Asymptotic behavior of the Laplacian quasi-maximum likelihood estimator of affine causal processes

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Outline

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   - Definition of causal affine time series
   - Existence and stationarity of causal affine models
2. Gaussian QMLE for causal affine models
   - Estimation and Gaussian QMLE
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   - Asymptotic behavior of Gaussian QMLE
3. Laplacian QMLE
   - Limit theorems of the Laplacian QMLE
   - Results of simulation
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Definition of causal affine time series

$(X_t)_{t\in\mathbb{Z}}$ is a process defined as a solution of the equation:

Causal affine models

$$X_t = M(X_{t-1}, X_{t-2}, \ldots)\,\xi_t + f(X_{t-1}, X_{t-2}, \ldots), \ \ \forall\, t \in \mathbb{Z}, \ \text{a.s.}.$$

- $M(\cdot)$ and $f(\cdot)$ are real valued function on $\mathbb{R}^{\mathbb{N}}$;
- $(\xi_t)_{t\in\mathbb{Z}}$ sequence of i.i.d.r.v. with $\mathbb{E}(\xi_0) = 0$ and $\mathbb{E}\big(|\xi_0|^r\big) < \infty$, $r \geq 1$.

Examples: Causal ARMA and one-sided linear processes

With $(\xi_t)_{t\in\mathbb{Z}}$ a sequence of centered i.i.d.r.v.,

- AR(p) processes $X_t = \sum_{i=1}^{p} a_i X_{t-i} + \xi_t$

  $\Longrightarrow$ AR(∞) processes $X_t = \sum_{i=1}^{\infty} a_i X_{t-i} + \xi_t$

- One-sided linear processes $X_t = \sum_{i=0}^{\infty} b_i \xi_{t-i}$ under certain conditions

  $\Longrightarrow$ Causal ARMA(p,q) processes $X_t + \sum_{i=1}^{p} a_i X_{t-i} = \xi_t + \sum_{i=1}^{q} b_i \xi_{t-i}$.

GARCH(p,q) and ARCH(∞) models

- GARCH(p,q) processes, (Bollersev, 1986)

$$\begin{cases} X_t &= \sigma_t \xi_t, \\ \sigma_t^2 &= b_0 + \sum_{j=1}^{p} b_j X_{t-j}^2 + \sum_{j=1}^{q} c_j \xi_{t-j}^2 \end{cases}$$

with $b_0 > 0$ and $b_j,\ c_j \geq 0$

$\Longrightarrow$ Particular case of:

- ARCH(∞) processes, (Robinson, 1991)

$$\begin{cases} X_t &= \sigma_t \xi_t, \\ \sigma_t^2 &= b_0 + \sum_{j=1}^{\infty} b_j X_{t-j}^2. \end{cases}$$

with $b_0 > 0$ and $b_j \geq 0$

Extensions of univariate ARCH models

- TGARCH(∞) processes, (Zakoïan, 1994)

$$\begin{cases} X_t &= \sigma_t \, \xi_t, \\ \sigma_t &= b_0 + \sum_{j=1}^{\infty} \left[b_j^+ \max(X_{t-j}, 0) - b_j^- \min(X_{t-j}, 0) \right] \end{cases}$$

with $b_0,\ b_j^+,\ b_j^- \geq 0$ for all $j \in \mathbb{N}^*$.

- APARCH(δ, p, q) processes, (Ding *et al.*, 1993)

$$\begin{cases} X_t &= \sigma_t \, \zeta_t, \\ \sigma_t^\delta &= \omega + \sum_{j=1}^{p} \alpha_i (|X_{t-i}| - \gamma_i X_{t-i})^\delta + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^\delta, \end{cases}$$

with $\delta \geq 1$, $\omega > 0$, $-1 < \gamma_i < 1$ and $\alpha_i,\ \beta_j \geq 0$.

Combinations of models

- ARMA-GARCH processes, (Ding *et al.*, 1993, Ling and McAleer, 2003)

$$\begin{cases} X_t &= \sum_{i=1}^{p} a_i X_{t-i} + \varepsilon_t + \sum_{j=1}^{q} b_j \varepsilon_{t-j}, \\ \varepsilon_t &= \sigma_t \zeta_t, \quad \text{with} \quad \sigma_t^2 = c_0 + \sum_{i=1}^{p'} c_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q'} d_j \sigma_{t-j}^2 \end{cases}$$

- ARMA-APARCH processes, (Ding *et al.*, 1993)

$$\begin{cases} X_t &= \sum_{i=1}^{p} a_i X_{t-i} + \varepsilon_t + \sum_{j=1}^{q} b_j \varepsilon_{t-j}, \\ \varepsilon_t &= \sigma_t \zeta_t, \quad \text{with} \quad \sigma_t^\delta = \omega + \sum_{j=1}^{p'} \alpha_i (|X_{t-i}| - \gamma_i X_{t-i})^\delta + \sum_{j=1}^{q'} \beta_j \sigma_{t-j}^\delta \end{cases}$$

Existence and stationarity of causal affine models

$$X_t = M(X_{t-1}, X_{t-2}, \ldots)\,\xi_t + f(X_{t-1}, X_{t-2}, \ldots), \quad \forall\, t \in \mathbb{Z},$$

We will assume that f and M satisfy Lipschitzian conditions:

$$\begin{cases} |f(x) - f(y)| & \leq & \sum_{j=1}^{\infty} \alpha_j^{(0)}(f)|x_j - y_j| \\ |M(x) - M(y)| & \leq & \sum_{j=1}^{\infty} \alpha_j^{(0)}(M)|x_j - y_j|. \end{cases}$$

for $x = (x_j)_{j\in\mathbb{N}}$ and $y = (y_j)_{j\in\mathbb{N}}$ two sequences of $\mathbb{R}^{\mathbb{N}}$.

Proposition (from Doukhan and Wintenberger, 2007)

If $\sum_{j=1}^{\infty} \alpha_j^{(0)}(f) + \big(\mathbb{E}(|X_0|^r)\big)^{1/r} \sum_{j=1}^{\infty} \alpha_j^{(0)}(M) < 1$, *there exists a unique causal (X_t is independent of $(\xi_i)_{i>t}$ for $t \in \mathbb{Z}$) solution $(X_t)_{t\in\mathbb{Z}}$ which is strictly stationary, ergodic and such as $\mathbb{E}(|X_0|^r) < \infty$.*

A useful lemma

$$X_t = M(X_{t-1}, X_{t-2}, \ldots)\,\xi_t + f(X_{t-1}, X_{t-2}, \ldots), \quad \forall\ t \in \mathbb{Z}.$$

Lemma

Let $\widetilde{X}$ be such as $\widetilde{X}_t = \Lambda(L)\ X_t$ for $t \in \mathbb{Z}$ with $\Lambda(L) = P^{-1}(L)\,Q(L)$ where (P, Q) coprime polynomials of a causal invertible ARMA(p, q). Then $\widetilde{X}$ is a.s. a causal stationary solution of the equation

$$\widetilde{X}_t = \widetilde{M}\big((\widetilde{X}_{t-i})_{i \geq 1}\big)\,\zeta_t + \widetilde{f}\big((\widetilde{X}_{t-i})_{i \geq 1}\big) \quad \text{for } t \in \mathbb{Z}, \quad \text{where}$$

- $\max\big(\alpha_j^{(0)}(f), \alpha_j^{(0)}(M)\big) = O(j^{-\beta}),\ \beta > 1 \implies \max\big(\alpha_j^{(0)}(\widetilde{f}), \alpha_j^{(0)}(\widetilde{M})\big) = O(j^{-\beta})$;
- $\max\big(\alpha_j^{(0)}(f), \alpha_j^{(0)}(M)\big) = O(\rho^j),\ |\rho| < 1 \implies \max\big(\alpha_j^{(0)}(\widetilde{f}), \alpha_j^{(0)}(\widetilde{M})\big) = O(\widetilde{\rho}^j),\ |\widetilde{\rho}| < 1.$

$\Longrightarrow$ Application to ARMA-GARCH, ARMA-ARCH(∞), ARMA-APARCH,...

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Semi-parametric models

With $\theta \in \mathbb{R}^d$ unknown but $\theta \mapsto f_\theta$, $\theta \mapsto M_\theta$ two known functions, suppose

$$X_t = M_\theta(X_{t-1}, X_{t-2}, \ldots)\, \xi_t + f_\theta(X_{t-1}, X_{t-2}, \ldots), \quad \forall\, t \in \mathbb{Z}.$$

Assume now that $(X_1, X_2, \cdots, X_n)$ is observed.

How to estimate θ when the distribution of (ξ_t) is unknown?

For instance:

- For a AR(p) process, $X_t = \xi_t + \sum_{j=1}^p a_j X_{t-j}$ how to estimate $\theta = (a_1, \cdots, a_p)$?
- For a ARCH(∞) process, $X_t = \xi_t \left(b_0(\theta) + \sum_{j=1}^\infty b_j(\theta) X_{t-j}^2\right)^{1/2}$ with explicit functions $b_j(\theta)$, $\theta \in \mathbb{R}^d$. For instance $b_j(\theta) = \lambda\, \mu^{-j}$ where $\lambda > 0$, $\mu > 1$ and $\theta = (\lambda, \mu)$.

Choice of the Quasi-Maximum Likelihood Estimation

- The Maximum Likelihood Estimator only possible knowing the distribution of $(\xi_t)_t$.
- The Whittle approximation of Maximum Likelihood Estimator:

$$\widehat{\theta} = \arg\min_{\theta \in \Theta} \Big\{ \int_{-\pi}^{\pi} \frac{I_n(\lambda)}{g_\theta(\lambda)}\, d\lambda + \int_{-\pi}^{\pi} \log(g_\theta(\lambda)) d\lambda \Big\}$$

with $I_n(\lambda) = \frac{1}{2\pi n} \Big| \sum_{k=1}^n X_k e^{-ik\lambda} \Big|^2$ periodogram, g_θ spectral density.

$\Longrightarrow$ For GARCH(p,q) or ARCH(∞) processes, this estimator has to be applied to $(X_t^2)_t$ instead on $(X_t)_t$ (Giraitis and Robinson, 2001)

Principle of the Gaussian QMLE

Let $(X_t)_t$ a GARCH(1,1) process: $\begin{cases} X_t &= \sigma_t \xi_t, \\ \sigma_t^2 &= b_0^* + b_1^* X_{t-1}^2 + c_1^* \sigma_{t-1}^2 \end{cases}$

$$\Longrightarrow X_t = \Big(\frac{b_0^*}{1-c_1^*} + b_1^* \sum_{j=1}^{\infty} c_1^{*j-1} X_{t-j}^2 \Big)^{1/2} \xi_t .$$

If $(\xi_t)_t$ standardized Gaussian i.r.v., with $\theta = (b_0, b_1, c_1)$, the conditional (w.r.t. $(X_t)_{t \leq 0}$) log-likelihood of $(X_1, \cdots, X_n)$,

$$L_n(\theta) = \sum_{t=1}^{n} q_t(\theta) \quad \text{for all } \theta \in \Theta \quad \text{with}$$

$$q_t(\theta) = -\frac{1}{2}\Big[\Big(\frac{b_0}{1-c_1} + b_1 \sum_{j=1}^{\infty} c_1^{j-1} X_{t-j}^2 \Big)^{-1} X_t^2 + \log \Big(\frac{b_0}{1-c_1} + b_1 \sum_{j=1}^{\infty} c_1^{j-1} X_{t-j}^2 \Big) \Big]$$

$\Longrightarrow$ $L_n(\theta)$ depends on $(X_t)_{t \leq 0}$: unknown!!

Principle the Gaussian QMLE (end)

- Idea: replace $q_t(\theta)$ by $\widehat{q}_t(\theta)$ with

$$\widehat{q}_t(\theta)=-\frac{1}{2}\Big[\Big(\frac{b_0}{1-c_1}+b_1\sum_{j=1}^{t-1}c_1^{j-1}X_{t-j}^2\Big)^{-1}X_t^2+\log\Big(\frac{b_0}{1-c_1}+b_1\sum_{j=1}^{t-1}c_1^{j-1}X_{t-j}^2\Big)\Big]$$

- Define the Gaussian QMLE $\widehat{\theta}_n = \arg\max_{\theta\in\Theta} \widehat{L}_n(\theta)$ with $\widehat{L}_n(\theta) = \sum_{t=1}^{n} \widehat{q}_t(\theta)$.
- Compute $\widehat{\theta}_n$ even if $(\xi_t)_t$ is not a Gaussian sequence.
- Consistency and Asymptotic Normality of $\widehat{\theta}_n$ (Lumsdaine, 1996, Berkes *et al.*, 2003, Francq and Zakoian, 2004)
- Extension of this method to causal affine models?

Gaussian QMLE of causal affine model

$$X_t = M_{\theta_0}(X_{t-1}, X_{t-2}, \ldots)\,\xi_t + f_{\theta_0}(X_{t-1}, X_{t-2}, \ldots), \quad \forall\, t \in \mathbb{Z}$$

- With $f_\theta^t = f_\theta(X_{t-1}, X_{t-2}, \ldots)$, $M_\theta^t = M_\theta(X_{t-1}, X_{t-2}, \ldots)$,

$$q_t(\theta) = -\frac{1}{2}\Big[\frac{(X_t - f_\theta^t)^2}{(M_\theta^t)^2} + \log\big((M_\theta^t)^2\big)\Big].$$

- Let $\widehat{f}_\theta^t = f_\theta(X_{t-1}, \ldots, X_1, 0, \cdots)$ and $\widehat{M}_\theta^t = M_\theta(X_{t-1}, \ldots, X_1, 0, \cdots)$

$$\widehat{q}_t(\theta) = -\frac{1}{2}\Big[\frac{(X_t - \widehat{f}_\theta^t)^2}{(\widehat{M}_\theta^t)^2} + \log\big((\widehat{M}_\theta^t)^2\big)\Big].$$

$\Longrightarrow$ Gaussian QMLE: $\widehat{\theta}_n = \arg\max_{\theta\in\Theta} \widehat{L}_n(\theta)$ with $\widehat{L}_n(\theta) = \sum_{t=1}^n \widehat{q}_t(\theta)$.

We assume:

- **C0**: $r \geq 2$ and $\mathbb{E}(\xi_0^2) = 1$;
- **C1**: Θ is a compact set included in

$$\Theta(r) = \Big\{ \theta \in \mathbb{R}^d \;\Big/\; \sum_{j=1}^{\infty} \alpha_j^{(0)}(f_\theta) + \big(\mathbb{E}(|\xi_0|^r)\big)^{1/r} \sum_{j=1}^{\infty} \alpha_j^{(0)}(M_\theta) < 1 \Big\}.$$

- **C2**: $\exists \underline{M} > 0$ such that $M_\theta(x) \geq \underline{M}$ for all $\theta \in \Theta$, $x \in \mathbb{R}^{\mathbb{N}}$.
- **C3**: M_θ and f_θ are such that for all $\theta_1, \ \theta_2 \in \Theta$, then:

$$\big(M_{\theta_1} = M_{\theta_2} \quad \text{and} \quad f_{\theta_1} = f_{\theta_2} \big) \quad \Longrightarrow \quad \theta_1 = \theta_2$$

- $\mathbf{A}_0(K, \Theta)$; There exists $\big(\alpha_j^{(0)}(K, \Theta)\big)_j$ such that $\forall x, \ y \in \mathbb{R}^{\mathbb{N}}$

$$\sup_{\theta \in \Theta} \big| K_\theta(x) - K_\theta(y) \big| \leq \sum_{j=1}^{\infty} \alpha_j^{(0)}(K, \Theta) |x_j - y_j|,$$

$$\text{with } \sum_{j=1}^{\infty} \alpha_j^{(0)}(K, \Theta) < \infty.$$

Strong consistency

Théorème (Bardet and Wintenberger, 2009)

Assume $r \geq 2$, $\Theta \subset \Theta(2)$, *Conditions C0-3 and* $A_0(f, \Theta)$ *and* $A_0(M, \Theta)$ *with*

$$\alpha_j^{(0)}(f, \Theta) + \alpha_j^{(0)}(M, \Theta) = O\big(j^{-\ell}\big) \quad \text{for some} \quad \ell > 2,$$

then the QMLE $\widehat{\theta}_n$ *is strongly consistent, i.e.* $\widehat{\theta}_n \xrightarrow[n\to\infty]{a.s.} \theta_0$.

Assumptions on the derivatives

$\mathbf{A}_k(K,\Theta)$: Function $\theta \to K_\theta(\cdot) \in \mathcal{C}^k(\Theta)$ satisfies $\sup_{\theta\in\Theta} \left\| \partial^k_\theta K_\theta(0) \right\| < \infty$ and there exists $\left(\alpha_j^{(k)}(K,\Theta)\right)_j$ a sequence such that $\forall x,\, y \in \mathbb{R}^{\mathbb{N}}$

$$\sup_{\theta\in\Theta} \left\| \partial^k_\theta K_\theta(x) - \partial^k_\theta K_\theta(y) \right\| \leq \sum_{j=1}^{\infty} \alpha_j^{(k)}(K,\Theta)|x_j - y_j|,$$

$$\text{with } \sum_{j=1}^{\infty} \alpha_j^{(0)}(K,\Theta) < \infty.$$

Asymptotic normality

Théorème (Bardet and Wintenberger, 2009)

Under conditions of SLLN, and if $r \geq 4$, *if* $\theta_0 \in \overset{\circ}{\Theta} \cap \Theta(4)$ *and if* $\mathbf{A}_k(K, \Theta)$ *for* $k = 1, 2$ *and*

$$\alpha_j^{(1)}(f, \Theta) + \alpha_j^{(1)}(M, \Theta) = O\big(j^{-\ell'}\big) \quad \text{for some} \quad \ell' > 3/2, \tag{1}$$

then the QMLE $\widehat{\theta}_n$ *is asymptotically normal, i.e., there exists matrix* $F(\theta_0)^{-1}$ *and* $G(\theta_0)$ *such that*

$$\sqrt{n}\big(\widehat{\theta}_n - \theta_0\big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_d\big(0\ ,\ F(\theta_0)^{-1} G(\theta_0) F(\theta_0)^{-1}\big). \tag{2}$$

- Could be applied to all cited processes ARMA, ARCH, APARCH,...
- But requires $r \geq 4$ and not very robust.

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Laplacian QMLE of causal affine model

$$X_t = M_{\theta_0}(X_{t-1}, X_{t-2}, \ldots)\,\xi_t + f_{\theta_0}(X_{t-1}, X_{t-2}, \ldots), \quad \forall\ t \in \mathbb{Z}$$

If $(\xi_t)_t$ standardized Laplacian i.r.v., *i.e.* $f_\xi(t) = \frac{1}{2}\exp(-|t|)$, then

$$\widehat{q}_t(\theta) = -\log|\widehat{M}_\theta^t| - |\widehat{M}_\theta^t|^{-1}|X_t - \widehat{f}_\theta^t|.$$

with $\widehat{f}_\theta^t = f_\theta(X_{t-1}, \ldots, X_1, 0, \cdots)$ and $\widehat{M}_\theta^t = M_\theta(X_{t-1}, \ldots, X_1, 0, \cdots)$

$\Longrightarrow$ Laplacian QMLE: $\widehat{\theta}_n = \arg\max_{\theta\in\Theta} \widehat{L}_n(\theta)$ with $\widehat{L}_n(\theta) = \sum_{t=1}^{n} \widehat{q}_t(\theta)$.

Strong consistency

Théorème (Bardet and Boularouk, 2016)

Assume $r \geq 1$, $\mathbb{E}(|\xi_0|) = 1$, $\Theta \subset \Theta(r)$, *conditions C1-3 and* $A_0(f, \Theta)$ *and* $A_0(M, \Theta)$ *with*

$$\alpha_j^{(0)}(f, \Theta) + \alpha_j^{(0)}(M, \Theta) = O\big(j^{-\ell}\big) \quad \text{for some} \quad \ell > \frac{2}{\min(r\,,\,2)},$$

then the QMLE $\widehat{\theta}_n$ *is strongly consistent, i.e.* $\widehat{\theta}_n \xrightarrow[n\to\infty]{a.s.} \theta_0$.

Sketch of the proof

The proof proceed in two (classical) steps.

1. $L(\theta) := \mathbb{E}[L_n(\theta)]$ has a unique maximum in θ_0.

2. A uniform law of large numbers on $(\widehat{q}_t)_{t\in\mathbb{N}^*}$ is established by proving

$$\frac{1}{n}\sup_{\theta\in\Theta}\left|\widehat{L}_n(\theta) - L(\theta)\right| \xrightarrow[n\to\infty]{a.s.} 0.$$

   This is obtained from Kounias and Weng (1969): $\exists s \in (0,1]$ such as

$$\sum_{t\geq 1}\frac{1}{t^s}\,\mathbb{E}\Big[\sup_{\theta\in\Theta}\left|\widehat{q}_t(\theta) - q_t(\theta)\right|^s\Big] < \infty.$$

Both those results lead to the strong consistency of $\widehat{\theta}_n$, Pfanzagl (1969).

Asymptotic normality

With
$$\begin{cases} \Gamma_F = \left(\mathbb{E}\left[(M^0_{\theta_0})^{-2} \left(\frac{\partial f^0_\theta}{\partial \theta_i} \right)_{\theta_0} \left(\frac{\partial f^0_\theta}{\partial \theta_j} \right)_{\theta_0} \right] \right)_{1\leq i,j\leq d} \\ \Gamma_M = \left(\mathbb{E}\left[\left(\frac{\partial \log(M^0_\theta)}{\partial \theta_i} \right)_{\theta_0} \left(\frac{\partial \log(M^0_\theta)}{\partial \theta_j} \right)_{\theta_0} \right] \right)_{1\leq i,j\leq d} \end{cases}$$

Théorème (Bardet and Boularouk, 2016)

Under conditions of SLLN, and if $r \geq 2$, if $\theta_0 \in \overset{\circ}{\Theta} \cap \Theta(2)$ and if $\mathbf{A}_k(K,\Theta)$ for $k = 1, 2$. Then, if the distribution function of ζ_0 is symmetric, $\mathcal{C}^1(\mathcal{V}_0)$ with derivative $g(0)$ and if Γ_F or Γ_M are definite positive symmetric matrix, then

$$\sqrt{n}\big(\widehat{\theta}_n - \theta_0\big) \overset{\mathcal{D}}{\underset{n\to\infty}{\longrightarrow}} \mathcal{N}_d\Big(0, \big(\Gamma_M + 2g(0)\,\Gamma_F\big)^{-1}\big((\sigma^2_\zeta - 1)\,\Gamma_M + \Gamma_F\big)\big(\Gamma_M + 2g(0)\,\Gamma_F\big)^{-1}\Big).$$

Comparison with existent results

- ARMA(p, q): Same results than in Davis and Dunsmuir (1997) for LAD estimator.
- ARCH(p): Same results than in Peng and Yao (2003).
- GARCH(p, q): Same results than in Berkes and Horvath (2004) and Franq and Zakoian (2013).
- ARCH(∞), APARCH(δ, p, q), ARMA-GARCH,...: New results.

Sketch of the proof

Let $v=\sqrt{n}(\theta-\theta_0)\in\mathbb{R}^d$. Maximizing $\widehat{L}_n(\theta)$ is equivalent to maximizing

$$\begin{aligned} W_n(v) &= -\sum_{t=1}^n \big(q_t(\theta_0+n^{-1/2}v)-q_t(\theta_0)\big)\\ &= \sum_{t=1}^n \log\Big(\frac{(M^t_{\theta_0+n^{-1/2}v})^{-1}}{(M^t_{\theta_0})^{-1}}\Big)+|X_t-f^t_{\theta_0}|\big((M^t_{\theta_0})^{-1}-(M^t_{\theta_0+n^{-1/2}v})^{-1}\big)\\ &\qquad +\sum_{t=1}^n (M^t_{\theta_0+n^{-1/2}v})^{-1}\big(|X_t-f^t_{\theta_0}|-\big|X_t-f^t_{\theta_0+n^{-1/2}v}\big|\big) \end{aligned}$$

Theorem (Extension of Theorem 2, Davis and Dunsmuir, 1997)

Let $(Z_t)_{t\in\mathbb{Z}}$ be i.i.d.r.v such as $Var(Z_0)=\sigma^2<\infty$ with symmetric distribution function $\mathcal{C}^1(\mathcal{V}_0)$ with derivative $g(0)$.
Denote $\mathcal{F}_t=\sigma(Z_t,Z_{t-1},\cdots)$ for $t\in\mathbb{Z}$ and let $(Y_t)_{t\in\mathbb{Z}}$ and $(V_t)_{t\in\mathbb{Z}}$ two stationary processes adapted to $(\mathcal{F}_t)_t$ and such as $\mathbb{E}\big[Y_0^2V_0^2\big]<\infty$. Then

$$\sum_{t=1}^n V_{t-1}\big(|Z_t-n^{-1/2}Y_{t-1}|-|Z_t|\big)\ \xrightarrow[n\to\infty]{\mathcal{D}}\ \mathcal{N}\Big(g(0)\,\mathbb{E}\big[V_0Y_0^2\big]\,,\,\mathbb{E}\big[V_0^2Y_0^2\big]\Big)$$

Simulation results

			$\mathcal{L}$		$\mathcal{N}$		t_3		$\mathcal{U}$		$\mathcal{M}$	
		n	$\widehat{\theta}_n^{GQL}$	$\widehat{\theta}_n^{LQL}$	$\widehat{\theta}_n^{GQL}$	$\widehat{\theta}_n^{LQL}$	$\widehat{\theta}_n^{GQL}$	$\widehat{\theta}_n^{LQL}$	$\widehat{\theta}_n^{GQL}$	$\widehat{\theta}_n^{LQL}$	$\widehat{\theta}_n^{GQL}$	$\widehat{\theta}_n^{LQL}$
ARMA(1,1)	θ	100	0.106	0.091	0.114	0.117	0.113	0.090	0.112	0.059	0.110	0.078
		1000	0.031	0.024	0.032	0.032	0.036	0.027	0.031	0.014	0.031	0.023
		5000	0.014	0.010	0.014	0.015	0.016	0.011	0.016	0.012	0.013	0.010
	ϕ	100	0.119	0.102	0.121	0.128	0.123	0.102	0.120	0.067	0.121	0.090
		1000	0.037	0.028	0.036	0.036	0.040	0.030	0.036	0.017	0.036	0.027
		5000	0.016	0.012	0.014	0.016	0.017	0.013	0.016	0.007	0.014	0.006
ARCH(1)	ω	100	0.068	0.061	0.048	0.049	0.254	0.085	0.035	0.025	0.062	0.052
		1000	0.020	0.018	0.015	0.015	0.134	0.049	0.011	0.016	0.036	0.018
		5000	0.010	0.009	0.006	0.006	0.115	0.044	0.005	0.015	0.031	0.008
	α	100	0.161	0.155	0.141	0.142	0.979	0.418	0.102	0.064	0.484	0.423
		1000	0.063	0.058	0.043	0.043	0.852	0.169	0.029	0.033	0.157	0.133
		5000	0.016	0.014	0.012	0.012	0.378	0.109	0.013	0.031	0.087	0.062
GARCH(1,1)	α_0	100	0.112	0.105	0.095	0.100	0.211	0.126	0.081	0.047	0.134	0.114
		1000	0.036	0.032	0.028	0.028	0.098	0.058	0.023	0.018	0.066	0.051
		5000	0.016	0.014	0.012	0.012	0.055	0.043	0.010	0.015	0.040	0.023
	α_1	100	0.162	0.157	0.149	0.150	0.453	0.364	0.115	0.070	0.507	0.429
		1000	0.061	0.056	0.449	0.449	0.333	0.150	0.030	0.033	0.160	0.136
		5000	0.029	0.026	0.020	0.020	0.193	0.095	0.013	0.030	0.086	0.058
	β	100	0.225	0.209	0.188	0.190	0.499	0.429	0.163	0.105	0.483	0.390
		1000	0.060	0.055	0.051	0.051	0.285	0.174	0.044	0.022	0.170	0.169
		5000	0.027	0.024	0.022	0.022	0.180	0.075	0.019	0.009	0.072	0.075

Simulation results

			$\mathcal{L}$		$\mathcal{N}$		t_3		$\mathcal{U}$		$\mathcal{M}$	
		n	$\widehat{\theta}_n^{GQL}$	$\widehat{\theta}_n^{LQL}$	$\widehat{\theta}_n^{GQL}$	$\widehat{\theta}_n^{LQL}$	$\widehat{\theta}_n^{GQL}$	$\widehat{\theta}_n^{LQL}$	$\widehat{\theta}_n^{GQL}$	$\widehat{\theta}_n^{LQL}$	$\widehat{\theta}_n^{GQL}$	$\widehat{\theta}_n^{LQL}$
ARMA$(1,1)$ -GARCH$(1,1)$	θ	100	0.120	0.097	0.107	0.107	0.121	0.098	0.097	0.067	0.123	0.087
		1000	0.035	0.024	0.028	0.028	0.048	0.029	0.024	0.015	0.035	0.026
		5000	0.016	0.010	0.012	0.012	0.023	0.012	0.015	0.011	0.011	0.007
	ϕ	100	0.135	0.109	0.117	0.119	0.141	0.116	0.110	0.077	0.132	0.102
		1000	0.044	0.030	0.033	0.033	0.063	0.035	0.029	0.023	0.053	0.046
		5000	0.020	0.014	0.015	0.015	0.029	0.015	0.013	0.012	0.019	0.014
	α_0	100	0.104	0.096	0.085	0.084	0.158	0.129	0.073	0.055	0.131	0.116
		1000	0.031	0.028	0.025	0.025	0.241	0.060	0.021	0.019	0.053	0.046
		5000	0.014	0.012	0.010	0.010	0.052	0.042	0.009	0.016	0.036	0.019
	α_1	100	0.179	0.177	0.166	0.167	0.469	0.385	0.134	0.107	0.494	0.405
		1000	0.064	0.060	0.045	0.045	0.328	0.161	0.031	0.046	0.160	0.137
		5000	0.031	0.027	0.020	0.020	0.182	0.096	0.013	0.038	0.090	0.062
	β	100	0.302	0.269	0.252	0.233	0.604	0.497	0.217	0.164	0.553	0.472
		1000	0.057	0.051	0.051	0.051	0.312	0.187	0.045	0.049	0.165	0.170
		5000	0.025	0.022	0.020	0.020	0.199	0.073	0.062	0.066	0.019	0.025
ARMA$(1,1)$ -APARCH$(1,1)$	θ	100	0.110	0.086	0.096	0.101	0.112	0.090	0.097	0.068	0.125	0.091
		1000	0.029	0.021	0.023	0.024	0.031	0.021	0.022	0.014	0.033	0.024
		5000	0.013	0.008	0.010	0.010	0.014	0.009	0.010	0.006	0.015	0.011
	ϕ	100	0.138	0.114	0.121	0.126	0.128	0.107	0.111	0.086	0.146	0.107
		1000	0.040	0.027	0.032	0.032	0.041	0.028	0.029	0.026	0.043	0.030
		5000	0.018	0.012	0.012	0.012	0.020	0.012	0.013	0.014	0.019	0.013
	ω	100	0.198	0.192	0.199	0.210	0.254	0.262	0.221	0.170	0.290	0.272
		1000	0.079	0.067	0.056	0.056	0.226	0.218	0.044	0.045	0.142	0.129
		5000	0.033	0.028	0.025	0.025	0.209	0.207	0.017	0.029	0.061	0.056
	α	100	0.206	0.201	0.183	0.184	0.464	0.449	0.167	0.131	0.352	0.327
		1000	0.060	0.053	0.041	0.041	0.447	0.432	0.029	0.043	0.143	0.134
		5000	0.025	0.023	0.018	0.018	0.421	0.414	0.012	0.027	0.071	0.059
	γ	100	0.413	0.386	0.346	0.356	0.439	0.426	0.310	0.233	0.613	0.601
		1000	0.105	0.094	0.071	0.070	0.101	0.092	0.057	0.041	0.217	0.220
		5000	0.042	0.039	0.029	0.029	0.045	0.038	0.024	0.018	0.086	0.089
	β	100	0.297	0.282	0.255	0.238	0.312	0.288	0.186	0.145	0.476	0.468
		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Outline

1. Causal affine models
   - Definition of causal affine time series
   - Existence and stationarity of causal affine models
2. Gaussian QMLE for causal affine models
   - Estimation and Gaussian QMLE
   - Gaussian QMLE of causal affine models
   - Asymptotic behavior of Gaussian QMLE
3. Laplacian QMLE
   - Limit theorems of the Laplacian QMLE
   - Results of simulation
4. Conclusion

Conclusion

Comparisons between Gaussian and Laplacian QMLE:

- Strong consistency obtained for $r = 1$ (Laplacian) vs $r = 2$ (Gaussian);
- Asympto. normality obtained for $r = 2$ (Laplacian) vs $r = 4$ (Gaussian);
- Simulations show more accurate and robust results for Laplacian than for Gaussian;
- But confidence intervals require the estimation of $g(0)$ and σ_ξ for Laplacian QMLE.