Overfitting of the Hurst index for a multifractional Brownian motion

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Workshop on Processes (CIRM, Marseille)
February 17, 2016

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Historical recall on some stochastic models

Stochastic model tries to better fit real datasets

- Brownian motion $(H = 1/2)$
  Einstein 1905, Bachelier 1901, Wiener 1930 ...
- Fractional Brownian motion $(0 < H < 1)$
  Kolmogorov 1940, Mandelbrot 1968.
- Multifractional Brownian motion ($H(t)$ is time-varying)
  Benassi, Jaffard, Roux 1997, Peltier, Levy-Vehel 1996, ...
- Different generalisations motivated by specific applications
  Many references since 2000.

Future ?

Stochastic model tries to better fit real datasets

- Brownian motion $(H = 1/2)$
- Fractional Brownian motion $(H \neq 1/2)$
- Multifractional Brownian motion $(H(t)$ is time-varying)
- Different generalisations motivated by specific applications

Future ?

Stochastic model tries to better fit real datasets

- Brownian motion $(H = 1/2)$
- Fractional Brownian motion $(H \neq 1/2)$
- Multifractional Brownian motion $(H(t)$ is time-varying$)$
- Different generalisations motivated by specific applications

What next ?

- Multifractional Brownian motion with a Hurst index $H(t, \omega)$ being itself a stochastic process ?
- A parcimonious model ?

A statistical artifact

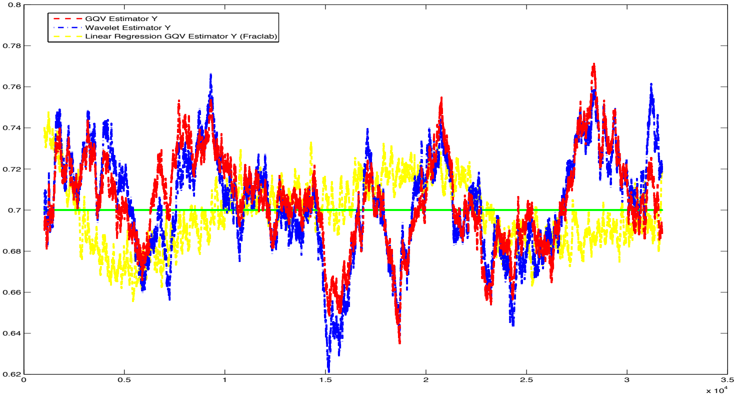

FIGURE: We have simulated a fBm with constant Hurst index $H = 0.7$ and estimated it as a time-varying Hurst index $\widehat{H}(t)$

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Recall on fractional Brownian motion

- The fractional Brownian motion (fBm), with Hurst index H and variance σ^2, is a zero mean Gaussian process with covariance

$$\begin{aligned} R_H(t_1, t_2) &= \operatorname{cov}\big(X(t_1), X(t_2)\big) \\ &= \frac{1}{2}\sigma^2 \left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}. \end{aligned}$$

- The Hurst index $H \in]0,1[$.
- When $H = 1/2$ et $\sigma = 1$, $B_{1/2}$ is a standard Brownian motion.

Fractional Brownian motion

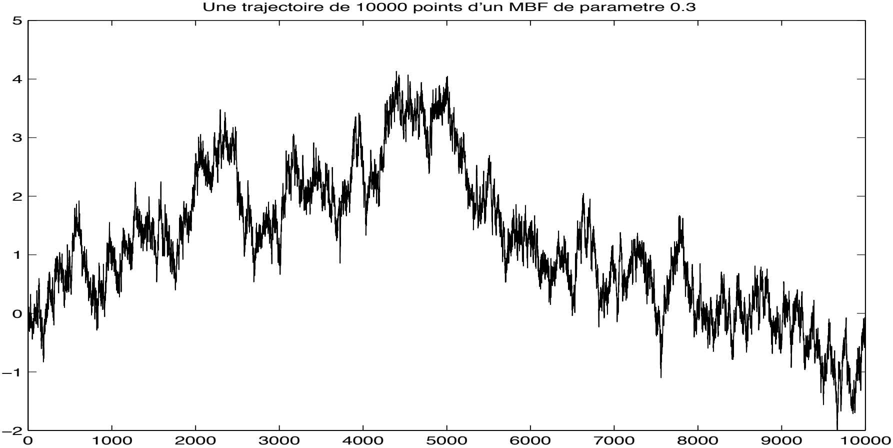


FIGURE: We have simulated a path of fBm with constant Hurst index $H = 0.3$

The Hurst index H drives 3 properties :

1. **Pathwise regularity** $\forall t,\ \alpha^*(t) = H$ *a.s.* where

$$\alpha^*(t) = \sup\left\{\alpha, limsup_{h\to 0}\frac{|X(t+h)-X(t)|}{h^{\alpha}} = 0\right\}$$

2. **Self-similarity :**

$$(B_H(\lambda t))_{t\in\mathbf{R}} \stackrel{(d)}{=} \left(\lambda^H B_H(t)\right)_{t\in\mathbf{R}}.$$

3. **Correlation of the increments :**

$$r(n) = cov\left(X(n+1)-X(n), X(1)-X(0)\right).$$

If $H > 1/2$, then $\sum_{k=-\infty}^{+\infty}|r(k)| = \infty$ **(Long memory)**

Three representations of fBm

1. **Moving average representation** (Mandelbrot & Van Ness, 1968)

$$B_H(t) \quad = \quad C\int_{-\infty}^{+\infty}\left[(t-s)_+^{H-1/2}-(-s)_+^{H-1/2}\right]dW_s.$$

2. **Harmonisable representation** (Kolmogorov, 1940)

$$B_H(t) \quad = \quad \int_{\mathbf{R}}\left(e^{it\xi}-1\right)\times|\xi|^{-(H+1/2)}\,\widehat{W}(d\xi)$$

where $\widehat{W}(d\xi)$ is the Fourier transform of the Wiener measure $W(dx)$.

Wavelet series expansion of fBm

1. **Wavelet series expansion** (Meyer, Sellan, Taqqu, 1999)

$$B(t,H) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-jH} \varepsilon_{j,k} \left\{ \Psi(2^j t - k, H) - \Psi(-k, H) \right\}, \quad (1)$$

- where $\left(\varepsilon_{j,k}\right)_{(j,k)\in\mathbf{Z}^2}$ is a family of independent Gaussian random variables $\mathcal{N}(0,1)$;
- $\{2^{j/2}\psi(2^j x - k) : (j,k) \in \mathbf{Z}^2\}$ is a Lemarié-Meyer wavelet basis ;
- and

$$\Psi(x,H) = \int_{\boldsymbol{R}} e^{ix\xi} \frac{\widehat{\psi}(\xi)}{|\xi|^{H+1/2}} \, d\xi. \quad (2)$$

The convergence of the series is uniform on every compact subset $I \times K \subset (0,1) \times \boldsymbol{R}$, almost surely (Ayache & Taqqu, 2003).

Recall on multifractional Brownian motion

- The multifractional Brownian motion (mBm) can be seen as a generalisation of the fBm
- The Hurst index $0 < H < 1$ is replaced by a time-varying function $t \mapsto H(t)$

$$X(t) = B(t, H(t))$$

  where $B(t, H) := B_H(t)$ is the wavelet series expansion of fBm, or another representation.

Applications in many fields

Models with a time-varying Hurst index can be encountered in many different fields

- In turbulence (see Papanicolaou and Solna, 2002) : the mBm with a regularly time-varying Hurst index is used for the air velocity.
- In statistical study on magnetic dynamics (see Wanliss and Dobias, 2007) : an abrupt change in Hurst index can be observed before a space storm in solar wind.

Behavioural economics. . .

Economic point of view is developed by Bianchi (2005) – Bianchi, Pantanella, Pianese (2015).

- Periods with significantly Hurst index $H \neq 1/2$ (independence of the increments = efficiency of the market) can be explained by behavioural economics :
  1. $H(t) < 1/2$ [increments negatively correlated] : the market is not confident in the past and it overreacts to new informations.
  2. $H(t) > 1/2$ [increments positively correlated] : the market is too confident in the past and it underreacts to new informations.
- In behavioural finance, underreaction is due to overconfidence of investors.

...against mainstream Finance

- Arbitrage opportunity for fBm is possible when the Hurst index H is constant and known by advance without transaction costs (Rogers 1997, Shyriaev 1998).
- However, arbitrage with fBm does no more exist with transaction costs (Cheridito 2003, Guasoni, 2006).
- Moreover, arbitrage opportunity is not possible for a stochastic Hurst index, even without transaction costs.

Estimating Hurst index

- Let X be a fBm or a mBm. We observe one path of size n of the process X with mesh $h_n = \frac{1}{n}$, namely $\big(X(0), X(t_1), \dots, X(t_n)\big)$.
- The standard method for estimating a time-varying Hurst index for mBm is to localise the estimation of a constant Hurst index on a small vicinity of each time t, namely on

$$\mathcal{V}(t, \varepsilon_n) = \big\{ t_k \text{ such that } |t_k - t| \le \varepsilon_n \big\},$$

where $\varepsilon_n = n^{-\alpha}$, with $0 < \alpha < 1$. Thus

$$\varepsilon_n \to 0 \qquad \textit{and} \qquad \frac{\varepsilon_n}{h_n} \to \infty \quad \textit{as} \quad n \to \infty.$$

Overfitting of localized estimator

Localization of Hurst index estimation implies overfitting as stated by the functional CLT of Coeurjolly (2005) for the GQV estimator $\widehat{H}_n(t)$:

Theorem (Coeurjolly, 2005–2006)

If $t \longmapsto H(t)$ is regular enough, then $\widehat{H}_n(t) \longrightarrow H$ and

$$\sqrt{2\varepsilon_n \cdot n} \times \left(\widehat{H}_n(t) - H(t)\right) \longrightarrow_{(\mathcal{L})} \mathbb{G}'(t)$$

where $\mathbb{G}'(t)$ a zero mean Gaussian process, with covariance structure :

$$\begin{aligned} var(\mathbb{G}'(t)) &= \gamma(H(t)) \quad \text{for all } t \in (0,1), \\ cov(\mathbb{G}'(t_1), \mathbb{G}'(t_2)) &= 0 \qquad \text{with } (t_1, t_2) \in (0,1)^2 \quad \text{for } t_1 \neq t_2, \end{aligned}$$

Technical details

- Assume $0 < H(t) < 1$ where $H \in \mathcal{C}^{\beta}\big([0,1],(0,1)\big)$, with $\beta > 0$.
- $X = B_{H(t)}$ is a mBm observed at times $(t_k = k/n)_{k=1,\dots,n}$.
- the Generalized Quadratic Variation associated to the filter $a = (1;-2;1)$ is

$$V_n(t,a) := \frac{1}{v_n} \sum_{t_k \in \mathcal{V}(t,\varepsilon_n)} \left|X(t_k) - 2X(t_{k-1}) + X(t_{k-2})\right|^2.$$

- The estimator is

$$\widehat{H}_n(t) = \frac{A^t}{2AA^t}\Big(\ln\big(V_n(t,a)\big)\Big)_{j=1,\dots,M}.$$

Technical details (continued)

- The variance of $\mathbb{G}'(t)$ is

$$\gamma(H) = \left(\frac{1}{\pi_H^a(0)^2}\sum_{k\in\mathbf{Z}}\pi_H^a(k)^2\right)\times\frac{A^t(UU^t)A}{4\|A\|^4} \qquad (3)$$

- where

$$\pi_H^a(k) := -\frac{1}{2}\sum_{q=0}^{2}\sum_{q'=0}^{2} a_q\, a_{q'}\,|q-q'+k|^{2H},$$

- A is the row vector $A_j = \ln(j) - \frac{1}{M}\sum_{\nu=1}^{M}\ln(\nu)$ for $j = 1,\ldots,M$ and $U = (1,\ldots,1)$.

Explanation of the statistical artifact

In this covariance structure, we have

$$\mathrm{cov}(\mathbb{G}'(t_1), \mathbb{G}'(t_2)) = 0$$

for all $(t_1, t_2) \in (0,1)^2$ such that $t_1 \neq t_2$. This explains the statistical artifact

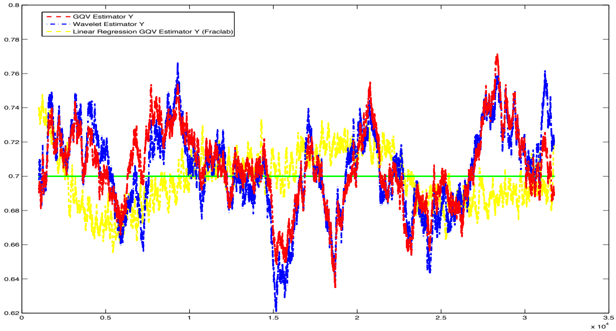
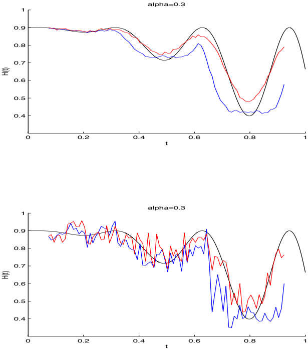
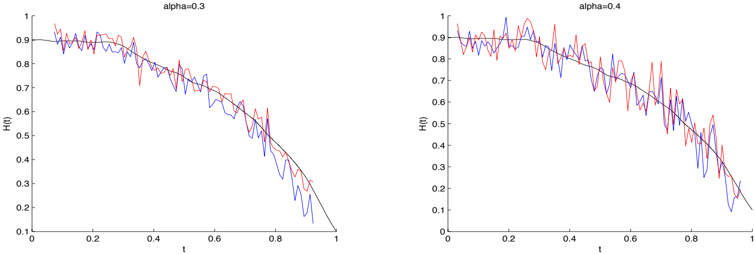


Illustration by Fig.1 p.1022 in Bardet-Surgailis, SPA (2013).



Estimates of the function $H_4(t) = 0.1 + 0.8(1-t)\sin^2(10t)$ with $t \in (0,1)$ for $n = 6000$ and $\alpha = 0.3$ and $\alpha = 0.4$ (from left to right). The top row represents the mean trajectories of $\widehat{H}(t)$ and $H^{(IR2)}(t)$ the localized IRS estimator obtained from 100 independent replications of MBM with the above function $H(\cdot)$.
The bottom row represents a trajectory of of $\widehat{H}(t)$ and $H^{(IR2)}(t)$ obtained from one trajectory of MBM with the above function $H(\cdot)$. The graphs of $H(t)$, $\widehat{H}(t)$ and $H^{(IR2)}(t)$ are in black, blue and red, respectively.

Fig.2 p.1023 in Bardet-Surgailis, SPA (2013).



- Trajectories of $\widehat{H}(t)$ and $H^{(IR2)}(t)$ for one of the 50 differentiable Hurst functions $H(\cdot) \in \mathcal{C}^{1.5-}$ for $n = 6000$ and $\alpha = 0.3$ and $\alpha = 0.4$ (from left to right).
- For $\alpha = 0.3$, we have $2\varepsilon_n = 882 \times h_n$.
  For $\alpha = 0.4$, we have $2\varepsilon_n = 370 \times h_n$.
- The graphs of $H(t)$, $\widehat{H}(t)$ and $H^{(IR2)}(t)$ are in black, blue and red, respectively.

Convergence of the normalized square error

From the previous functional CLT, we deduce the convergence of normalized square error

$$\lim_{n\to\infty} \boldsymbol{E}\left[\frac{1}{n}\sum_{k=1}^{n}\left|\widehat{H}_n(t_k)-H(t_k)\right|^2\right]=\int_0^1 \gamma\big(H(t)\big)\,dt.$$

We also get the CLT

$$\frac{(2n\varepsilon_n)\times\left[\frac{1}{n}\sum_{k=1}^{n}\left|\widehat{H}_n(t_k)-H(t_k)\right|^2\right]-\int_0^1\gamma\big(H(t)\big)\,dt}{\left[\left(\frac{2}{n}\right)\times\int_0^1\gamma\big(H(t)\big)^2\,dt\right]^{1/2}} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0,1),$$

where $\gamma(H)$ is given by (3).

A fitting test for time-varying Hurst index

We want to test if a time-varying Hurst index $\widetilde{H}(\cdot)$ is an admissible model, that is

$$(H_0) : \widetilde{H}(\cdot) = H(\cdot) \qquad \textit{versus} \qquad (H_1) : \widetilde{H}(\cdot) \neq H(\cdot).$$

We use the test statistic

$$T_n(\widetilde{H}) \quad = \quad \frac{(2n\varepsilon_n) \times \left[\frac{1}{n}\sum_{k=1}^{n} |\widehat{H}_n(t_k) - \widetilde{H}(t_k)|^2\right] - \int_0^1 \gamma\big(\widetilde{H}(t)\big)\, dt}{\left(\left(\frac{2}{n}\right) \times \int_0^1 \gamma\big(\widetilde{H}(t)\big)^2\, dt\right)^{1/2}}.$$

A fitting test for time-varying Hurst index

- Under the null hypothesis, we have

$$T_n(\widetilde{H}) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

- On the other hand, we cannot calculate the power of the test since $H(\cdot) \in \mathcal{C}([0,1])$ which is an infinite dimensional vector space.

1st application to model rejection

The naive time-varying estimator of the Hurst index could not be chosen as a valid model. Let

$$\widetilde{H}(t) = \lim_{n\to\infty} \widehat{H}_n(t)$$

Then

$$\begin{aligned} T_n(\widetilde{H}(t)) &\simeq \frac{-\int_0^1 \gamma_{\widetilde{H}(t)}\,dt}{\left(\frac{2}{n}\int_0^1 (\gamma_{\widetilde{H}(t)})^2\,dt\right)^{1/2}} \\ &\simeq -\sqrt{\frac{n}{2}} \times \frac{\|\gamma_{\widetilde{H}(t)}\|_{L^1(]0;1[)}}{\|\gamma_{\widetilde{H}(t)}\|_{L^2(]0;1[)}} \longrightarrow \infty \text{ as } n \to \infty \end{aligned}$$

The null hypothesis (H_0) is asymptotically rejected.

Application to model selection

Next idea :
Determine the simplest possible function $\widetilde{H}(t)$, that is eligible for the test, to describe the theoretical Hurst index $H(t)$

This model selection is a kind of Portemanteau test :

1. $\mathcal{M}_0$ the family of constant models $\widetilde{H}(t) = H$, obtained as the empirical mean of $\widehat{H}_n(t_k)$.
2. $\mathcal{M}_1$ the family of affine models $\widetilde{H}(t)$, obtained by linear regression of $\widehat{H}_n(t_k)$.
3. $\mathcal{M}_2$ the family of quadratic models $\widetilde{H}(t)$,
4. ...
5. $\mathcal{M}_k$ the family of polynomial function of order k.

Conclusion

1. We have explained the statistical artifact.
2. We propose a fitting test for admissible time-varying Hurst index $H(t)$.
3. Selection of the best model should be enhanced.

Thanks

Thank for your attention ...

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