Phantom distribution functions for dependent random vectors

Thematic Month “Statistics”
Week 3 “Processes”
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The talk

The first part of the talk is
a joint work with
Paul Doukhan and Gabriel Lang
The second part is a joint work with
Natalia Soja-Kukieła

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- Following Tippett, Fischer, Gnedenko, Gumbel, de Haan,... people used to look for conditions on F guaranteeing existence of sequences a_n and b_n such that

$$P(M_n \leqslant a_n x + b_n) \to K(x), x \in \mathbb{R}^1,$$

  where K is non-degenerate.

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- We claim that the asymptotics of $1 - F(v_n)$ along a single sequence $v_n \to F_*-$ determines everything.

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- We claim that the asymptotics of $1 - F(v_n)$ along a single sequence $v_n \to F_*-$ determines everything.
- Here $F_* = \sup\{x\,;\ F(x) < 1\}$.

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- It is an observation made long time ago by O'Brien (1974), that for a given distribution function G one can find a sequence $\{v_n = v_n(\gamma)\}$ such that

$$G^n(v_n) \to \gamma \in (0, 1),$$

if, and only if, G satisfies the relations

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- We will say that G is regular (in the sense of O'Brien) if the above conditions hold.

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- We will modify it slightly, by saying that the tails of two distribution functions G and H with right ends G_* and H_* are strictly tail-equivalent if

$$G_* = H_* \quad \text{and} \quad \frac{1 - H(x)}{1 - G(x)} \to 1, \text{ as } x \to G_*-.$$

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- $$\sup_{x \in \mathbb{R}^1} \left| G^n(x) - H^n(x) \right| \to 0, \text{ as } n \to \infty.$$

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$$\sup_{x \in \mathbb{R}^1} \left| G^n(x) - H^n(x) \right| \to 0, \text{ as } n \to \infty.$$
- H is regular and strictly tail-equivalent to G.

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- It is obvious that G is not uniquely determined.
- If H is another phantom distribution function, then

$$\sup_{x \in \mathbb{R}^1} \left| G^n(x) - H^n(x) \right| \to 0, \text{ as } n \to \infty.$$

  and G and H are strictly tail-equivalent.

The extremal index

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- Suppose that $\{X_j\}$ admits a phantom distribution function G of the form $G(x) = F^\theta(x)$, for some $\theta \in (0,1]$, i.e.

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- In many cases the extremal index is the reciprocal of the mean size of clusters of big values occurring in the sequence $\{X_j\}$.
- The extremal index attracted a lot of attention over years.
- But there are models, in which the extremal index is uninformative, while the phantom distribution function brings some light.

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- Following Leadbetter (1983) we say that $\{X_j\}$ has the extremal index $\theta = 0$ if

$$\mathbb{P}(M_n \leqslant u_n(\tau)) \to 1$$

whenever $\{u_n(\tau)\}$ is such that

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- Intuitively this means that the partial maxima M_n increase much slower comparing with the independent case and that information on F alone cannot determine the limit behavior of laws of M_n.

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- Let

$$X_{j+1} = (X_j + Z_j)^+, \;\; j = 1, 2, \ldots,$$

where $Z_1, Z_2, \ldots$ are i.i.d. with a distribution function H and mean $-m < 0$ and X_0 is independent of $\{Z_j\}$ and distributed according to the unique stationary distribution F.

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- Suppose that H is subexponential, i.e. strictly tail-equivalent to a distribution function $B(x)$ concentrated on $(0, \infty)$ and such that
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- Then $\{X_j\}$ has the extremal index zero.
- We can show that it admits a continuous phantom distribution function.

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- Let $f(x)$ be the target probability density.
- We consider the random walk Metropolis algorithm given by the recursive equation

$$X_{j+1} = X_j + Z_{j+1}\mathbf{1}\{U_{j+1} \leqslant \psi(X_j, X_j + Z_{j+1})\},$$

where $\psi(x,y)$ is defined as

$$\psi(x,y) = \begin{cases} \min\{f(y)/f(x), 1\} & \text{if } f(x) > 0, \\ 1 & \text{if } f(x) = 0. \end{cases}$$

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- G.O. Roberts, J.S. Rosenthal, J. Segers and B. Sousa (2006, Extremes) showed the following result.

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Then for every real sequence $\{u_n\}$ such that $\sup_n n(1 - F(u_n)) < +\infty$, we have

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In particular, the extremal index does exist and is equal to zero.

- Thus heavy tails imply $\theta = 0$.
- We can show that still a continuous phantom distribution function exists.

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- The above theorem is a direct consequence of a more general result.

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- The sequence $\{X_j\}$ admits a *continuous* phantom distribution function.
- There exists a sequence $\{v_n\}$ and $\gamma \in (0, 1)$ such that

$$\mathbb{P}(M_n \leqslant v_n) \to \gamma,$$

and the following Condition $B_\infty(v_n)$ holds: as $n \to \infty$

$$\sup_{p,q \in \mathbb{N}} \left| \mathbb{P}(M_{p+q} \leqslant v_n) - \mathbb{P}(M_p \leqslant v_n)\mathbb{P}(M_q \leqslant v_n) \right| \to 0.$$

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Condition $B_\infty(v_n)$ does not mean “asymptotic independence of maxima"!

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- The above sequence can be chosen in such a way, that it has the extremal index $\theta = 0$.

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- The definition is immediate: G is a phantom distribution function for a stationary sequence of random vectors

$$(X_1^{(1)}, X_1^{(2)}), (X_2^{(1)}, X_2^{(2)}), (X_3^{(1)}, X_3^{(2)}), \ldots$$

with partial maxima

$$\mathbf{M}_n = (M_n^{(1)}, M_n^{(2)}) = (\max_{1 \leqslant j \leqslant n} X_j^{(1)}, \max_{1 \leqslant j \leqslant n} X_j^{(2)}),$$

if

$$\sup_{\mathbf{u}=(u_1,u_2)\in\mathbb{R}^2} \Big| P(\mathbf{M}_n \leqslant \mathbf{u}) - G^n(\mathbf{u}) \Big| \to 0, \text{ as } n \to \infty.$$

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- In fact, it is more convenient to take sup over $\overline{\mathbb{R}}^2$!

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- Find $v_n^{(i)}$, $i = 1, 2$, such that

$$P(M_n^1 \leqslant v_n^{(1)}) \to \rho_1 \in (0,1), P(M_n^2 \leqslant v_n^{(2)}) \to \rho_2 \in (0,1).$$

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- Assume $B_\infty(v_n^{(1)})$ for $\{X_1^{(1)}, X_2^{(1)}, \ldots\}$ and similarly $B_\infty(v_n^{(2)})$ for $\{X_1^{(2)}, X_2^{(2)}, \ldots\}$.

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- Then for $i = 1, 2$

$$P(M_n^{(i)} \leqslant v_{[ns_i]}^{(i)}) \to \rho_i^{1/s_i}$$

if $s_i \in [0, +\infty]$.

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- Assume that for some $\rho : \mathcal{L} \to (0, 1)$

$$P\big(\mathbf{M}_n \leqslant v_n(\mathbf{s})\big) \to \rho(\mathbf{s}), \quad \mathbf{s} \in \mathcal{L}.$$

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$$\mathcal{L} = \{\mathbf{s} \in [1, +\infty)^2 \,;\, s_1 \wedge s_2 = 1\}.$$

- Assume that for some $\rho : \mathcal{L} \to (0, 1)$

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- Assume that $B_\infty(v_n(\mathbf{s}))$ holds for every $\mathbf{s} \in \mathcal{L}$.

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Theorem

- Condition $B_\infty(v_n(\mathbf{s}))$ holds for every $\mathbf{s} \in [0, +\infty]$.
- There exists $H : [0, +\infty]^2 \to [0, 1]$ such that

$$P\big(\mathbf{M}_n \leqslant v_n(\mathbf{s})\big) \to H(\mathbf{s}), \quad \mathbf{s} \in [0, +\infty]^2.$$

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Moreover, if $H^{(1)}$ and $H^{(2)}$ are the marginal cumulative distribution functions, then

$$H^{(i)}((-\log\rho_i)s) = G_{2,1}(s), i = 1,2,$$

where $G_{2,1}(s)$ is the CDF of the standard Fréchet extreme value distribution.

Phantom distribution function for random vectors

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Theorem

$$G(\mathbf{x}) = H(\mathbf{n}(\mathbf{x})),$$

where

$$n_i(\mathbf{x}) = \sup\{n \in \mathbb{N}\,;\, v_n^{(i)} \leqslant x_i\}, \quad i = 1, 2,$$

is a phantom distribution function for $\mathbf{X}_1, \mathbf{X}_2, \ldots$.