_{n\in\mathbb{N}}$ possesses a local spectral density $f(u, \lambda)$, $f \in L_2([0, 1] \times [-\pi, \pi])$ such that $\lambda(A) > 0$ where $A = \{u : f(u, \lambda) \neq g(\lambda)\} \subseteq [0, 1]$. In this case,

$$T_n \xrightarrow{P} \int_0^1 \int_{-\pi}^{\pi} \Big(\frac{f(u, \lambda)}{g(\lambda)} - 1\Big)^2 d\lambda du,$$

that is $N\sqrt{Mb}T_n - \mu_n \to \infty$ as $n \to \infty$.

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  that is $N\sqrt{Mb}T_n - \mu_n \to \infty$ as $n \to \infty$.
- The asymptotic distribution of T_n under fixed (locally stationary) alternatives has been also established. While under H_0 the variance of T_n is of order $O(N^{-2}M^{-1}b^{-1})$, under H_1 and, in particular, under fixed locally stationary alternatives, the variance of T_n is of order $O(N^{-1}M^{-1}) = O(n^{-1})$.

- The asymptotic results concerning the limiting distribution of T_n under fixed alternatives allow for an approximative expression of the power function of the test T_n. In particular,

$$P(T_n \text{ rejects } H_0 \mid \mathbf{X}_n \text{ is loc. stat.}) \approx 1 - \Phi\Big(-\frac{\sqrt{NM}}{\tau_1} D_n^2\Big),$$

where $N^{-1}M^{-1} = O(n^{-1})$,

$$D_n^2 = \frac{1}{M}\sum_{s=1}^{M}\int_{-\pi}^{\pi}\Big(\frac{f(u_s,\lambda)}{g(\lambda)} - 1\Big)^2 d\lambda,$$

$f(u,\lambda)$ the local spectral density of the locally stationary process $g(\lambda) = \int_0^1 f(u,\lambda)du$. and τ_1 is the variance of the limiting distribution of T_n under fixed locally stationary alternatives which depends on $f(u,\lambda)$, $g(\lambda)$, κ_4 ... .

Data example 1 (Tremor Series continued): Consider again the $n = 3072$ observations of tremor data (first differences) recorded in the Cognitive Neuroscience Laboratory, Univ. of Quebec, Montreal.

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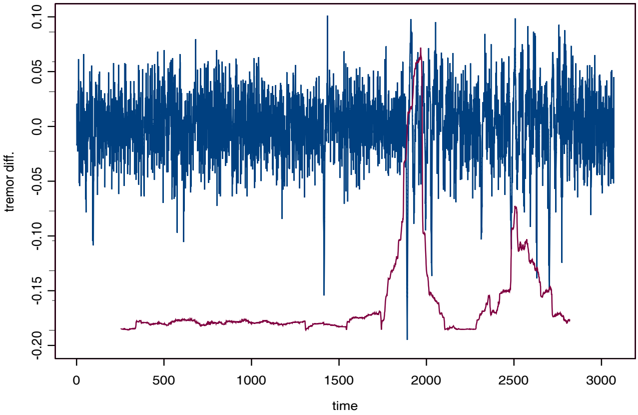
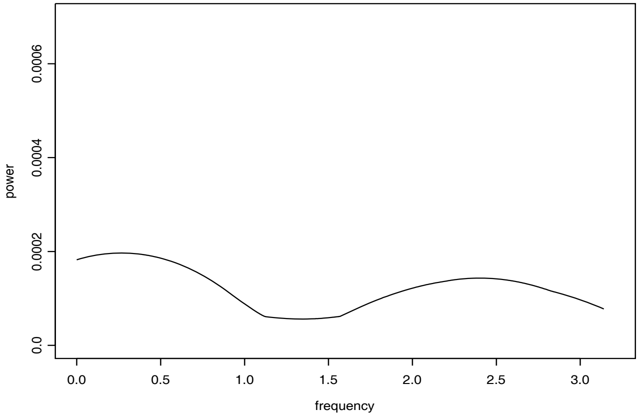
Value of the test statistic $(N\sqrt{Mb}T_n - \widehat{\mu}_n)/\tau_0 = 21.16$ which leads to a rejection of the null hypothesis of autocovariance stationarity. A window size of $N = 256$ observations (which implies $S = N$ and $M = 12$) and the Bartlett-Priestley kernel, have been used).

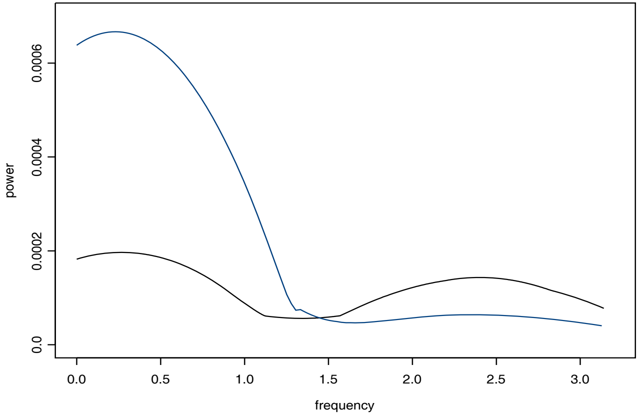
Data example 1 (con.): Tremor series and statistic $Q_n(\cdot)$ (red line). Time window width $N = 256$, Bandwidths h, b chosen by CV.



Data example 1 (con.): Estimated spectral density: Whole series.

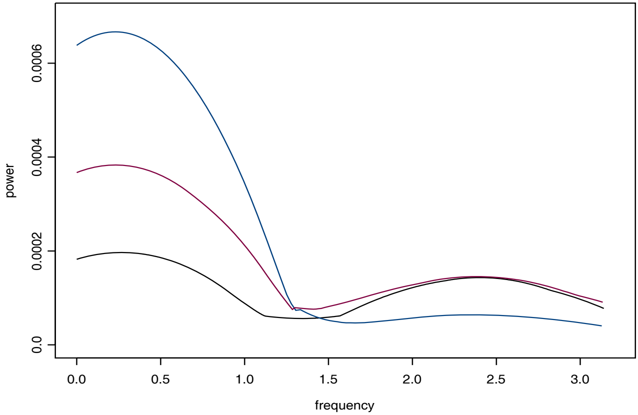


Data example 1 (con.): Estimated spectral densities: Whole series (black) and first segment (blue).



Blue: Refers to the observations X_t, $t \in \{1760, 1761, \ldots, 2170\}$.

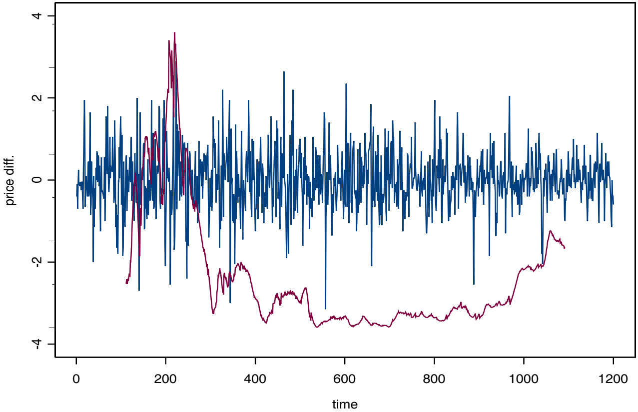
Data example 1 (con.): Estimated spectral densities: Whole series (black), first segment (blue), second segment (red).



Blue: Refers to the observations X_t, $t \in \{1760, 1761, \ldots, 2170\}$
Red: Refers to the observations X_t, $t \in \{2350, 2351, \ldots, 2840\}$

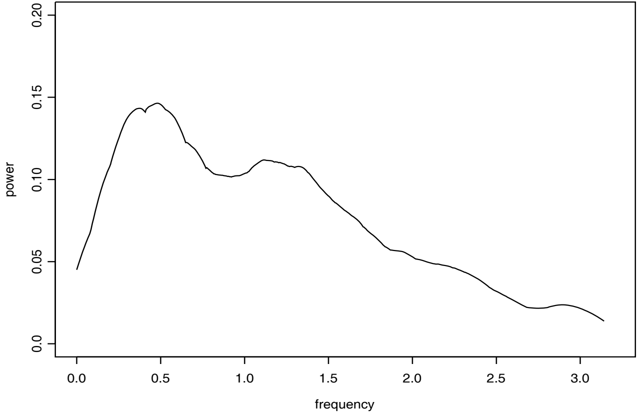
Data example 2 (Egg price series continued) : n=1201 observations of weekly egg prices (first differences) at a German agriculture market between April 1967 and May 1990 (Fan and Yao (2003)).

Data example 2 (con.): Egg price series: Test results.

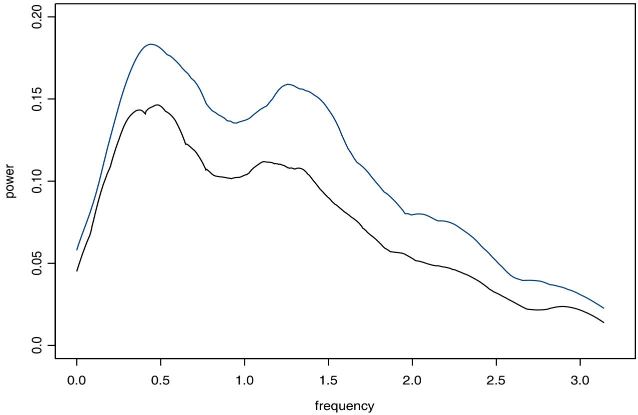


Value of the test statistic $(N\sqrt{Mb}T_n - \widehat{\mu}_n)/\tau_0 = 12.95$ leads to rejection of the null hypothesis ($N = 128$, $M = 9$, Bartlett-Priestley kernel).

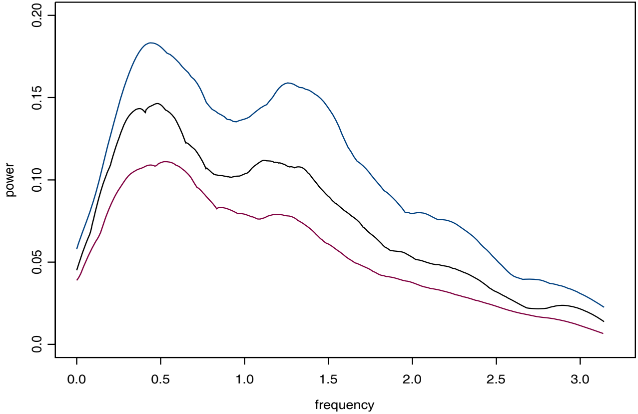
Data example 2 (con.): Estimated spectral densities of Egg-Price Data: Whole series $1 \leq t \leq 1200$, black.



Data example 2 (con.): Estimated spectral densities of Egg-Price Data: Whole series $1 \leq t \leq 1200$, black. First segment $1 \leq t \leq 350$, blue.



Data example 2 (con.): Estimated spectral densities of Egg-Price Data: Whole series $1 \leq t \leq 1200$, black. First segment $1 \leq t \leq 350$, blue. Last segment $500 \leq t \leq 1200$, red.



Alternative Tests and Power Comparisons

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(i) L_2-type tests based on integrated local periodograms without smoothing; Dette, Preuss and Vetter (2011).

(ii) Kolmogorov-Smirnov type tests based on integrated local periodograms; Dahlhaus (2009), Preuss, Vetter and Dette (2013).

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$$D^2 = \int_0^1 \int_{-\pi}^{\pi} \big(f(u,\lambda) - g(\lambda)\big)^2 d\lambda du$$
$$= \int_0^1 \int_{-\pi}^{\pi} f^2(u,\lambda) d\lambda du - \int_{-\pi}^{\pi} g^2(\lambda) d\lambda.$$

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where

$$\widehat{F}_{1,n} = n^{-1} \sum_{s=1}^{M} \sum_{j=1}^{[N/2]} I_N(u_s, \lambda_j)^2$$

and

$$\widehat{F}_{2,n} = N^{-1} \sum_{j=1}^{[N/2]} \left(M^{-1} \sum_{s=1}^{M} I_N(u_s, \lambda_j)\right)^2.$$

- A test is then constructed by using the test statistic

$$\widetilde{D}_n = \sqrt{n}\frac{\widehat{D}_n^2 + 2\pi N/n\widehat{F}_{1,n}}{\widehat{\tau}_0},$$

where $\widehat{\tau}_0^2 = 4\pi^2(6n)^{-1}\sum_{s=1}^{M}\sum_{j=1}^{[N/2]} I_N^4(u_s,\lambda_j)$ and the property that under H_0 and some technical conditions, $\widetilde{D}_n \xrightarrow{d} N(0,1)$.
See Dette, Preuss and Vetter (2011). Notice that no kernel smoothing is used.

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  An estimator of KS is given by

$$KS_n = \sup_{(v,w)\in[0,1]^2} \Big| \frac{2\pi}{n} \sum_{s=1}^{[vM]} \sum_{j=1}^{[wN/2]} I_N(u_s,\lambda_j) - \frac{[vM]}{M} \frac{2\pi}{n} \sum_{s=1}^{M} \sum_{j=1}^{w[N/2]} I_N(u_s,\lambda_j) \Big|.$$

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- Under H_0, $KS_n \xrightarrow{d} \sup_{v,w\in[0,1]^2} |G_0(v,w)|$, where G_0 a zero mean Gaussian process on $[0,1]^2$ with covariance function

$$Cov(G_0(v_1,w_1), G_0(v_2,w_2)) = 2\pi[\min\{v_1,v_2\} - v_1 v_2] \int_0^{\pi \min\{w_1,w_2\}} g^2(\lambda) d\lambda;$$

  see Dahlhaus (2009) and Preuss, Vetter and Dette (2013).

- All three tests considered, i.e., T_n, D_n and KS_n are consistent. We compare them by investigating their local power properties, P. and Preuss (2015).

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- That is we consider sequences of (Gaussian) locally stationary processes $\{\mathbf{X}_n\}_{n\in\mathbb{N}} = \{X_{t,n}, t = 1, 2, \ldots, n\}_{n\in\mathbb{N}} \in \mathcal{F}_{LS}$ that "converge" to a stationary process $\mathbf{X} \in \mathcal{F}_S$ (at some controlled rate and in some appropriate manner) as the time series length $n \to \infty$.

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  - Global in time local alternatives to stationarity. In this case, $\mathbf{X}_n$ possesses a local spectral density

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  - Local in time local alternatives to stationarity. In this case, $\mathbf{X}_n$ possesses a local spectral density

    $$f_n(u,\lambda) \approx f(\lambda) + c_n w\Big(\frac{u-u_0}{\gamma_n},\lambda\Big), \quad \text{and} \quad c_n,\ \gamma_n \to 0.$$

    (Deviations from null become "more concentrated" around the time point $u_0 \in (0,1)$ as n increases to infinity.)

- The aim is to identify the maximal rate at which

$$c_n \sim n^{-\kappa} \quad \text{for some} \quad \kappa > 0,$$

resp.

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  are allowed to converge to zero such that the test T_n has power which is bounded away from the level and from unity.
- The following table summarizes our theoretical findings:

	Global in Time $c_n \sim n^{-\kappa}$	Local in Time $c_n \sim n^{-\kappa}$ and $\gamma_n \sim n^{-\zeta}$
T_n-Test $h \sim n^{-\rho}$, $M \sim n^{\delta}$	$\kappa = \frac{1}{4} + \frac{1}{4}\delta(1-\rho)$	$2\kappa + \zeta = \frac{1}{2} + \frac{1}{2}\delta(1-\rho)$
D_n-Test $M \sim n^{\delta}$	$\kappa = \frac{1}{4}$	$2\kappa + \zeta = \frac{1}{2}$
KS_n-Test $M \sim n^{\delta}$	$\kappa = \frac{1}{2}$	$\kappa + \zeta = \frac{1}{2}$

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- For global in time local alternatives and because

$$\frac{1}{4} < \frac{1}{4} + \frac{1}{4}\delta(1-\rho) < \frac{1}{2},$$

the D_n-Test is the worst and the KS_n-Test the best. The KS_n-Test detects deviations converging at the so-called "parametric rate", i.e., $n^{-1/2}$ while the T_n-test at a slower rate $(0 < \delta(1-\rho) < 1)$ but larger than that of the D_n test.

- For local in time local alternatives and because

$$\frac{1}{2} < \frac{1}{2} + \frac{1}{2}\delta(1-\rho),$$

the T_n-test is the best detecting deviations converging at a rate which is even faster than the parametric rate $n^{-1/2}$ while for this class of alternatives the D_n-test is the worst (since $2\kappa + \gamma > 1/2$ for $\kappa + \gamma = 1/2$, and therefore in this case the power of the D_n-test converges against its size).

Some Numerical Results

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$$Y_{t,n} = \sigma_n(t/n)\varepsilon_t,$$

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where $\varepsilon_t \sim I.I.D. - N(0,1)$,

$$\sigma_n^2(u) = 0.5 + n^{-0.45}1.5u, \qquad u \in [0,1]$$

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$$a_n(u) = 0.5n^{-0.05}e^{-n^{0.5}(u-0.5)^2}sin(4\pi u), \qquad u \in [0,1].$$

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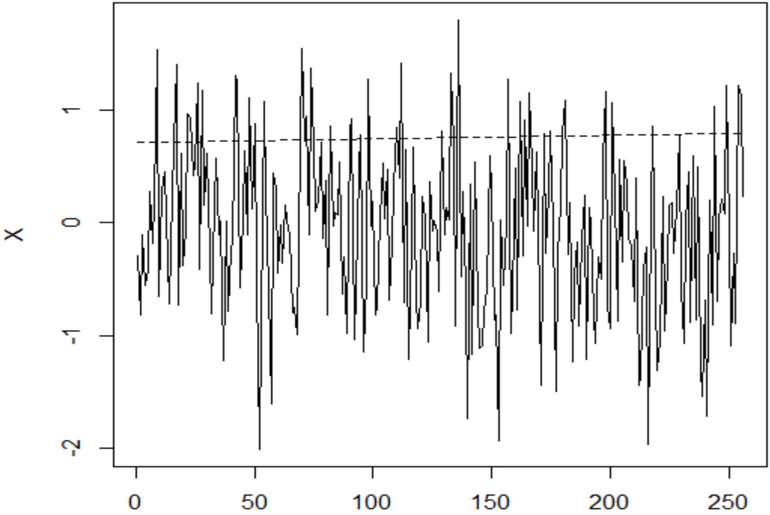
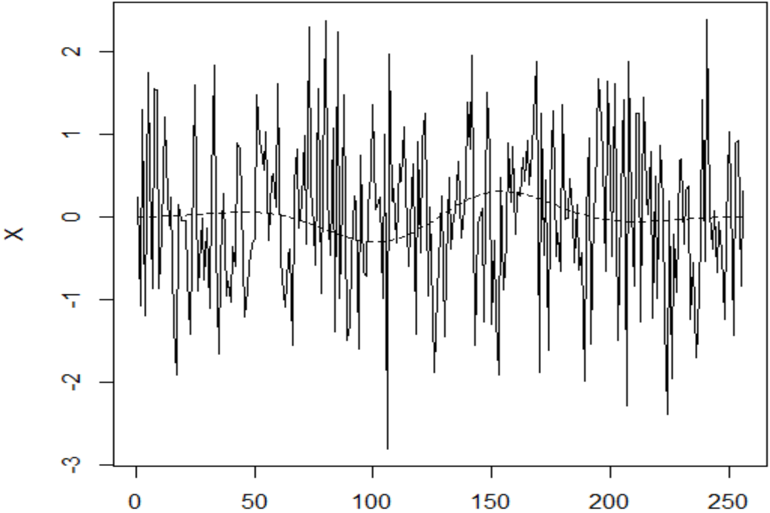
- Figure 1 and Figure 2 show plots of a realization of length $n = 256$ of the above processes together with the corresponding time varying functions.

Time series generated from the first process:
$Y_{t,n} = \sigma_n(t/n)\varepsilon_t$.



Time series generated from the second process:
$X_{t,n} = a_n(t/n)X_{t-1,n} + \varepsilon_t$.



- Different sample sizes have been considered. Smoothing parameters have been chosen by cross-validation and $N = [n/8]$. Critical points of all tests have been obtained by the autoregressive-sieve bootstrap with AR-order selected by AIC. $R = 500$ simulation runs have been used and the rejection frequencies have been calculated.

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- The numerical results obtained, $(\alpha = 5\%)$, coincides with those predicted by our asymptotic analysis:

	$Y_{t,n}$			$X_{t,n}$		
n	T_n	D_n	KS_n	T_n	D_n	KS_n
64	0.072	0.094	0.086	0.090	0.080	0.048
256	0.102	0.070	0.202	0.146	0.118	0.076
1024	0.156	0.070	0.266	0.294	0.104	0.084
2048	0.196	0.062	0.320	0.550	0.086	0.082

Conclusions and Outlook

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- We considered tests of (second order) stationarity based on comparing local with global periodogram properties of a time series. Three different types of tests have been discussed.
- To better understand the power properties of these tests, their behavior for different types of local alternatives to stationarity has been investigated.
- The Kolmogorov-Smirnov-type test, KS_n, seems to have certain advantages when global (in time) deviations from stationarity are present, while the smoothed L_2-type test T_n, seems to be more powerful for time localized type of alternatives. The L_2-type test D_n is the worst under both scenarios of local alternatives considered. Our asymptotic results parallel similar results obtained in the context of testing the distribution (see Bickel and Rosenblatt (1973) and Rosenblatt (1975)) or of testing the form of the regression function (see Härdle and Mammen (1993)), in the i.i.d. set-up.

- It will be interesting to look at other types of local alternatives and compare the power behavior of the different tests.

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  For instance, consider the simple TvMA(q)- process $X_{t,n} = \varepsilon_t + c_n \varepsilon_{t-q}$, where $q \in \mathbb{N}$. Then,

$$f_n(u,\lambda) = \frac{\sigma_\varepsilon^2}{2\pi}\Big(1 + c_n \cdot 2\cos(\lambda \cdot q)\Big) + O(c_n^2),$$

  which belongs to the class of global in time local alternatives. However, the deviation from stationarity generated by such a process is located at different frequencies depending on the value of q. How does this affect the power of the tests?

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Thank you for your Attention!