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Semi-parametric dynamic factor models for non-stationary time series

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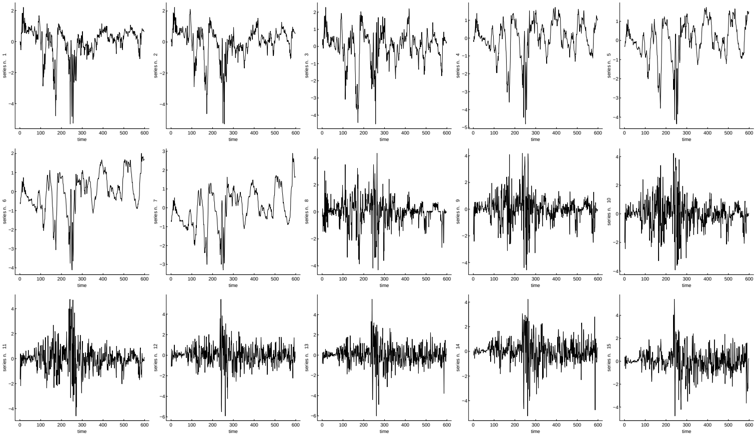
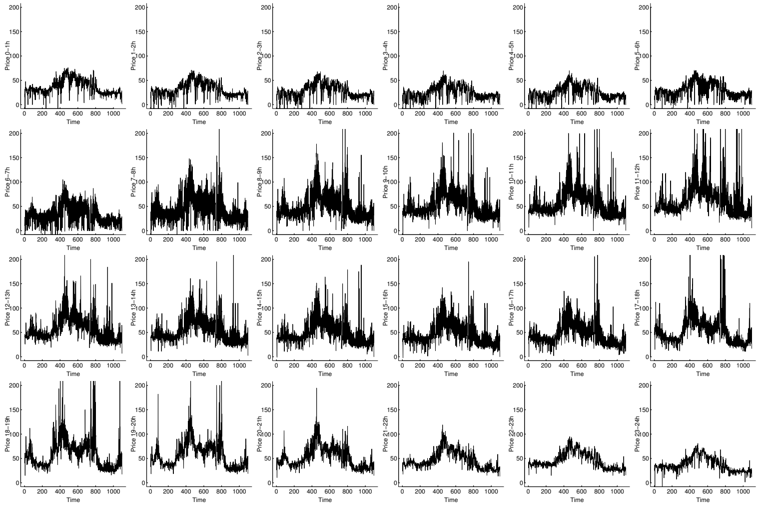
Motivation

Interest rates and spreads



Motivation

Hourly electricity spot prices from EEX



Motivation

Factor structure:

Often a small number q of latent factors $\boldsymbol{f}(t)$ is sufficient to explain the common behaviour of a large panel of N time series $\boldsymbol{Y}_N(t)$:

$$\begin{aligned}\boldsymbol{Y}_N(t) &= \boldsymbol{X}_N(t) + \boldsymbol{e}_N(t) \\ &= \boldsymbol{\Lambda}\boldsymbol{f}(t) + \boldsymbol{e}_N(t), \qquad t = 1, \ldots, T\end{aligned}$$

where

- $\boldsymbol{Y}_N(t) = (Y_1(t), \ldots, Y_N(t))'$
- $\boldsymbol{f}(t) = (f_1(t), \ldots, f_q(t))'$ common factors
- $\boldsymbol{Z}_N(t) = (Z_1(t), \ldots, Z_N(t))'$ idiosyncratic components

Advantages:

- $\boldsymbol{X}_N(t)$ contains all relevant joint information;
- $\boldsymbol{Z}_N(t)$ explains measurement errors/sectoral specific dynamics, usually allowed to be mildly serially and cross-correlated.

Motivation

Non-stationarity:

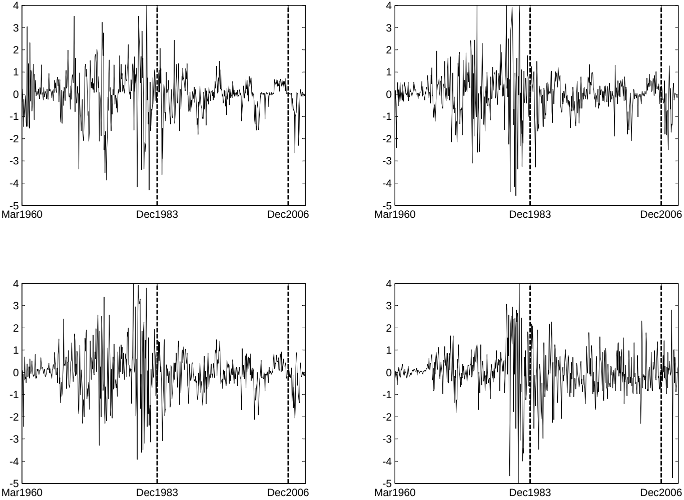
The data exhibit some time variation in their serial variance-covariance structure.

Example: industrial production data

- There is evidence of regime shifts, e.g. in the early 1980s we observe a decrease in variance of the majority of macroeconomic indicators (the Great Moderation); the introduction of the Euro in 1999; the recent financial crisis.
- It is difficult to detect the exact point in time of change in regime.

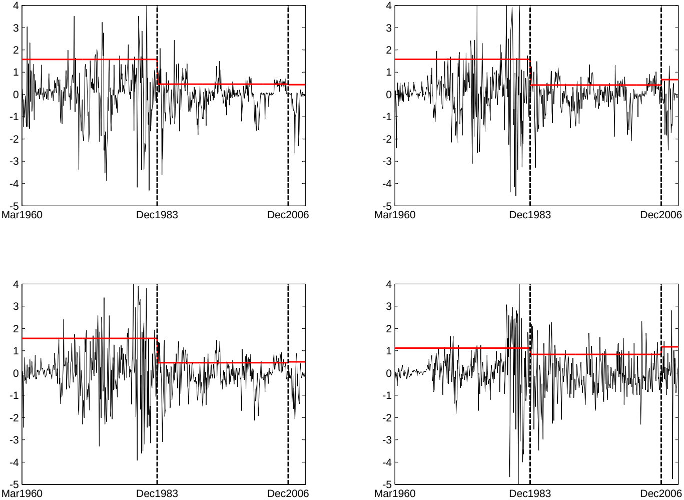
We opt for a slowly changing dynamics, e.g. the covariance matrix is a smooth function of time.

Motivation
Interest rates

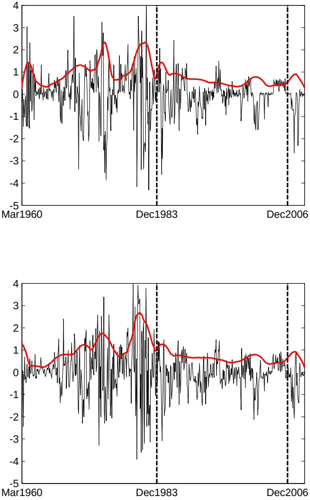


Motivation

Interest rates with structural breaks



Motivation
Interest rates with smooth volatilities



Outline

- Motivation
- Stationary and non-stationary factor models
- A semi-parametric dynamic factor model
- Estimation
- Simulations and applications
- Conclusions/Work to be done

Stationary factor models

Static case (Chamberlain and Rotschild 1983; Bai 2003):

$$\boldsymbol{Y}(t) = \boldsymbol{\Lambda}\boldsymbol{u}(t) + \boldsymbol{Z}(t), \quad t = 1, \ldots, T$$

with $\boldsymbol{u}(t)$ white noise. It is too simple.

Dynamic case (Forni, Hallin, Lippi and Reichlin 2000):

$$\boldsymbol{Y}(t) = \boldsymbol{\Psi}(B)\boldsymbol{u}(t) + \boldsymbol{Z}(t), \quad t = 1, \ldots, T$$

with $\boldsymbol{u}(t)$ white noise. It delivers two-sided filters.

Steps of estimation in static case:

- estimate covariance matrix

$$\hat{\boldsymbol{\Sigma}}_N$$

- obtain eigenvectors:

$$\hat{\mathbf{P}}_N = (\hat{\mathbf{P}}_{1,N}, \ldots, \hat{\mathbf{P}}_{N,N})$$

- obtain projection filter:

$$\hat{\boldsymbol{\Phi}}_N = \hat{\mathbf{P}}_N \, \mathrm{Q}_q \, \hat{\mathbf{P}}_N{}^*$$

- apply filter:

$$\hat{\boldsymbol{X}}_N(t) = \hat{\boldsymbol{\Phi}}_N \boldsymbol{Y}_N(t)$$

Stationary factor models

Estimation

Steps of estimation in dynamic case:

- estimate spectral density matrix (nonparametrically):

$$\hat{\boldsymbol{\Sigma}}_N(\omega)$$

- obtain dynamic eigenvectors:

$$\hat{\mathbf{P}}_N(\omega) = (\hat{\mathbf{P}}_{1,N}(\omega), \ldots, \hat{\mathbf{P}}_{N,N}(\omega))$$

- obtain projection filter:

$$\hat{\boldsymbol{\Phi}}_N(\omega) = \hat{\mathbf{P}}_N(\omega)\,\mathrm{Q}_q\,\hat{\mathbf{P}}_N(\omega)^*$$

- apply filter:

$$\hat{\boldsymbol{X}}_N(t) = \hat{\boldsymbol{\Phi}}_N(B)\boldsymbol{Y}_N(t)$$

Non-stationary factor models

Static case (Motta, Hafner and von Sachs 2011):

$$\boldsymbol{Y}(t) = \boldsymbol{\Lambda}(t)\,\boldsymbol{u}(t) + \boldsymbol{Z}(t), \qquad t = 1, \ldots, T$$

with $\boldsymbol{u}(t)$ white noise.

Dynamic case (Eichler, Motta and von Sachs 2011):

$$\boldsymbol{Y}(t) = \boldsymbol{\Psi}(t, B)\,\boldsymbol{u}(t) + \boldsymbol{Z}(t), \qquad t = 1, \ldots, T$$

with $\boldsymbol{u}(t)$ white noise. It is hard to estimate.

Non-stationary factor models
Estimation

Steps of estimation in evolutionary static case:

- estimate time-varying covariance matrix

$$\hat{\boldsymbol{\Sigma}}_N(u), \qquad u \in [0,1]$$

- obtain eigenvectors:

$$\hat{\mathbf{P}}_N(u) = \left(\hat{\mathbf{P}}_{1,N}(u), \ldots, \hat{\mathbf{P}}_{N,N}(u)\right)$$

- obtain projection filter:

$$\hat{\boldsymbol{\Phi}}_N(u) = \hat{\mathbf{P}}_N(u)\, \mathrm{Q}_q\, \hat{\mathbf{P}}_N(u)^*$$

- apply filter:

$$\hat{\boldsymbol{X}}_{NT}(t) = \hat{\boldsymbol{\Phi}}_N\left(\tfrac{t}{T}\right) \boldsymbol{Y}_{NT}(t)$$

Non-stationary factor models
Estimation

Steps of estimation in evolutionary dynamic case:

- estimate time-varying spectral density matrix (nonparametrically):

$$\hat{\boldsymbol{\Sigma}}_N(u,\omega), \qquad u \in [0,1]$$

- obtain dynamic eigenvectors:

$$\hat{\mathbf{P}}_N(u,\omega) = \left(\hat{\mathbf{P}}_{1,N}(u,\omega), \ldots, \hat{\mathbf{P}}_{N,N}(u,\omega)\right)$$

- obtain projection filter:

$$\hat{\boldsymbol{\Phi}}_N(u,\omega) = \hat{\mathbf{P}}_N(u,\omega)\, \mathrm{Q}_q\, \hat{\mathbf{P}}_N(u,\omega)^*$$

- apply filter:

$$\hat{\boldsymbol{X}}_{NT}(t) = \hat{\boldsymbol{\Phi}}_N\left(\tfrac{t}{T}, B\right) \boldsymbol{Y}_{NT}(t)$$

Non-stationary factor models

Stationary factor models:

- Global parametrization, with parameters fixed over time.
  - principal components (Bai & Ng, Stock & Watson, Forni et al.)
  - fully parametric model (ML) (Doz et al. 2008)

Non-stationary factor models:

- Global parametrization, with hyper-parameters fixed over time.
- Localization, with stationary models fitted locally at every point.
  - evolutionary-static principal components (Motta et al. 2011)
  - evolutionary-dynamic principal components (Eichler et al. 2011)
  - evolutionary-static principal components with dynamic factors (Motta et al. 2012, Barigozzi & Motta 2012)

Non-stationary factor models

Stationary factor models:

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Non-stationary factor models:

- Global parametrization, with hyper-parameters fixed over time.
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  - evolutionary-dynamic principal components (Eichler et al. 2011)
  - evolutionary-static principal components with dynamic factors (Motta et al. 2012, Barigozzi & Motta 2012)
- ***Semi-parametric approach:*** only some parameters are time-varying
  - loadings are constant over time, estimated parametrically
  - low-dimensional time-varying parameters, estimated locally

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Semi-parametric approach

Consider non-stationary generalized dynamic factor model:

$$\boldsymbol{Y}_N(t) = \boldsymbol{\Psi}_N(t,B)\,\boldsymbol{u}(t) + \boldsymbol{e}_N(t)$$

Assumptions:

- $\boldsymbol{\Psi}_N(B,t) = \boldsymbol{\Lambda}_N\,\mathbf{G}(B,t)$
- $\mathbf{G}(B,t) = (\mathbf{I} - \mathbf{A}(t)\,B)^{-1}\,\mathbf{V}(t)$
- $\text{var}(\boldsymbol{e}_N(t)) = \boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \ldots, \sigma_N^2)$
- $\text{var}(\boldsymbol{u}(t)) = \mathbf{I}$

Semi-parametric dynamic factor model:

$$\boldsymbol{Y}_N(t) = \boldsymbol{\Lambda}_N\,\boldsymbol{X}(t) + \boldsymbol{Z}_N(t), \qquad \boldsymbol{Z}_N(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \boldsymbol{\Sigma})$$

$$\boldsymbol{f}(t) = \mathbf{A}(t)\,\boldsymbol{f}(t-1) + \mathbf{V}(t)\,\boldsymbol{u}(t), \qquad \boldsymbol{u}(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \mathbf{I})$$

Semi-parametric approach

Semi-parametric dynamic factor model:

$$\boldsymbol{Y}_N(t) = \boldsymbol{\Lambda}_N \boldsymbol{X}(t) + \boldsymbol{Z}_N(t), \qquad \boldsymbol{Z}_N(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \boldsymbol{\Sigma})$$

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Advantages:

- exploit full length to estimate the (fixed) $N \times q$ loadings $\boldsymbol{\Lambda}$;
- estimate locally the (evolutionary) $q \times q$ coefficients $\mathbf{A}(t)$.

Semi-parametric approach

Semi-parametric dynamic factor model:

$$\mathbf{Y}_N(t) = \mathbf{\Lambda}_N \mathbf{X}(t) + \mathbf{Z}_N(t), \qquad \mathbf{Z}_N(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \mathbf{\Sigma})$$

$$\mathbf{f}(t) = \mathbf{A}(t)\mathbf{f}(t-1) + \mathbf{V}(t)\mathbf{u}(t), \qquad \mathbf{u}(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \mathbf{I})$$

Advantages:

- exploit full length to estimate the (fixed) $N \times q$ loadings $\mathbf{\Lambda}$;
- estimate locally the (evolutionary) $q \times q$ coefficients $\mathbf{A}(t)$.

Two situations:

- ***Exact factor model:***
  N finite, $\text{var}(\mathbf{Z}_N)$ is diagonal
- ***Approximate factor model:***
  $N \to \infty$
  largest eigenvalue of $\text{var}(\mathbf{Z}_N)$ is uniformly bounded in $N \in \mathbb{N}$

Two-step estimation

Step 1: Estimation of the loadings and the latent factors

Let

- $\bar{\boldsymbol{\Gamma}}^f = \int_0^1 \boldsymbol{\Gamma}^f(u,0)\,du$
- $\bar{\boldsymbol{\Gamma}}_N = \int_0^1 \boldsymbol{\Gamma}(u,0)\,du = \boldsymbol{\Lambda}_N \bar{\boldsymbol{\Gamma}}^f \boldsymbol{\Lambda}_N' + \boldsymbol{\Gamma}_N^Z.$

For fixed N and large T:

$$\sqrt{T}\left\|\tfrac{1}{N}\left(\widehat{\boldsymbol{\Gamma}}_{NT} - \bar{\boldsymbol{\Gamma}}_N\right)\right\| = O_p(1), \qquad \text{where } \widehat{\boldsymbol{\Gamma}}_{NT} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{Y}_N(t)\boldsymbol{Y}_N(t)',$$

For large N:

$$\sqrt{N}\left\|\tfrac{1}{N}\left(\bar{\boldsymbol{\Gamma}}_N - \boldsymbol{\Lambda}_N \bar{\boldsymbol{\Gamma}}^f \boldsymbol{\Lambda}_N'\right)\right\| = O(1), \qquad \text{since } \left\|\boldsymbol{\Gamma}_N^Z\right\| \le |\boldsymbol{\Gamma}_N^Z| \le \sqrt{N} e_1^Z.$$

Error decomposition:

$$\tfrac{1}{N}\left[\widehat{\boldsymbol{\Gamma}}_{NT} - \boldsymbol{\Lambda}_N \bar{\boldsymbol{\Gamma}}^f \boldsymbol{\Lambda}_N'\right] = \underbrace{\tfrac{1}{N}\left[\bar{\boldsymbol{\Gamma}}_N - \boldsymbol{\Lambda}_N \bar{\boldsymbol{\Gamma}}^f \boldsymbol{\Lambda}_N'\right]}_{O\left(\frac{1}{\sqrt{N}}\right)} + \underbrace{\tfrac{1}{N}\left[\widehat{\boldsymbol{\Gamma}}_{NT} - \bar{\boldsymbol{\Gamma}}_N\right]}_{O_p\left(\frac{1}{\sqrt{T}}\right)}$$

Two-step estimation

Step 1: Estimation of the loadings and the latent factors

Let

- $\mathbf{\Gamma}_N^{\Lambda} := \frac{\mathbf{\Lambda}_N' \mathbf{\Lambda}_N}{N}$
- $\mathbf{\Gamma}_N^{\Lambda} = \text{diag}\{\gamma_{N,1}^{\Lambda}, \dots, \gamma_{N,q}^{\Lambda}\}$.

Then:

- eigenvalue are time-varying eigenvalues
- eigenvectors are ***time-invariant***

  $\mathbf{\Gamma}_N^X(u) = \mathbf{P}_N \mathbf{D}_N(u) \mathbf{P}_N', \qquad \text{for all } N \geq q \qquad \text{and all } u \in (0,1),$

where $\mathbf{P}_N = \pm \mathbf{\Lambda}_N \left(\mathbf{\Gamma}_N^{\Lambda}\right)^{-\frac{1}{2}}$, and $\mathbf{D}_N(u) = \mathbf{\Gamma}_N^{\Lambda} \mathbf{\Gamma}^f(u)$.

Two-step estimation
Step 1: Estimation of the loadings and the latent factors

Define

$$\mathbf{\Gamma}^{\Lambda} := \lim_{N\to\infty} \frac{\mathbf{\Lambda}_N' \mathbf{\Lambda}_N}{N}, \qquad \ell_N := \left\| \frac{\mathbf{\Lambda}_N{}' \mathbf{\Lambda}_N}{N} - \mathbf{\Gamma}^{\Lambda} \right\|, \qquad \mathbf{R} := [\mathbf{\Gamma}^{\Lambda}]^{-\frac{1}{2}}$$

$$\widehat{\mathbf{D}}_{NT} := \widehat{\mathbf{P}}_{NT}' \widehat{\mathbf{\Gamma}}_{NT} \widehat{\mathbf{P}}_{NT}, \qquad \widehat{\mathbf{\Lambda}}_{NT} := \sqrt{N}\, \widehat{\mathbf{P}}_{NT}, \qquad \widehat{\mathbf{F}}_{NT} := \frac{1}{N} \mathbf{Y} \widehat{\mathbf{\Lambda}}_{NT}$$

Result:

Assume $\ell_N \to 0$ as $N \to \infty$, and $\mathbf{\Gamma}^{\Lambda} = \mathrm{diag}\{\gamma_1^{\Lambda}, \ldots, \gamma_q^{\Lambda}\}$.

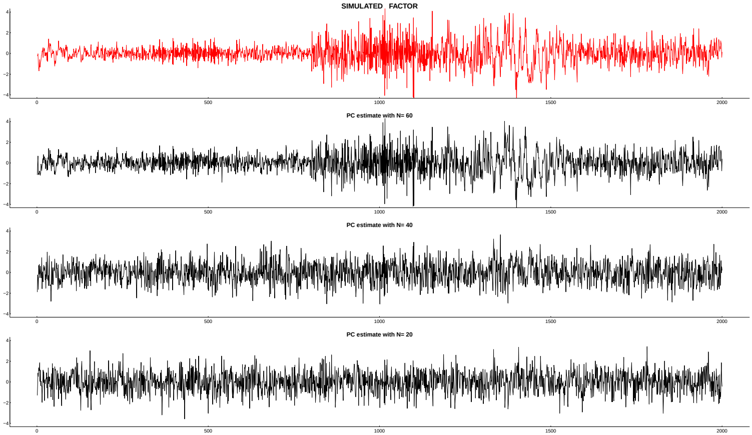
Then as $T \to \infty$ and $N \to \infty$

$$\min(\sqrt{T}, \sqrt{N}, \ell_N) \left\| \frac{1}{\sqrt{N}} \left(\widehat{\mathbf{\Lambda}}_{NT} - \mathbf{\Lambda}_N \mathbf{R} \right) \right\| = O_p(1);$$

$$\min(\sqrt{T}, \sqrt{N}, \ell_N) \left\| \widehat{\boldsymbol{f}}_{NT}(t) - \mathbf{R}^{-1} \boldsymbol{f}(t) \right\| = O_p(1).$$

Two-step estimation
Step 1: Estimation of the loadings and the latent factors

Latent factor ($q = 1$) estimated by the first PC



Pre-covariance

Let $\{X(t),\, 1 \le t \le T\}$, be the observed time series and define

$$\mathrm{g}^x(u,k) = X\Big(\lfloor uT - \tfrac{k}{2} \rfloor\Big) X\Big(\lfloor uT + \tfrac{k}{2} \rfloor\Big),$$

where $\lfloor y \rfloor$ is the largest integer less than or equal to y.

The pre-covariance $\mathrm{g}^x(u,k)$ is such that, for all $k \in \mathbb{Z}$,

- $\mathrm{g}^x(u,k) = \mathrm{g}^x(u,-k)$ for a fixed $u \in (0,1)$,
- $\frac{1}{T} \sum_{t=1+\lfloor \frac{k+1}{2} \rfloor}^{T-\lfloor \frac{k}{2} \rfloor} \mathrm{g}^x(\tfrac{t}{T},k) = \widehat{\gamma}^x(k)$,

where $\widehat{\gamma}^x(k)$ is the sample auto-covariance

$$\widehat{\gamma}^x(k) = \sum_{t=1}^{T-k} X(t)X(t+k) = \sum_{t=k+1}^{T} X(t-k)X(t).$$

Two-step estimation

Step 2: Local polynomials for the latent factors

Notice that

- $\lfloor uT + \frac{k}{2} \rfloor - \lfloor uT - \frac{k}{2} \rfloor = k$ for all $u \in (0,1)$ and all $k \in \mathbb{N}$.
- for all $u \in [\frac{t}{T}, \frac{t+1/2}{T}[$ and for all k

$$g^x(u,k) = X\left(t - \lfloor \tfrac{k+1}{2} \rfloor\right) X\left(t + \lfloor \tfrac{k}{2} \rfloor\right) = X\left(t - \lceil \tfrac{k}{2} \rceil\right) X\left(t + \lceil \tfrac{k-1}{2} \rceil\right)$$

Two-step estimation

Step 2: Local polynomials for the latent factors

Notice that $\mathrm{g}^x(u,k) = \mathrm{g}^x(u,-k)$ for all $k \in \mathbb{Z}$, but we need to fix u:

$$\mathrm{g}^x(u,0) = X(t)^2 \qquad u \in [\tfrac{t-1/2}{T}, \tfrac{t+1/2}{T}[$$

$$\mathrm{g}^x(u,1) = X(t-1)X(t) \qquad u \in [\tfrac{t-1/2}{T}, \tfrac{t+1/2}{T}[$$

$$\mathrm{g}^x(u,2) = \begin{cases} X(t-2)X(t) & u \in [\tfrac{t-1/2}{T}, \tfrac{t}{T}[\\ X(t-1)X(t+1) & u \in [\tfrac{t}{T}, \tfrac{t+1/2}{T}[\end{cases}$$

$$\mathrm{g}^x(u,3) = X(t-2)X(t+1) \qquad u \in [\tfrac{t-1/2}{T}, \tfrac{t+1/2}{T}[$$

$$\mathrm{g}^x(u,4) = \begin{cases} X(t-3)X(t+1) & u \in [\tfrac{t-1/2}{T}, \tfrac{t}{T}[\\ X(t-2)X(t+2) & u \in [\tfrac{t}{T}, \tfrac{t+1/2}{T}[\end{cases}$$

Two-step estimation

Step 2: Local polynomials for the latent factors

Local auto-covariance

For $j = 1, \ldots, q$,

$$\begin{aligned} c_j^f(u,k) &= \mathbb{E}\left[g_j^f(u,k)\right] = \mathbb{E}\left[f_j\left(\lfloor uT + \tfrac{k}{2}\rfloor\right) f_j\left(\lfloor uT - \tfrac{k}{2}\rfloor\right)\right] \\ &= \int_{-\pi}^{\pi} \exp(\mathrm{i}\omega k) S_{j,T}^0\left(\lfloor uT + \tfrac{k}{2}\rfloor, \omega\right) S_{j,T}^0\left(\lfloor uT - \tfrac{k}{2}\rfloor, -\omega\right) d\omega \\ &= \gamma_j^f(u,k) + O\left(\tfrac{1}{T}\right) \end{aligned}$$

Local auto-covariance

For $j = 1, \ldots, q$,

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where

$$\gamma_j^f(u,k) = \int_{-\pi}^{\pi} \sigma_j^f(u,\omega) \exp(\mathrm{i}\omega k) d\omega$$

Two-step estimation

Step 2: Local polynomials for the latent factors

Local auto-covariance

For $j = 1, \dots, q$,

$$
\begin{aligned}
c_j^f(u,k) &= \mathbb{E}\left[g_j^f(u,k)\right] = \mathbb{E}\left[f_j\left(\lfloor uT + \tfrac{k}{2}\rfloor\right) f_j\left(\lfloor uT - \tfrac{k}{2}\rfloor\right)\right] \\
&= \int_{-\pi}^{\pi} \exp(\mathrm{i}\omega k) S_{j,T}^0\left(\lfloor uT + \tfrac{k}{2}\rfloor, \omega\right) S_{j,T}^0\left(\lfloor uT - \tfrac{k}{2}\rfloor, -\omega\right) d\omega \\
&= \gamma_j^f(u,k) + O(\tfrac{1}{T})
\end{aligned}
$$

where

$$
\gamma_j^f(u,k) = \int_{-\pi}^{\pi} \sigma_j^f(u,\omega) \exp(\mathrm{i}\omega k) d\omega
$$

Localized Estimator of the Auto-Covariance

$$
\tilde{\gamma}_j^f(u,k;b) = \frac{1}{T} \sum_{t=1+\lfloor \frac{k+1}{2} \rfloor}^{T-\lfloor \frac{k}{2} \rfloor} W(u,t;b)\, g^x(\tfrac{t}{T}, k)
$$

Two-step estimation

Step 2: Local polynomials for the latent factors

Main idea: approximate $\gamma_j(t)$ locally about t by a polynomial

$$\gamma_j(s) \approx \sum_{k=0}^{d} (s-t)^k \, \breve{\gamma}_j^{(k)}(t),$$

and minimize the kernel-weighted local-loss function

$$\sum_{s=1}^{T} \Big[f_j(s-1) f_j(s) - \sum_{k=0}^{d} (s-t)^k \, \breve{\gamma}_j^{(k)}(t) \Big]^2 K_b(s-t) \tag{1}$$

with respect to $[\breve{\gamma}_j^{(0)}(t), \ldots, \breve{\gamma}_j^{(k)}(t), \ldots, \breve{\gamma}_j^{(d)}(t)]$, where $\breve{\gamma}_j^{(k)}(t) = \frac{\gamma_j^{(k)}(t)}{k!}$.

Two-step estimation
Step 2: Local polynomials for the latent factors

Example: Locally Stationary AR(p)

$$\widetilde{\boldsymbol{\Gamma}}_j^f(u;b) = \begin{bmatrix} \widetilde{\gamma}_j^f(u,0) & \widetilde{\gamma}_j^f(u,1) & \dots & \widetilde{\gamma}_j^f(u,p-1) \\ \widetilde{\gamma}_j^f(u,1) & \widetilde{\gamma}_j^f(u,0) & \dots & \widetilde{\gamma}_j^f(u,p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\gamma}_j^f(u,p-1) & \widetilde{\gamma}_j^f(u,p-2) & \dots & \widetilde{\gamma}_j^f(u,0) \end{bmatrix}, \qquad \widetilde{\boldsymbol{\gamma}}_j^f(u;b) = \begin{bmatrix} \widetilde{\gamma}_j^f(u,1) \\ \widetilde{\gamma}_j^f(u,2) \\ \vdots \\ \widetilde{\gamma}_j^f(u,p) \end{bmatrix}$$

Localized Estimator of the AR Coefficients:

$$\underset{p\times 1}{\widetilde{\boldsymbol{\alpha}}_j(u;b)} = [\underset{p\times p}{\widetilde{\boldsymbol{\Gamma}}_j^f(u;b)}]^{-1} \underset{p\times 1}{\widetilde{\boldsymbol{\gamma}}_j^f(u;b)}, \qquad j = 1,\dots,q$$

Result:

Let $\nu = \int K(x)^2 dx$ and set $d = 1$.

$$\sqrt{\frac{Tb}{\nu}}[\widetilde{\boldsymbol{\alpha}}_j(u;b) - \boldsymbol{\alpha}_j(u)] \sim \mathcal{N}(\mathbf{0}, [\boldsymbol{\Gamma}_j(u)]^{-1}).$$

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Local ML estimation

Semiparametric dynamic factor model:

$$Y_N(t) = \Lambda_N(t) X(t) + e_N(t), \qquad e_N(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \Sigma(t))$$

$$X(t) = \mathbf{A} X(t-1) + u(t) \qquad u(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \mathbf{I})$$

Local ML estimation

Log-likelihood function of $\mathbf{Y}_{NT}$ and $\mathbf{X}_{NT}$:

$$-2\,\ell(\boldsymbol{\theta}|\mathbf{Y}_{NT},\mathbf{X}_{NT}) = \log(\boldsymbol{\Sigma}) + \sum_{t=1}^{T} \left\| \boldsymbol{Y}_N(t) - \boldsymbol{\Lambda}_N \boldsymbol{X}_N(t) \right\|^2_{\boldsymbol{\Sigma}^{-1}} + \sum_{t=1}^{T} \left\| \boldsymbol{X}_N(t) - \mathbf{A}(t) \boldsymbol{X}_N(t-1) \right\|^2$$

with $\mathbf{Y}_{NT} = (\boldsymbol{Y}_N(1),\ldots,\boldsymbol{Y}_N(T))$, $\mathbf{X}_{NT} = (\boldsymbol{X}_N(1),\ldots,\boldsymbol{X}_N(T))$, $\boldsymbol{\theta} = (\Lambda_{ij}, \sigma_i^2, a_{kl}(t))$

Two components:

- conditional likelihood of $\mathbf{Y}_{NT}$ given $\mathbf{X}_{NT}$

$$-2\,\ell(\boldsymbol{\theta}|\mathbf{Y}_{NT};\mathbf{X}_{NT}) = \log(\boldsymbol{\Sigma}) + \sum_{t=1}^{T} \left\| \boldsymbol{Y}_N(t) - \boldsymbol{\Lambda}_N \boldsymbol{X}_N(t) \right\|^2_{\boldsymbol{\Sigma}^{-1}}$$

- marginal likelihood of $\mathbf{X}_{NT}$

$$-2\,\ell(\boldsymbol{\theta}|\mathbf{X}_{NT}) = \sum_{t=1}^{T} \left\| \boldsymbol{X}_N(t) - \mathbf{A}(t) \boldsymbol{X}_N(t-1) \right\|^2$$

Local ML estimation

EM algorithm:

- Expectation step:

$$Q(\boldsymbol{\theta}|\boldsymbol{\theta}^*) = -2\,\mathbb{E}_{\boldsymbol{\theta}^*}\left(\ell(\boldsymbol{\theta}|\mathbf{Y}_{NT},\mathbf{X}_{NT})|\mathbf{Y}_{NT}\right)$$

- Maximization step:

$$\boldsymbol{\theta}^{**} = \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{\operatorname{argmax}}\, Q(\boldsymbol{\theta}|\boldsymbol{\theta}^*)$$

Local ML estimation

E-step:

Take conditional expectation given $\bar{Y}_T$ (and $\boldsymbol{\theta}^*$):

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\theta}^*}&\left[\left(Y_i(t) - \boldsymbol{\Lambda}_i \boldsymbol{X}_N(t)\right)^2 | \mathbf{Y}_{NT}\right] \\ &= Y_i(t)^2 - 2\, Y_i(t)\boldsymbol{\Lambda}_i\, \mathbb{E}_{\boldsymbol{\theta}^*}(\boldsymbol{X}_N(t)|\mathbf{Y}_{NT}) + \boldsymbol{\Lambda}_i \mathbb{E}(\boldsymbol{X}_N(t)\boldsymbol{X}_N(t)'|\mathbf{Y}_{NT})\boldsymbol{\Lambda}_i'\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\theta}^*}&\left[\left\|\boldsymbol{X}_N(t) - \mathbf{A}(t)\boldsymbol{X}_N(t-1)\right\|^2 | \mathbf{Y}_{NT}\right] \\ &= \mathbb{E}_{\boldsymbol{\theta}^*}\Big[\mathrm{tr}\Big(\boldsymbol{X}_N(t)\boldsymbol{X}_N(t)' - 2\,\mathbf{A}(t)\boldsymbol{X}_N(t-1)\boldsymbol{X}_N(t)' \\ &\qquad + \mathbf{A}(t)\boldsymbol{X}_N(t-1)\boldsymbol{X}_N(t-1)'\mathbf{A}(t)'\Big)|\mathbf{Y}_{NT}\Big] \\ &= \mathbb{E}_{\boldsymbol{\theta}^*}\left[\boldsymbol{X}_N(t)\boldsymbol{X}_N(t)'|\mathbf{Y}_{NT}\right] - 2\,\mathbf{A}(t)\,\mathbb{E}_{\boldsymbol{\theta}^*}\left[\boldsymbol{X}_N(t-1)\boldsymbol{X}_N(t)'|\mathbf{Y}_{NT}\right] \\ &\qquad + \mathbf{A}(t)\mathbb{E}_{\boldsymbol{\theta}^*}\left[\boldsymbol{X}_N(t-1)\boldsymbol{X}_N(t-1)'|\mathbf{Y}_{NT}\right]\mathbf{A}(t)'\end{aligned}$$

Local ML estimation

E-step (contd):

Note that:

$$\mathbb{E}_{\boldsymbol{\theta}^*}\left[\boldsymbol{X}_N(t-1)\boldsymbol{X}_N(t-1)'|\mathbf{Y}_{NT}\right]$$
$$= \text{var}_{\boldsymbol{\theta}^*}\left(\boldsymbol{X}_N(t-1)|\mathbf{Y}_{NT}\right) + \mathbb{E}_{\boldsymbol{\theta}^*}(\boldsymbol{X}_N(t)|\mathbf{Y}_{NT})\mathbb{E}_{\boldsymbol{\theta}^*}(\boldsymbol{X}_N(t)|\mathbf{Y}_{NT})'$$
$$\mathbb{E}_{\boldsymbol{\theta}^*}\left[\boldsymbol{X}_N(t-1)\boldsymbol{X}_N(t)'|\mathbf{Y}_{NT}\right]$$
$$= \text{cov}_{\boldsymbol{\theta}^*}\left(\boldsymbol{X}_N(t-1),\boldsymbol{X}_N(t)|\mathbf{Y}_{NT}\right) + \mathbb{E}_{\boldsymbol{\theta}^*}(\boldsymbol{X}_N(t)|\mathbf{Y}_{NT})\mathbb{E}_{\boldsymbol{\theta}^*}(\boldsymbol{X}_N(t)|\mathbf{Y}_{NT})'$$

The quantities

- $\mathbb{E}_{\boldsymbol{\theta}^*}(\boldsymbol{X}_N(t)|\mathbf{Y}_{NT})$
- $\text{var}_{\boldsymbol{\theta}^*}\left(\boldsymbol{X}_N(t-1)|\mathbf{Y}_{NT}\right)$
- $\text{cov}_{\boldsymbol{\theta}^*}\left(\boldsymbol{X}_N(t-1),\boldsymbol{X}_N(t)|\mathbf{Y}_{NT}\right)$

can be computed by application of the Kalman filter and smoother.

Local ML estimation

M-step:

Maximization of the log-likelihood is accomplished in two steps:

- maximize conditional likelihood of Y_t given X_t with respect to $\mathbf{\Lambda}_N$ and $\mathbf{\Sigma}$;
- maximize marginal likelihood of X_t with respect to $\mathbf{A}(t)$ locally.

Local ML estimation

M-step: $\mathbf{\Lambda}_N$ and $\mathbf{\Sigma}$

We have the usual ML estimators:

- $\mathbf{\Lambda}_N = \mathbf{Y}_{NT}\mathbf{X}'_{NT}(\mathbf{X}_{NT}\mathbf{X}'_{NT})^{-1}$
- $\sigma_n^2 = \frac{1}{T}\left\|\boldsymbol{Y}_{nT} - \mathbf{\Lambda}_n\mathbf{X}_{NT}\right\|^2, n = 1, \ldots, N$

Local ML estimation

M-step: $\mathbf{A}(t)$

Idea:

- approximate $\mathbf{A}(t)$ locally about $t = t_0$ by polynomial of order p:

$$\mathbf{A}(t) \approx \mathbf{A}_0 + \mathbf{A}_1 (t - t_0) + \ldots + \mathbf{A}_p (t - t_0)^p = \tilde{\mathbf{A}}(t)$$

- minimize the local kernel-weighted (-2) log-likelihood function

$$\sum_{t=1}^{T} \mathbb{E}_{\boldsymbol{\theta}^*}\left(\left\|\boldsymbol{X}_N(t) - \tilde{\mathbf{A}}(t)\boldsymbol{X}_N(t-1)\right\|^2 |\mathbf{Y}_{NT}\right) K_h(t - t_0)$$

  with respect to $\mathbf{A}_0, \ldots, \mathbf{A}_p$ to obtain $\hat{\mathbf{A}}(t_0)$
- here $K_h(t)$ is a kernel function with bandwidth h
- smoothness of the estimate $\hat{\mathbf{A}}(t_0)$ depends on
  - order p of the approximating polynomial
  - bandwidth h of the kernel function K_h

Local ML estimation

Consider case of one factor $\mathbf{X}(t) = X(t)$:

- $\mathbf{P}_T(t_0) = \begin{pmatrix} 1 & 1-t_0 & \cdots & (1-t_0)^p \\ \vdots & \vdots & \ddots & \vdots \\ 1 & T-t_0 & \cdots & (T-t_0)^p \end{pmatrix}$
- $\tilde{\mathbf{X}}_T = \mathrm{diag}(X(0), \ldots, X(T-1))$
- $\mathbf{W}_T(t_0) = \mathrm{diag}(K_h(1-t_0), \ldots, K_h(T-t_0))$
- $\hat{\boldsymbol{\alpha}}(t_0) = (a_0(t_0), \ldots, a_p(t_0))'$

Then the local (-2) log-likelihood can be written as

$$\mathbb{E}_{\boldsymbol{\theta}^*}\left(\left\| \boldsymbol{X}_T - \tilde{\mathbf{X}}_T \mathbf{P}_T(t_0) \boldsymbol{\alpha}(t_0) \right\|^2_{\mathbf{W}_T(t_0)} | \mathbf{Y}_{NT} \right)$$

Local ML estimation

The local (-2) log-likelihood

$$\mathbb{E}_{\boldsymbol{\theta}^*}\left(\left\|\boldsymbol{X}_T - \tilde{\mathbf{X}}_T \mathbf{P}_T(t_0)\boldsymbol{\alpha}(t_0)\right\|^2_{\mathbf{W}_T(t_0)} | \mathbf{Y}_{NT}\right)$$

is minimized by

$$\hat{\boldsymbol{\alpha}}(t_0) = \left(\mathbf{P}_T(t_0)'\mathbf{Q}_T(t_0)\mathbf{P}_T(t_0)\right)^{-1}\mathbf{P}_T(t_0)'\boldsymbol{R}_T(t_0)$$

where

- $\mathbf{Q}_T(t_0) = \operatorname{diag}\left(\mathbb{E}_{\boldsymbol{\theta}^*}(X(t-1)^2|\mathbf{Y}_{NT})K_h(t-t_0), t = 1, \ldots, T\right)$
- $\boldsymbol{R}_T(t_0) = \left(\mathbb{E}_{\boldsymbol{\theta}^*}(X(t-1)X(t)|\mathbf{Y}_{NT})K_h(t-t_0), t = 1, \ldots, T\right)$

Local ML estimation

Convergence of EM algorithm: consider weighted log-likelihood

$$
\begin{aligned}
&L^{\mathrm{w}}(\theta|\mathbf{Y}_{NT},\mathbf{X}_{NT}) \\
&\quad = \tfrac{1}{T}\log(\boldsymbol{\Sigma}) + \tfrac{1}{T}\sum_{t=1}^{T}\left\|\boldsymbol{Y}_N(t) - \boldsymbol{\Lambda}_N \boldsymbol{X}_N(t)\right\|^2_{\boldsymbol{\Sigma}^{-1}} \\
&\qquad\qquad + \tfrac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left\|\boldsymbol{X}_N(s) - \tilde{\mathbf{A}}(s;t)\boldsymbol{X}_N(t-1)\right\|^2 K_b(t-s) \\
&\quad = \tfrac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} L\Big[\boldsymbol{Y}_N(s),\boldsymbol{X}_N(s)|\boldsymbol{X}_N(s-1);\boldsymbol{\Lambda}_N,\boldsymbol{\Sigma},\tilde{\mathbf{A}}(s,t)\Big]K_b(t-s)
\end{aligned}
$$

Then the EM-algorithm iteratively maximizes

$$Q(\theta|\hat{\theta}^{(i-1)}) = \mathbb{E}\Big(L^{\mathrm{w}}(\theta|\mathbf{Y}_{NT},\mathbf{X}_{NT})|\mathbf{Y}_{NT};\hat{\theta}^{(i)}\Big)$$

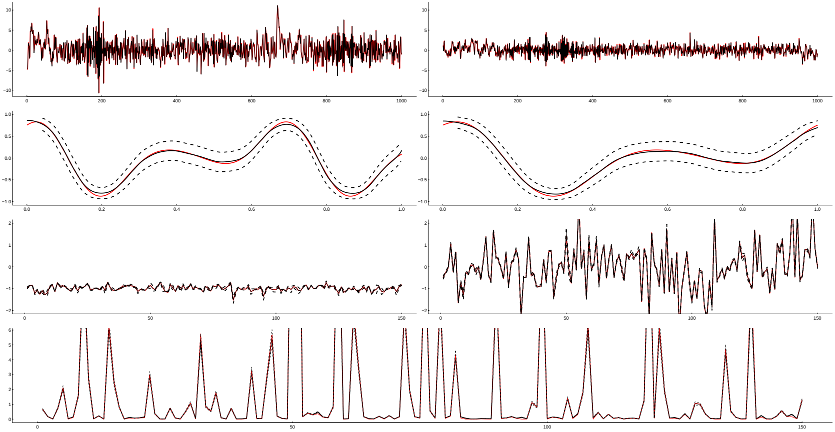
and thus

$$\lim_{i\to\infty}\hat{\theta}^{(i)} = \operatorname{argmax} L^{\mathrm{w}}(\theta|\mathbf{Y}_{NT})$$

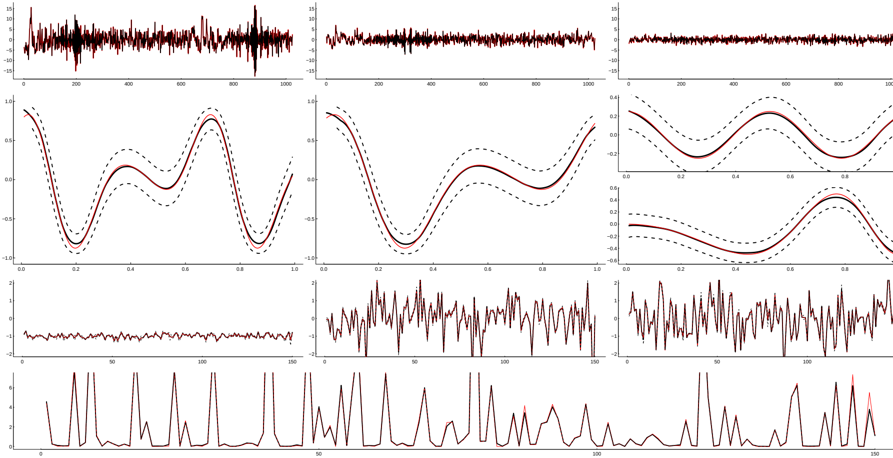
Outline

- Motivation
- Stationary and non-stationary factor models
- A semi-parametric dynamic factor model
- Estimation
- Simulations and applications
- Conclusions/Work to be done

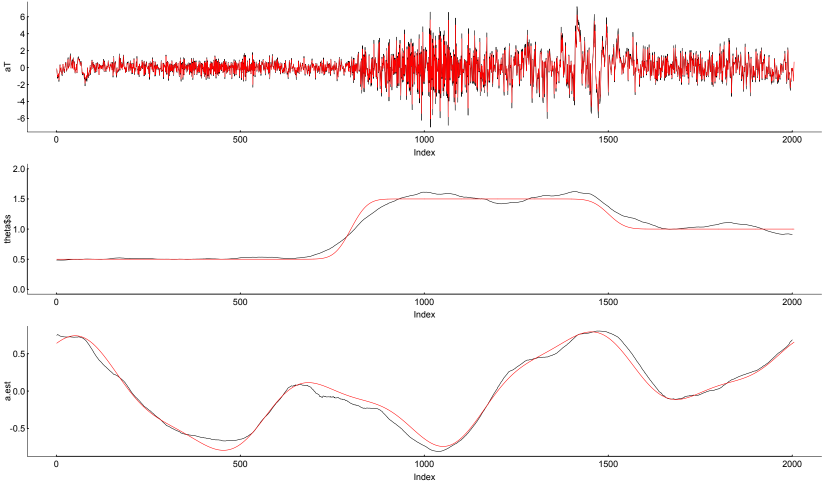
Estimated $f(t)$, $A(t)$, Λ, and Γ^Z



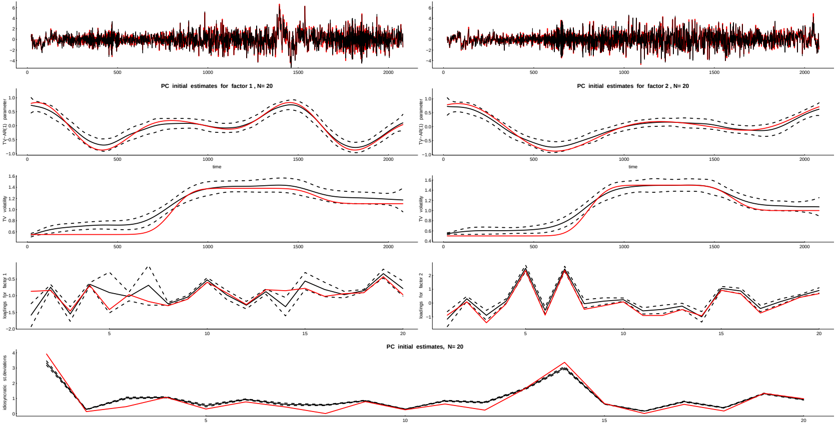
Estimated $f(t)$, $A(t)$, Λ, and Γ^Z



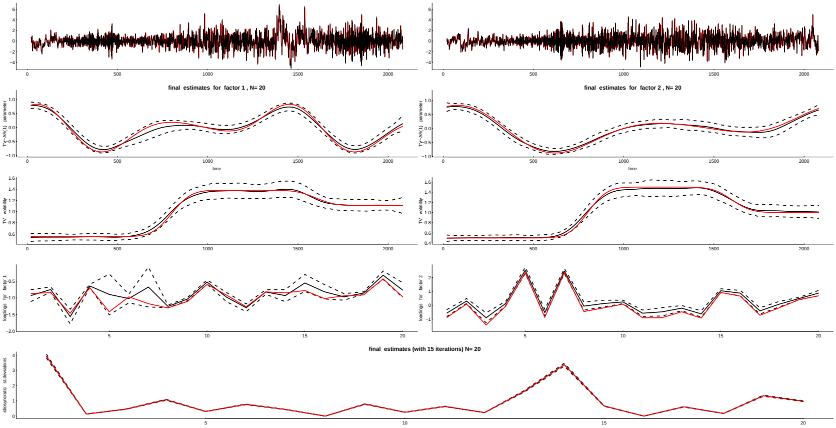
Estimated $f(t)$, v(t) and $\alpha(t)$



Estimated $f(t)$, $A(t)$, v(t), Λ, and Γ^Z



Estimated $f(t)$, $\mathbf{A}(t)$, $\mathrm{v}(\mathrm{t})$, Λ, and Γ^Z



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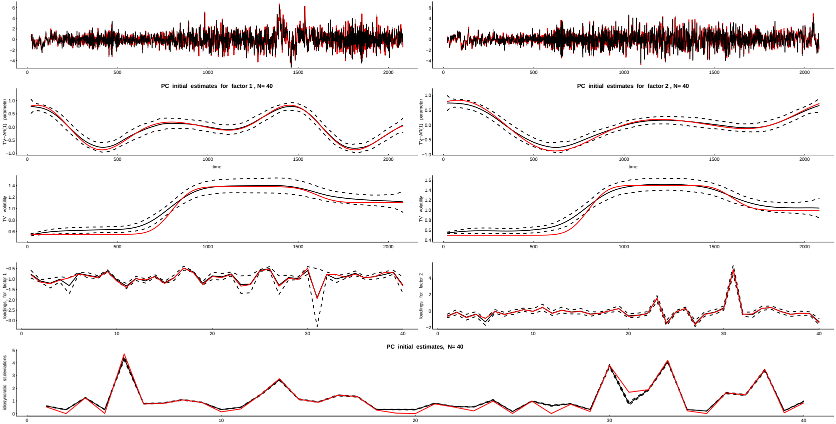

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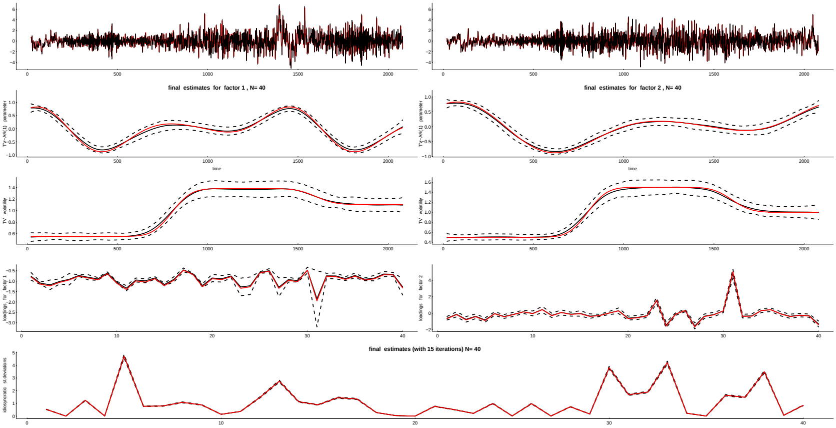
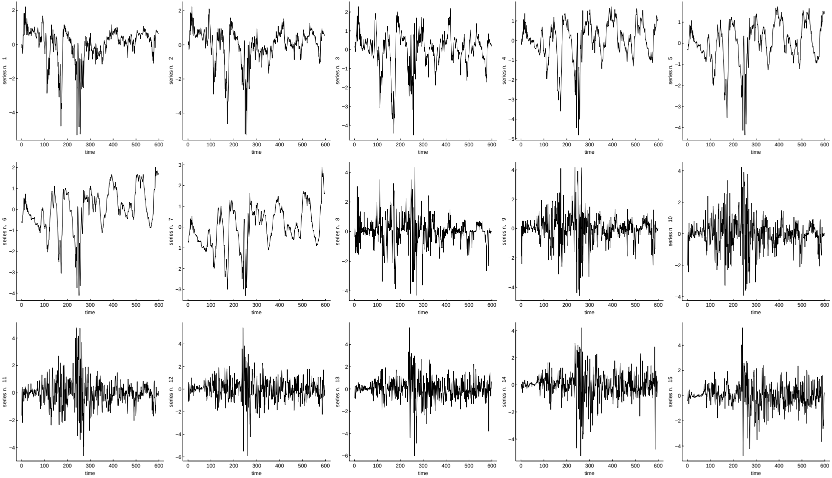


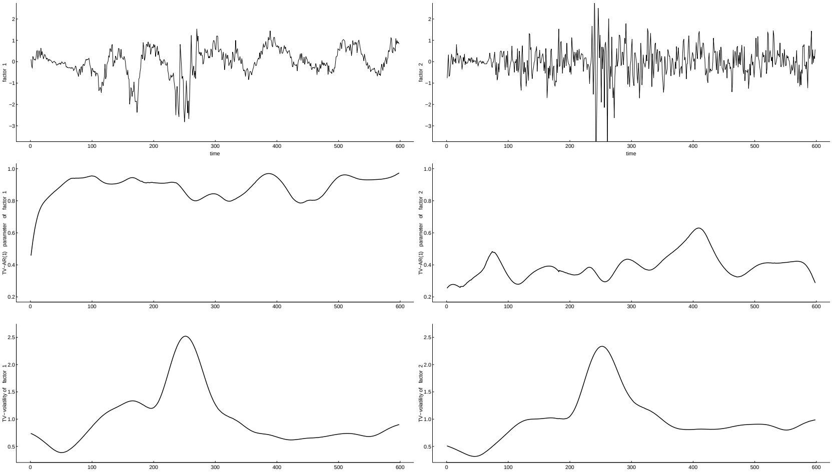
Simulation results: summary

- The estimators are indexed by N to stress their dependence on the cross-section size (T is fixed).
- For all N, the EM-localML estimators perform better than the PCEs.
- The MSE is decreasing over N, especially for the PCEs.
- For both PCE and EM-localML, the MSE of the estimators of $\mathbf{F}$, $\mathbf{A}$ and $\mathbf{V}$ is pretty much stable for $N \geq 60$. This is due to the fact that we average over T the MSE of:
  - the *non-stationary* factors, and
  - the *time-varying* functions $\mathbf{A}$ and $\mathbf{V}$.
- The MSE of $\widetilde{\mathbf{\Gamma}}$ is small (compared to that of $\widehat{\mathbf{\Gamma}}$) for all N. This is due to the fact that the EM-localML estimators exploits that $\mathbf{\Gamma}^Z$ is diagonal.

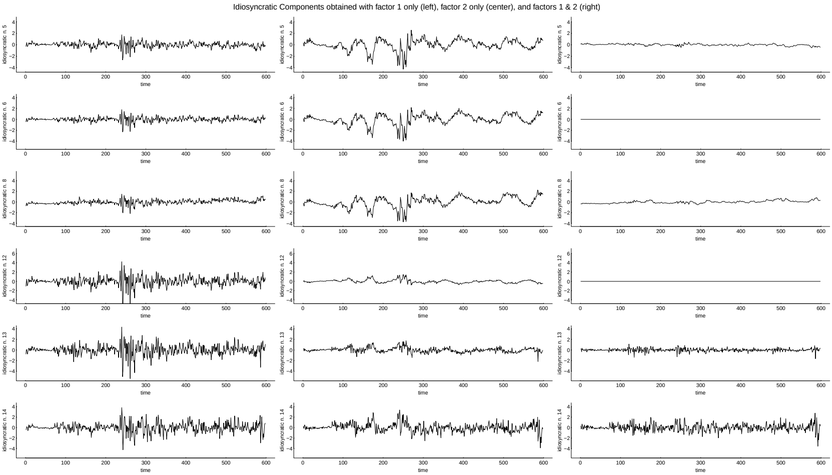
Exchange and interest rates



Exchange and interest rates



Exchange and interest rates



Summary

Our contribution

- We introduce a semi-parametric non-stationary factor model.
- Non-stationarity explained by *smooth, time-varying parameters*.
- The time-varying parameters modelled locally by polynomials.
- For ***large*** N, the factors can be recovered by PCs and the T-V parameters can be estimated locally from the extracted factors.
- Refined Estimation: EM-KF algorithm and the Kalman filter, local ML.
- Compared to the PCA-based approach, the second approach produces superior results, in particular for ***small*** N.

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Future research

- Properties of the semi-parametric EM-ℓML algorithm
- Semi-parametric hypotheses testing on the dynamics of the latent factors
- Prediction