Quasi-MLE for Quadratic ARCH model with long memory

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Joint work with Donatas Surgailis (Vilnius) and Andrius Škarnulis (Vilnius)

CIRM, Luminy, 15-19 February 2016

Outline:

1. Generalized Quadratic ARCH (GQARCH) model
2. Properties of GQARCH: stationarity, long memory and leverage
3. QMLE of long memory GQARCH
4. Simulation study
5. Some proofs

The talk is based on recent work:

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The GQARCH model

Definition 1 (The GQARCH model)

$$\begin{aligned} r_t &= \zeta_t \sigma_t, \\ \sigma_t^2 &= \omega^2 + (a + \sum_{j=1}^{\infty} b_j r_{t-j})^2 + \gamma \sigma_{t-1}^2, \end{aligned} \quad (1)$$

where:

- $\{\zeta_t\}$: *standardized* $(0,1)$ *i.i.d. innovations*
- $\omega \geq 0$, a, $0 \leq \gamma < 1$: *parameters*
- $b_j, j \geq 1$: *coefficients*

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GQARCH: the role of parameters

- $\omega > 0$: nonvanishing volatility ($\sigma_t \geq \omega$)
- hyperbolically decaying $b_j \sim cj^{d-1}$, $0 < d < 1/2$ allow modelling of long memory in volatility
- $a \neq 0$: allow modelling of the leverage effect: $\mathrm{Cov}(r_{t-j}, \sigma_t^2) < 0$ (past returns are negatively correlated with future volatility)

GQARCH: particular case of Sentana's QARCH

- By iterating (1) σ_t^2 can be written as a quadratic form in lagged variables $r_{t-1}, r_{t-2}, \cdots$:

$$\sigma_t^2 \;=\; \sum_{\ell=0}^{\infty} \gamma^{\ell} \Big\{ \omega^2 + \Big(a + \sum_{j=1}^{\infty} b_j r_{t-\ell-j} \Big)^2 \Big\}$$

- Hence (1) represents a particular case of Sentana's (1995) Quadratic ARCH with $p = q = \infty$:

$$\sigma_t^2 = \theta + \sum_{i=1}^{p} \psi_i r_{t-i} + \sum_{i=1}^{p} a_{ii} r_{t-i}^2 + 2 \sum_{i=1}^{q} \sum_{j=i+1}^{q} a_{ij} r_{t-i} r_{t-j}$$

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Two particular cases of GQARCH:

- Engle's (1990) Asymmetric GARCH(1,1):

$$\sigma_t^2 = c^2 + (a + br_{t-1})^2 + \gamma\sigma_{t-1}^2$$

(proposed to capture the leverage effect)

- The Linear ARCH (LARCH) (Robinson, 1991):

$$\sigma_t = a + \sum_{j=1}^{\infty} b_j r_{t-j} \qquad (2)$$

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- LARCH (properties): Giraitis, Robinson and Surgailis (2000), Berkes and Horváth (2003), Giraitis, Leipus, Robinson and Surgailis (2004), (estimation): Beran and Schützner (2009), Francq and Zakoian (2010), Truquet (2014)
- The squared stationary solution $\{r_t^2\}$ of the LARCH model with b_j decaying as $j^{d-1}, 0 < d < 1/2$ may have covariance long memory (Giraitis *et al.* (2000))
- For the LARCH model, $ab_j < 0$ implies the leverage effect (Giraitis *et al.* (2004))
- The main drawback of the LARCH model: *volatility σ_t may assume negative values*
- Because of the last fact, QMLE for the LARCH model may be *inconsistent* (Francq and Zakoian, 2010)

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Properties of GQARCH: stationary solution

$r_t = \zeta_t \sigma_t, \ \sigma_t^2 = \omega^2 + (a + \sum_{j=1}^{\infty} b_j r_{t-j})^2 + \gamma \sigma_{t-1}^2 \quad (1)$

Theorem 2

Let $\gamma \geq 0$. Then

$$\gamma + \sum_{j=1}^{\infty} b_j^2 \ < \ 1 \qquad (3)$$

is a necessary and sufficient condition for the existence of a stationary solution of (1) with $\mathrm{E} r_t^2 < \infty$.
In the latter case, this solution $\{r_t\}$ is unique and a martingale difference sequence with
$\mathrm{E}[r_t|\zeta_s, s < t] = 0, \mathrm{E}[r_t^2|\zeta_s, s < t] = \sigma_t^2.$

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Let $|\mu|_p := \mathrm{E}|\zeta_0|^p, \ p \geq 1$

Theorem 3

Let $p = 2, 4, \cdots$ be even, $\gamma > 0$ and

$$\sum_{j=2}^{p} \binom{p}{j} |\mu_j| \sum_{k=1}^{\infty} |b_k|^j < (1-\gamma)^{p/2}. \tag{4}$$

Then the stationary solution of (1) satisfies $\mathrm{E}|r_t|^p < \infty$.

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Properties of GQARCH: long memory

Theorem 4

Let $\{r_t\}$ *be stationary solution of* (1) *with* $\mathrm{E}r_t^4 < \infty$ *and*

$$b_j \sim c\, j^{d-1}, \quad j \to \infty \tag{5}$$

for some $0 < d < 1/2,\ c > 0$. *Then*

$$\mathrm{Cov}(r_0^2, r_t^2) \ \sim\ \kappa^2 t^{2d-1}, \quad t \to \infty\ (\exists\ \kappa > 0).$$

Moreover, normalized partial sums $\sum_{s=1}^{[nt]}(r_s^2 - \mathrm{E}r_s^2)$ *tend to a fractional Brownian motion with Hurst parameter* $H = d + 1/2$.

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Properties of GQARCH: leverage

Definition 5 (Giraitis et al., 2004)

We say that $\{r_t\}$ *has leverage of order* $k \geq 1$ *if*

$$h_j := \mathrm{Cov}(\sigma_j^2, r_0) < 0, \qquad \forall\ 1 \leq j \leq k.$$

Theorem 6

Let $\{r_t\}$ *be a stationary solution of* (1) *with* $\mathrm{E}r_t^4 < \infty$.
Assume in addition that $\sum_{j=1}^{\infty} b_j^2 < (1-\gamma)/5$ *and* $\mathrm{E}\zeta_0^3 = 0$.
Then:

(i) if $ab_j < 0$, $j = 1, \cdots, k$, *then* $\{r_t\}$ *has leverage of order* k;
(ii) if $ab_j > 0$, $j = 1, \cdots, k$, *then* $h_j > 0, j = 1, \cdots, k$.

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Extension of GQARCH:

- Theorems 2 (stationary solution) and 3 (higher moments) can be extended (see Doukhan et al. (2015), Grublytė and Škarnulis (2015)) to a more general model:

$$r_t = \zeta_t \sigma_t, \quad \sigma_t^2 = Q\Big(a + \sum_{j=1}^{\infty} b_j r_{t-j}\Big) + \gamma \sigma_{t-1}^2, \quad (6)$$

  where $\{\zeta_t\}, a, b_j, \gamma$ are as in (1) and $Q(x)$ is a Lipschitz function of real variable $x \in \mathbb{R}$.

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QMLE of 5-parametric long memory GQARCH

Aim: quasi-maximum likelihood estimation (QMLE) of 5-parametric GQARCH model:

$$\sigma_t^2(\theta) \;=\; \sum_{\ell=0}^{\infty} \gamma^\ell \Big\{ \omega^2 + \Big(a + c \sum_{j=1}^{\infty} j^{d-1} r_{t-\ell-j} \Big)^2 \Big\}, \qquad (7)$$

depending on unknown $\theta = (\gamma, \omega, a, d, c) \in \mathbb{R}^5$

- $c \neq 0$ and $d \in (0, 1/2)$: long memory parameters
- $a \neq 0$: asymmetry
- $\omega > 0$: lower volatility 'threshold' ($\sigma_t(\theta) \geq \omega > 0$)

QMLE minimizes the QML function over $\theta \in \Theta_0$:

$$L_n(\theta) \;:=\; \frac{1}{n} \sum_{t=1}^{n} \Big(\frac{r_t^2}{\sigma_t^2(\theta)} + \log \sigma_t^2(\theta) \Big). \qquad (8)$$

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Related work (Beran and Schützner, 2009):

'Modified QMLE' of the 3-parametric LARCH model:

$$\sigma_t(\theta) \;=\; a + c\sum_{j=1}^{\infty} j^{d-1} r_{t-j}, \tag{9}$$

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- The parametric form $b_j = c\, j^{d-1}$ of moving-average coefficients in (7) and (9) are the same
- Because of the degeneracy of σ_t^{-2} in the LARCH case, Beran and Schützner (2009) minimize the *modified* QML

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$$L_{n,\epsilon}(\theta) := \frac{1}{n}\sum_{t=1}^{n}\Big(\frac{r_t^2+\epsilon}{\sigma_t^2(\theta)+\epsilon} + \log(\sigma_t^2(\theta)+\epsilon)\Big),$$

  where $\epsilon > 0$ is small but *fixed*
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- Numeric simulations of Beran and Schützner (2009) show that $L_{n,\epsilon}(\theta)$ exhibits many local minima as $\epsilon \to 0$

- Optimal choice of $\epsilon > 0$ is unknown and seems very difficult

QMLE of 5-parametric GQARCH: assumptions

Assumption (A) $\{\zeta_t\}$ is a standardized i.i.d. sequence with $\mathrm{E}\zeta_t = 0, \mathrm{E}\zeta_t^2 = 1$.

Assumption (B) $\Theta \subset \mathbb{R}^5$ is a compact set of parameters $\theta = (\gamma, \omega, a, d, c)$ defined by:

(i) $\gamma \in [\gamma_1, \gamma_2], \; 0 < \gamma_1 < \gamma_2 < 1$;

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(iii) $a \in [a_1, a_2], \; -\infty < a_1 < a_2 < \infty$;

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(v) $c \in [c_1, c_2]$ with $0 < c_i = c_i(d, \gamma) < \infty$, $c_1 < c_2$ such that

$$\sum_{j=1}^{\infty} b_j^2 = c^2 \sum_{j=1}^{\infty} j^{2(d-1)} < 1 - \gamma$$

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5-parametric GQARCH: QML functions

We assume that the observations $\{r_t, 1 \le t \le n\}$ follow the model in (1) with the true parameter $\theta_0 = (\gamma_0, \omega_0, a_0, d_0, c_0)$ belonging to the interior Θ_0 of Θ in Assumption (B).

Similarly to Beran and Schützner (2009), we discuss two QML estimates: a 'theoretical QMLE' given infinite past $r_s, -\infty \le s < n$, and a 'realistic QMLE' depending only on $r_s, 1 \le s < n$

QMLE given infinite past: The estimator of $\theta \in \Theta$ is defined as

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$\sigma_t(\theta)$ and $Y_t(d)$ depend on infinite past $r_s, -\infty < s < t$

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Following Beran and Schützner (2009) we define 'finite past' QMLE:

$$\widetilde{\theta}_n^{(\beta)} := \arg\min_{\theta\in\Theta} \widetilde{L}_n^{(\beta)}(\theta) = \arg\min_{\theta\in\Theta} \frac{1}{[n^\beta]} \sum_{n-[n^\beta]<t\leq n} \widetilde{\ell}_t(\theta)$$

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QMLE of 5-parametric GQARCH: main results

Everywhere below we assume the stationary 5-parametric GQARCH model $r_t = \zeta_t \sigma_t$ with σ_t in (7) satisfying Assumptions (A) and (B)

Theorem 7 ('Infinite past QMLE')

(i) Let $\mathrm{E}|r_t|^3 < \infty$. *Then* $\widehat{\theta}_n$ *is a strongly consistent estimator of* θ_0, *i.e.*

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where $\Sigma(\theta_0)$ *is the same as in Theorem 7.*

- Thms 7 and 8 are similar to (Beran and Schützner, 2009, Thms 1-4) for the LARCH model and the modified QLME
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Simulation study

Goal: finite-sample accuracy (RootMSE) of QML estimates
$\hat{\theta}_n = (\hat{\gamma}_n, \hat{\omega}_n, \hat{a}_n, \hat{c}_n, \hat{d}_n)$

- Two sample sizes: $n = 1000$ (medium) and $n = 5000$ (large), with $N = 100$ independent replications each
- GQARCH data was generated for $-n \le t \le n$ using the recurrent equation

$$r_t = \zeta_t \sigma_t, \quad \sigma_t^2 = \omega^2 + \Big(a + c \sum_{j=1}^{n \wedge (t+n)} j^{d-1} r_{t-j}\Big)^2 + \gamma \sigma_{t-1}^2, \quad -n \le t \le n$$

  with i.i.d. $\zeta_t \sim N(0,1)$ and zero initial condition $\sigma_{-n-1} = 0$
- The numerical optimization using MATLAB software minimized the QML function:

$$L_n = \frac{1}{n} \sum_{t=1}^{n} \left(\frac{r_t^2}{\sigma_t^2} + \log \sigma_t^2 \right)$$

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Goal: finite-sample accuracy (RootMSE) of QML estimates $\hat{\theta}_n = (\hat{\gamma}_n, \hat{\omega}_n, \hat{a}_n, \hat{c}_n, \hat{d}_n)$

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- Optimization constraints (the set Θ):

$$0.001 \le \gamma \le 0.9, \quad 0 \le \omega \le 2, \quad -2 \le a \le 2, \quad 0 \le d \le 0.5,$$
$$(0.05-\gamma) \vee (\gamma/999) \le c^2 \zeta(2(1-d)) \le (0.99-\gamma) \wedge (99\gamma),$$

where $\zeta(z) = \sum_{j=1}^{\infty} j^{-z}$ is the Riemann zeta function.

- RMSE's reported for fixed $\gamma_0 = 0.7, a_0 = -0.2, c_0 = 0.2$ and several different values of ω_0 and d_0:

$$\omega_0 = 0.1, 0.01, 0.001, \qquad d_0 = 0.1, 0.2, 0.3, 0.4$$

- The above choices of $\theta_0 = (\gamma_0, \omega_0, a_0, c_0, d_0)$ in the numerical experiment can be explained by the observation that the QML estimation of γ_0, a_0, c_0 is more accurate and stable compared to the estimation of ω_0 and d_0

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R(oot)MSE, $\omega_0 = 0.1$

		ω_0=0.1				
n	d_0	$\widehat{\gamma}_n$	$\widehat{\omega}_n$	$\widehat{a}_n$	$\widehat{d}_n$	$\widehat{c}_n$
1000	0.1	0.091	0.057	0.035	0.103	0.035
	0.2	0.083	0.047	0.045	0.109	0.031
	0.3	0.071	0.045	0.047	0.094	0.043
	0.4	0.073	0.029	0.054	0.097	0.036
5000	0.1	0.031	0.021	0.012	0.047	0.015
	0.2	0.030	0.015	0.015	0.041	0.014
	0.3	0.028	0.011	0.025	0.042	0.013
	0.4	0.031	0.014	0.053	0.059	0.018

R(oot)MSE, $\omega_0 = 0.01$

		ω_0=0.01				
n	d_0	$\widehat{\gamma}_n$	$\widehat{\omega}_n$	$\widehat{a}_n$	$\widehat{d}_n$	$\widehat{c}_n$
1000	0.1	0.070	0.049	0.030	0.103	0.029
	0.2	0.061	0.043	0.035	0.089	0.024
	0.3	0.066	0.040	0.045	0.106	0.044
	0.4	0.055	0.042	0.056	0.105	0.038
5000	0.1	0.025	0.032	0.011	0.035	0.013
	0.2	0.022	0.028	0.013	0.032	0.013
	0.3	0.025	0.028	0.025	0.046	0.016
	0.4	0.031	0.031	0.046	0.096	0.034

R(oot)MSE, $\omega_0 = 0.001$

		ω_0=0.001				
n	d_0	$\widehat{\gamma}_n$	$\widehat{\omega}_n$	$\widehat{a}_n$	$\widehat{d}_n$	$\widehat{c}_n$
1000	0.1	0.086	0.058	0.026	0.095	0.037
	0.2	0.056	0.043	0.027	0.084	0.031
	0.3	0.053	0.039	0.046	0.080	0.029
	0.4	0.055	0.047	0.060	0.122	0.041
5000	0.1	0.022	0.033	0.009	0.031	0.012
	0.2	0.020	0.030	0.012	0.028	0.012
	0.3	0.022	0.032	0.024	0.038	0.014
	0.4	0.032	0.037	0.046	0.098	0.031

Observations from simulations

• All RMSEs decrease as n increases. The convergence rate of estimates seems quite good overall.

• Parameter γ_0 is estimated rather accurately. E.g., for $n = 5000$ RMSE($\widehat{\gamma}_n$) is very stable for all values of ω_0 and d_0.

• The previous conclusion generally applies also to the QML estimates $\widehat{a}_n, \widehat{c}_n$ and $\widehat{d}_n$ except that their RMSE markedly increases when $d_0 = 0.4$.

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Parameter estimates from real data

For estimation we used daily stock returns from slightly different time window. Three examples:

Airbus Group SE	Estimates				
	$\hat{\gamma}$	$\hat{\omega}$	$\hat{a}$	$\hat{d}$	$\hat{c}$
2004.01.01-2006.12.29	0,172	0,012	-0,009	0,251	0,496
2003.10.01-2006.12.29	0,168	0,013	-0,013	0,320	0,464
2004.01.01-2007.03.30	0,163	0,013	-0,010	0,268	0,472

Nordea Bank AB	Estimates				
	$\hat{\gamma}$	$\hat{\omega}$	$\hat{a}$	$\hat{d}$	$\hat{c}$
2004.01.01-2006.12.29	0,7314	0.0048	-0.0044	0.1313	0.2563
2003.10.01-2006.12.29	0.6466	0.0058	-0.0073	0.3112	0.2800
2004.01.01-2007.03.30	0.6203	0.0061	-0.0051	0.1543	0.2751

Ford Motor Co	Estimates				
	$\hat{\gamma}$	$\hat{\omega}$	$\hat{a}$	$\hat{d}$	$\hat{c}$
2004.01.01-2006.12.29	0,7856	0,0069	0,0023	0,2591	0,2117
2003.10.01-2006.12.29	0,6053	0,0100	0,0015	0,1424	0,2740
2004.01.01-2007.03.30	0,8049	0,0066	0,0023	0,3124	0,1880

Some proofs. Notation

- $L(\theta) := \mathrm{E}L_n(\theta) = \mathrm{E}\ell_t(\theta)$
- $A(\theta) := \mathrm{E}\left[\nabla^T \ell_t(\theta)\nabla\ell_t(\theta)\right]$, $B(\theta) \ := \ \mathrm{E}\left[\nabla^T \nabla\ell_t(\theta)\right]$
- $\nabla = (\partial/\partial\theta_1, \cdots, \partial/\partial\theta_5)$

Some proofs

Lemma 9

The function $L(\theta), \theta \in \Theta$ is bounded and continuous. Moreover, it attains its unique minimum at $\theta = \theta_0$.

- $L(\theta) - L(\theta_0) = \mathrm{E}\left[\frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)} - \log \frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)} - 1\right]$.
- the function $f(x) := x - 1 - \log x > 0$ for $x > 0, x \neq 1$ and $f(x) = 0$ if and only if $x = 1$
- therefore $L(\theta) \geq L(\theta_0), \forall\, \theta \in \Theta$ while $L(\theta) = L(\theta_0)$ is equivalent to

$$\sigma_t^2(\theta) = \sigma_t^2(\theta_0) \qquad (\mathrm{P}_{\theta_0} - \text{a.s.}) \tag{11}$$

- it remains to show that (11) implies $\theta = \theta_0 = (\gamma_0, \omega_0, a_0, d_0, c_0)$.

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Some proofs

- using the 'projection'

$$P_s\xi = \mathrm{E}[\xi|\mathcal{F}_s] - \mathrm{E}[\xi|\mathcal{F}_{s-1}]$$

of r.v. $\xi, \mathrm{E}|\xi| < \infty$, where $\mathcal{F}_s = \sigma(\zeta_u, u \le s)$,

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$$P_s\sigma_t^2(\theta) = P_s\sigma_t^2(\theta_0) \qquad (\mathrm{P}_{\theta_0} - \text{a.s.})$$

- with $s = t - 1$ we obtain

$$C_1(\theta,\theta_0)\zeta_{t-1}^2 + 2C_2(\theta,\theta_0)\zeta_{t-1} - C_1(\theta,\theta_0) = 0.$$

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$$\begin{aligned} C_1(\theta,\theta_0) &:= (c^2 - c_0^2)\sigma_{t-1}(\theta_0), \\ C_2(\theta,\theta_0) &:= (ac - a_0c_0) + \sum_{u<t-1} (c^2(t-u)^{d-1} - c_0^2(t-u)^{d_0-1})r_u. \end{aligned}$$

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Lemma 10

(i) Let $\mathrm{E}|r_t|^3 < \infty$. *Then*

$$\sup_{\theta\in\Theta} |L_n(\theta) - L(\theta)| \overset{a.s.}{\to} 0 \qquad \textit{and} \qquad \mathrm{E}\sup_{\theta\in\Theta} |L_n(\theta) - \widetilde{L}_n(\theta)| \to 0.$$

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Some proofs

For multi-index

$$\begin{aligned} \boldsymbol{i} &= (i_1, \cdots, i_5) \in \mathbb{N}^5,\ \boldsymbol{i} \neq \boldsymbol{0} = (0, \cdots, 0), \\ |\boldsymbol{i}| &:= i_1 + \cdots + i_5, \end{aligned}$$

denote partial derivative $\partial^{\boldsymbol{i}} := \partial^{|\boldsymbol{i}|} / \prod_{j=1}^{5} \partial^{i_j}\theta_j$.

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Let $\mathrm{E}|r_t|^{2+p} < \infty$*, for some integer* $p \geq 1$*. Then for any* $\boldsymbol{i} \in \mathbb{N}^5$*,* $0 < |\boldsymbol{i}| \leq p$*,*

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Moreover, if $\mathrm{E}|r_t|^{2+p+\epsilon} < \infty$ *for some* $\epsilon > 0$ *and* $p \in \mathbb{N}$ *then for any* $\boldsymbol{i} \in \mathbb{N}^5$, $0 \leq |\boldsymbol{i}| \leq p$

$$\mathrm{E} \sup_{\theta \in \Theta} |\partial^{\boldsymbol{i}} (\ell_t(\theta) - \widetilde{\ell}_t(\theta))| \to 0, \qquad t \to \infty.$$

Some proofs

- using Faà di Bruno differentiation rule, Holders inequality and $\mathrm{E}|r_t|^{2+p} \leq C$, the statement of the lemma follows from

$$\mathrm{E} \sup_{\theta \in \Theta} \left(|\partial^{\boldsymbol{j}} \sigma_t^2(\theta)| / \sigma_t(\theta) \right)^{(2+p)/|\boldsymbol{j}|} \quad < \quad \infty$$

for any multi-index $\boldsymbol{j} \in \mathbb{N}^5$, $1 \leq |\boldsymbol{j}| \leq p$.

- there exist $C > 0, 0 < \bar{\gamma} < 1$ such that

$$\sup_{\theta \in \Theta} \left| \frac{\partial_i \sigma_t^2(\theta)}{\sigma_t(\theta)} \right| \leq C(1 + J_{t,0} + J_{t,1}), \quad i = 1, \cdots, 5, \quad \text{where}$$

$$J_{t,0} := \sum_{\ell=0}^{\infty} \bar{\gamma}^{\ell} \sup_{d \in [d_1, d_2]} |Y_{t-\ell}(d)|, \quad J_{t,1} := \sum_{\ell=0}^{\infty} \bar{\gamma}^{\ell} \sup_{d \in [d_1, d_2]} |\partial_d Y_{t-\ell}(d)|.$$

- $\mathrm{E} \sup_{d \in [d_1, d_2]} |Y_{t-\ell}(d)|^m$ is bounded by a linear combination involving $\sup_{d \in [d_1, d_2]} \mathrm{E}|Y_{t-\ell}(d)|^m$ and $\sup_{d \in [d_1, d_2]} \mathrm{E}|\partial_d Y_{t-\ell}(d)|^m$ (we use Lemma from Beran and Schützner (2009)).

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Some proofs

Lemma 12

Let $\mathrm{E}r_0^4 < \infty$*. Then matrices* $A(\theta)$ *and* $B(\theta)$ *are well-defined and strictly positive definite for any* $\theta \in \Theta$*.*

- $\nabla \sigma_t^2(\theta)\lambda^T = 0$ for some $\theta \in \Theta$ and $\lambda \in \mathbb{R}^5, \lambda \neq 0$ leads to a contradiction
- use projections to obtain

$$D_1(\lambda)\zeta_{t-1}^2 + 2D_2(\lambda)\zeta_{t-1} - D_1(\lambda) = 0 \tag{12}$$

- $D_1(\lambda) := 2\lambda_5\sigma_{t-1}(\theta)$
- $D_2(\lambda) := \lambda_3 c + \lambda_5 a + 2\lambda_5 c \sum_{u<t-1}(t-u)^{d-1} r_u + \lambda_4 c^2 \sum_{u<t-1}(t-u)^{d-2}\log(t-u) r_u$

Some proofs

Lemma 12

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References

Beran, J., Schützner, M., 2009. On approximate pseudo-maximum likelihood estimation for LARCH-processes. Bernoulli, 15, 1057–1081.

Berkes, I., Horvath, L., 2003. Asymptotic results for long memory LARCH sequences. Ann. Appl. Probab., 13, 641-668.

Doukhan, P., Grublytė, I., Surgailis, D., 2014. A nonlinear model for long memory conditional heteroscedasticity. Preprint. Available at arXiv:1502.00095 [math.ST]

Engle, R.F., 1990. Stock volatility and the crash of '87. Discussion. Rev. Financial Studies 3, 103–106.

Francq, C., Zakoian, J.-M., 2010. Inconsistency of the MLE and inference based on weighted LS for LARCH models. J. Econometrics 159, 151–165.

Giraitis, L., Leipus, R., Robinson, P.M., Surgailis, D., 2004. LARCH, leverage and long memory. J. Financial Econometrics, 2, 177-210.

Giraitis, L., Robinson, P.M., Surgailis, D., 2000. A model for long memory conditional heteroskedasticity. Ann. Appl. Probab., 10, 1002–1024.

Grublytė, I., Škarnulis, A., 2015. A generalized nonlinear model for long memory conditional heteroscedasticity. Preprint.

Robinson, P.M., 1991. Testing for strong serial correlation and dynamic conditional heteroskedasticity in multiple regression. J. Econometrics, 47, 67–84.

Sentana, E., 1995. Quadratic ARCH models. Rev. Econom. Stud. 3, 77–102.

Truquet, L., 2014. On a family of contrasts for parametric inference in degenerate ARCH models. Econometric Th., 30, 1165–1206.

Thank you!