Various Approaches to the relative Fundamental group

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Contents

The Framework

Good reduction problem or, in general, describing the special fiber for a family may have several *shades*.

Suppose we have a family over a complex disk

$$f : X \to \Delta$$

where we assume that f is proper, flat, X is smooth and f is smooth outside the origin (suppose some control on the degeneration at 0....semistable). In this case how can we detect the shape of the special fiber? Of course the same can be done for a scheme which is defined over $\mathcal{V}$ a discrete valuation ring in mixed or equi characteristic.

In this talk we would like to address these kinds of problems.

Previous Results

Here the situation is $f : X \to \Delta$ i.e. in $\mathbb{C}$. Which kind of methods one would like to use? the relative cohomology groups $\mathbf{R}^i f_* \mathbb{C}_X$ give local systems on $\Delta \setminus \{0\}$, then one would like to study the action of monodromy around zero in them. In this setting several results have been obtained.

That the action on cohomology was not enough, in general,has been clear for long time now.

For abelian varieties cohomology and H^1 (in particular) was enough: we have the results of Grothendieck, Serre -Tate (ℓ), Coleman-Iovita (p)

In the arithmetic case: we study the action $Gal(\overline{K}/K)$ on the ℓ-adic etale cohomology of the generic fiber ($\ell \neq p$, where p is the characteristic of the residual field of $\mathcal{V}$).

The first cohomology group is, in general, linked to the abelianization of the fundamental group.

If we want deal with other relative varieties even with a family of curves? Here we have that it is not enough to study its first cohomology group i.e. the abelianization. In fact for the *classical geometric situation over* $\mathbb{C}$ we have

Theorem

*(**Oda 85**) Consider a family of curves of genus $g > 1$ with stable degeneration $f : X \to \Delta$. If the dual graph of the special fiber has* **loop** *then the monodromy action on the first cohomology group of the generic fiber X_t is non trivial. Moreover* **in the other cases** *the good reductions are observed by the action of monodromy rapresentation on the whole $\pi_1(X_t)$ (not only in its abelianization...H_1 or H^1.)*

As a matter of fact one needs to study the action of the monodromy on the quotient of the fundamental group up to **the fourth level of the lower central series**!! Hence the *unipotent fundamental group* ! Of course Oda used trascendental methods to prove such a theorem, and he applied this to the the arithmetic case i.e. If $\mathcal{V}$ is a complete discrete valuation ring in mixed ch. p and K its fraction field:

Proposition

Take a proper curve X over $\mathcal{V}$, suppose that X_K has genus $g \geq 2$, then X_K has good reduction if and if for some (all) prime integer $\ell \neq p$ the outer representations $\pi_1^{(\ell)}/\pi_1^{(\ell)}[n]$ (on the lower central series) are unramified for all $n \geq 1$. In fact it is sufficient up to level 4.

he translated the problem by deformation in a stable complex family hence to his previous result.

Later Andreatta, Iovita and Kim proved a p-adic analogue of the previous result: a criterium for the p-adic etale cohomology of the generic fiber

They proved the fact by p-adic Hodge theory. They first construct an universal unipotent object as in Hadian to obtain the relative fundamental group: such a object obeys to p-adic hodge formalism. In this way they can transform the problem in a (log-)-crystalline framework where they can associate the monodromy to a Gauss-Manin connection in a deformation. Being over curves, the deformations are un-obstructed: then they can reduce the problem to Oda's complex case.In the end they use Oda's trascendental methods.

In our joint work with **Di Proietto** and **Shiho** we have a strategy to obtain Oda's results in the complex case (a family of curves) but without using trascendental methods/topological methods. All in algebraic terms: hence to obtain Andreatta-Iovita-Kim without using trascendental methods.

We will consider the special fiber of a family as a log-scheme. If we have a semistable/stable reduction we will consider **the central fiber over the punctured point** as a log-smooth scheme over a point which is not anymore trivial log-scheme....hence **we work in log-relative terms**. we will need to have at hands several kinds of definitions of fundamental group....

For a morphism $h : Y \to Z$ of fine log schemes of finite type, we denote the category of locally free $\mathcal{O}_Y$-modules of finite rank with integrable connection by $\mathrm{MIC}(Y/Z)$

For an object (E, ∇) in $\mathrm{MIC}(Y/Z)$ and a geometric point y of Y, the fiber $\nabla : E \longrightarrow E \otimes_{\mathcal{O}_Y} \Omega^1_{Y/Z}$ induces a $k(y)$-linear map $\nabla_y : E|_y \longrightarrow E|_y \otimes_{\mathbb{Z}} \mathcal{M}^{\mathrm{gp}}_{Y,y}/\mathcal{M}^{\mathrm{gp}}_{Z,h(y)}$, where $E|_y := E \otimes_{\mathcal{O}_Y} k(y)$. For a $\mathbb{Z}$-linear map $\psi : \mathcal{M}^{\mathrm{gp}}_{Y,y}/\mathcal{M}^{\mathrm{gp}}_{Z,h(y)} \longrightarrow \mathbb{Z}$, the composite $\psi \circ \nabla_y : E|_y \longrightarrow E|_y$ is called the residue map of (E, ∇) at y along ψ. We say that (E, ∇) has **nilpotent residue** if, for any y and any $\mathbb{Z}$-linear map ψ as above, the residue map $\psi \circ \nabla_y$ is **nilpotent as endomorphism** on $E|_y$.

$\mathrm{MIC}^{\mathrm{n}}(Y/Z)$: the full subcategory of $\mathrm{MIC}(Y/Z)$ which consists of objects having nilpotent residue.
We need to linearize the connections: this is done via stratification. We have such a notion also in log-terms. We denote the category of locally free $\mathcal{O}_Y$-modules of finite rank (resp. coherent $\mathcal{O}_Y$-modules) endowed with stratification (with respect to h) by $\mathrm{Str}(Y/Z)$ (resp. $\widetilde{\mathrm{Str}}(Y/Z)$).

We may also define the log **infinitesimal site** $(Y/Z)_{\mathrm{inf}}$ and **stratifying site** $(Y/Z)_{\mathrm{str}}$ of Y/Z: the nilpotent closed immersions are exact with respect the log-structure.

We call a crystal E on $(Y/Z)_*$ $(* \in \{\mathrm{inf}, \mathrm{str}\})$ locally free if E_T is a locally free $\mathcal{O}_T$-module of finite rank for any (U, T, i). We denote the category of locally free crystals on $(Y/Z)_*$ by $\mathrm{Crys}((Y/Z)_*)$.

We have the following relations among integrable log connection, stratifications and crystals.

Proposition

Let k, $h : Y \longrightarrow Z$, be as before and $(U, T, i) \in (Y/Z)_*$ $(* \in \{\mathrm{inf}, \mathrm{str}\})$. Assume that $U = Y$ and that $\Omega^1_{T/Z}$ is a locally free $\mathcal{O}_T$-module of finite rank. Then we have equivalences of categories

$$\mathrm{Crys}((Y/Z)_{\mathrm{str}}) \xrightarrow{\cong} \mathrm{Str}(T/Z) \xrightarrow{\cong} \mathrm{MIC}(T/Z), \tag{1}$$

(2)

Furthermore, when $T \longrightarrow Z$ is log smooth, for $* = \mathrm{inf}$, we have the equivalences of categories

$$\mathrm{Crys}((Y/Z)_{\mathrm{inf}}) \xrightarrow{\cong} \mathrm{Str}(T/Z) \tag{3}$$

Here it is the situation we want to deal with. **Let k be a field of characteristic zero** and consider the diagram of fine log schemes

$$X \underset{\iota}{\overset{f}{\rightleftarrows}} S \xrightarrow{g} \operatorname{Spec} k \tag{4}$$

satisfying the following conditions.

(A) f is a proper log smooth integral morphism and g is a separated morphism of finite type. ι is a section of f.

(B) $\Omega^1_{S/k}$ is a locally free $\mathcal{O}_S$-module of finite rank.

Later, for our application to Oda's result: S will be the punctured point: i.e. $\operatorname{Spec} k$ endowed with the canonical log-structure whose chart $\mathbb{N}$ is $1 \to 0$.

Then, in this framework,

$$\mathrm{MIC}(S/k) \cong \mathrm{Str}(S/k) \tag{5}$$

Because $\Omega^1_{X/k}$ is locally free, there exist equivalences

$$\mathrm{MIC}(X/k) \cong \mathrm{Crys}((X/k)_{\mathrm{str}}), \tag{6}$$

Also, since f is log smooth, there exist equivalences

$$\mathrm{MIC}(X/S) \cong \mathrm{Crys}((X/S)_{\mathrm{inf}}) \tag{7}$$

On each step of $X/S/\mathrm{Spec}\, k$ we may **introduce different** *differential structures*. We want to linearize objects in $\mathrm{MIC}(X/k)$. I.e. objects in X/S but with more structure.
In fact we introduce $\mathrm{StrCrys}(X/k)$. Here we want to put a structure on X/S and S/k. This is done as follows

Definition

Let the notations be as above. Then we define $\mathrm{StrCrys}(X/k)$ *to be the category of pairs* $(E, \{\epsilon^m\}_m)$, *where* $E \in \mathrm{Crys}((X/S)_{\mathrm{inf}})$ *and* $\{\epsilon^m : p_2^{m*}E \longrightarrow p_1^{m*}E\}_m$ *is a compatible family of isomorphisms in* $\mathrm{Crys}((X/S^m(1))_{\mathrm{inf}})$ $(m \in \mathbb{N})$ *with* $\epsilon^0 = \mathrm{id}$ *and* $p_{1,3}^{m}{}^*(\epsilon^m) = p_{1,2}^{m}{}^*(\epsilon^m) \circ p_{2,3}^{m}{}^*(\epsilon^m)$.

For $m, j \in \mathbb{N}$, we have $S^m(j)$ to be the m-th log infinitesimal neighborhood of S in $S \times_k \cdots \times_k S$ ($(j+1)$ times). Let $p_j^m : S^m(1) \longrightarrow S\,(j = 1, 2)$, $p_{j,j'}^m : S^m(2) \longrightarrow S^m(1)\,(1 \leq j < j' \leq 3)$ be projections. For X/S we have a crystal and for the part S/k we have the linearized part.

We have "linearized" our objects.

Proposition

Let the notations be as above. Then we have equivalences of categories $\mathrm{MIC}(X/k) \cong \mathrm{StrCrys}(X/k)$.

We want to introduce the "Gauss-Manin" connection in our framework: with a more crystalline flavour. Let (E, ∇) be an object in $\mathrm{MIC}(X/k)$. Now we use our interpretation in terms of $\mathrm{StrCrys}(X/k)$. Thanks to our hypotheses (B) we know that p_j^m is flat then **we can construct isomorphisms** between $p_1^{m*}R^i f_{\mathrm{dR}*}\overline{E}$ and $p_2^{m*}R^i f_{\mathrm{dR}*}\overline{E}$

then defining an object in $\widetilde{\mathrm{Str}}(S/k) \cong \widetilde{\mathrm{MIC}}(S/k)$ (the coherent ..... they are not locally free, a priori). But they have a log-connection...
This is our definition of the Gauss-Manin connection on relative de Rham cohomology.

Note that our definition of Gauss-Manin connection here, which has crystalline flavor, is not a priori the same as the usual Katz-Oda

Berthelot proved in [LNM 407, V Proposition 3.6.4] the equivalence of two definitions when the base scheme is killed by a power of some prime number p. We can prove the coincidence of two definitions in certain cases, partly using Berthelot's result. We expect that the two definitions coincide in general, but we don't have a proof.

In the sequel, we assume the following conditions on the morphism f .

(C) For any i, $R^i f_{\mathrm{dR}*}(\mathcal{O}_X, d)$ (endowed with Gauss-Manin connection) belongs to $\mathrm{MIC}^{\mathrm{n}}(S/k)$.

(D) $f_{\mathrm{dR}*}(\mathcal{O}_X, d) = (\mathcal{O}_S, d)$, $g_{\mathrm{dR}*}(\mathcal{O}_S, d) = k$.

Let

$$\mathrm{N}_f\mathrm{MIC}(X/k) \quad (\textit{resp}.\mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k))$$

be the full subcategory of $\mathrm{MIC}(X/k)$ consisting of iterated extensions of objects in $f_{\mathrm{dR}}^*\mathrm{MIC}(S/k)$ (resp. $f_{\mathrm{dR}}^*\mathrm{MIC}^{\mathrm{n}}(S/k)$): our **unipotent objects**.

Note that, since $\mathrm{MIC}(S/k)$ is not necessarily in general an abelian category due to the possible existence of non-trivial log structure on S.

The linearized contre-part:

Definition

Let the notations be as above.
(1) *Let* $\mathrm{N}_f\mathrm{StrCrys}(X/k)$ *be the full subcategory of* $\mathrm{StrCrys}(X/k)$ *consisting of the objects which are iterated extensions of objects in* $f^*\mathrm{Str}(S/k)$.
(2) *Let* $\mathrm{N}_f\mathrm{StrCrys}^{\mathrm{n}}(X/k)$ *be the full subcategory of* $\mathrm{N}_f\mathrm{StrCrys}(X/k)$ *consisting of the objects which are iterated extension of objects in the essential image of*

$$\mathrm{MIC}^{\mathrm{n}}(S/k) \hookrightarrow \mathrm{MIC}(S/k) \cong \mathrm{Str}(S/k) \xrightarrow{f^*} \mathrm{StrCrys}(X/k).$$

It follows from the definition that

$$\mathrm{N}_f\mathrm{StrCrys}(X/k) \cong \mathrm{N}_f\mathrm{MIC}(X/k)$$

(resp. $\mathrm{N}_f\mathrm{StrCrys}^{\mathrm{n}}(X/k) \cong \mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k)$) via the equivalence $\mathrm{MIC}(X/k) \cong \mathrm{StrCrys}(X/k)$.

We fix a diagramm as before satisfying the conditions $(\mathrm{A}), (\mathrm{B}), (\mathrm{C}), (\mathrm{D})$ above and

(E) $\mathrm{MIC^n}(S/k)$ form an abelian category (morphisms of connection)

Remember:

$$X \underset{\iota}{\overset{f}{\rightleftarrows}} S \xrightarrow{g} \operatorname{Spec} k \tag{8}$$

Also, let $s \hookrightarrow S$ be an exact closed immersion over k of a closed point s. We denote the composite $\iota \circ s$ by $x : s \hookrightarrow X$. We denote the fiber product $s \times_S X$ by X_s and the projection $X_s \longrightarrow s$ by f_s. Then x induces the closed immersion $s \hookrightarrow X_s$, which we denote also by x. Here we choose a point in the fiber X_s.....

After our hypotheses: $\mathrm{MIC^n}(S/k)$ is actually a neutral Tannakian category. By our assumptions (D) and (E) we have that $\mathrm{N}_f\mathrm{MIC^n}(X/k)$ **is a neutral Tannakian category.**

Then we can consider their analogous absolute terms: $\mathrm{N}_{f_s}\mathrm{MIC}(X_s/s)$ a neutral Tannakian category.

Relative Tannakian theory. First definition

In this section, we recall the definition of relatively unipotent log de Rham fundamental group following Lazda which, in turn, is based on the theory of Tannakian category due to Deligne.

Notation as before. We have the functors between Tannakian categories

$$f_{\mathrm{dR}}^{*} : \mathrm{MIC}^{\mathrm{n}}(S/k) \longrightarrow \mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k), \quad \iota_{\mathrm{dR}}^{*} : \mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k) \longrightarrow \mathrm{MIC}^{\mathrm{n}}(S/k).$$

we obtain the first definition of relatively unipotent de Rham fund. group, which is due to Lazda:

Definition

We define the relatively unipotent de Rham fundamental group $\pi_1^{\mathrm{dR}}(X/S,\iota)$ *by*

$$\pi_1^{\mathrm{dR}}(X/S,\iota) := G(\mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k), \iota_{\mathrm{dR}}^{*}).$$

This is a group scheme in $\mathrm{MIC}^{\mathrm{n}}(S/k)$.

The construction is linked to the restriction from X/k to S/k via ι: in fact $\iota_{\mathrm{dR}}^{*} \circ f_{\mathrm{dR}}^{*} = id$.

We can restrict to the absolute case in the sense that we can fix a point s.

We define $\pi_1^{\mathrm{dR}}(S,s)$, $\pi_1^{\mathrm{dR}}(X,x)$, $\pi_1^{\mathrm{dR}}(X_s,x)$ We have the split exact sequence

$$1 \longrightarrow s_{\mathrm{dR}}^{*}\pi_1^{\mathrm{dR}}(X/S,\iota) \longrightarrow \pi_1^{\mathrm{dR}}(X,x) \underset{\iota_*}{\overset{f_*}{\rightleftarrows}} \pi_1^{\mathrm{dR}}(S,s) \longrightarrow 1,$$

(Di Proietto/Shiho).

The morphism $\pi_1^{\mathrm{dR}}(X_s, x) \longrightarrow \pi_1^{\mathrm{dR}}(X, x)$ induced by the exact closed immersion $X_s \hookrightarrow X$ gives

$$\pi_1^{\mathrm{dR}}(X_s, x) \cong s_{\mathrm{dR}}^* \pi_1^{\mathrm{dR}}(X/S, \iota). \tag{9}$$

So we can replace $s_{\mathrm{dR}}^* \pi_1^{\mathrm{dR}}(X/S, \iota)$ with $\pi_1^{\mathrm{dR}}(X_s, x)$. This is due to Lazda when the log structures on X, S are trivial and S is smooth over k, and the same proof works also in our case.

This will be the monodromy action on the relative fundamental group. In fact from

$$1 \longrightarrow s_{\mathrm{dR}}^* \pi_1^{\mathrm{dR}}(X/S, \iota) \longrightarrow \pi_1^{\mathrm{dR}}(X, x) \underset{\iota_*}{\overset{f_*}{\rightleftarrows}} \pi_1^{\mathrm{dR}}(S, s) \longrightarrow 1,$$

when we will have S just one punctured point we will obtain the definition of monodromy.

Construction of Hadian, Andreatta-Iovita-Kim and Lazda. Second definition.

We give the second definition of relatively unipotent de Rham fundamental group following the construction of Hadian, Andreatta-Iovita-Kim and Lazda, and we prove the coincidence of it with the one given in the previous section. Their proof is based on the existence of an *"universal unipotent object"*.

There exists a projective system $(W, e) := \{(W_n, e_n)\}_{n\geq 1}$ consisting of
$W_n := (\overline{W}_n, \{\epsilon_n^m\}_m) \in \mathrm{N}_f\mathrm{StrCrys}^{\mathrm{n}}(X/k) \cong \mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k)$, and
A morphism $e_n : (\mathcal{O}_S, d) \longrightarrow \iota_{\mathrm{dR}}^* W_n$ in $\mathrm{MIC}^{\mathrm{n}}(S/k)$
which satisfies the following conditions:

$*$ For any $n \geq 2$, the morphism $f_{\mathrm{dR}*}(\overline{W}_{n-1}^{\vee}) \longrightarrow f_{\mathrm{dR}*}(\overline{W}_n^{\vee})$ induced by the **transition map** $\overline{W}_n \longrightarrow \overline{W}_{n-1}$ is an isomorphism.

$**$ For any $E \in \mathrm{N}_f\mathrm{MIC}(X/S)$ **of index of unipotence** $\leq n$ and a morphism $v : \mathcal{O}_S \longrightarrow \iota_{\mathrm{dR}}^* E$ in $\mathrm{MIC}(S/S)$, **there exists a unique morphism** $\varphi : \overline{W}_n \longrightarrow E$ in $\mathrm{N}_f\mathrm{MIC}(X/S)$ with $\iota_{\mathrm{dR}}^*(\varphi) \circ \overline{e}_n = v$, where $\overline{e}_n : \mathcal{O}_S \longrightarrow \iota_{\mathrm{dR}}^* \overline{W}_n$ is the underlying morphism of e_n in $\mathrm{N}_f\mathrm{MIC}(S/S)$.

and we ask that the same universality holds after base change by any morphism $S' \longrightarrow S$ **of finite type**: this will allow us to put a stratification!!

$***$ For any n, **there exists an exact sequence**

$$0 \longrightarrow f^*_{\mathrm{dR}} R^1 f_{\mathrm{dR}*}(W_n^\vee)^\vee \longrightarrow W_{n+1} \longrightarrow W_n \longrightarrow 0 \qquad (10)$$

in $\mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k)$, where on $f^*_{\mathrm{dR}} R^1 f_{\mathrm{dR}*}(W_n^\vee)^\vee$ we put the log connection induced by the Gauss-Manin connection.

We prove that this new definition coincides with Lazda's one and Andreatta-Kim-Iovita's one. In the Lazda's case the equivalence can be rephrased to the claim that the two definitions of Gauss-Manin connections on $R^1 f_{\mathrm{dR}*}(W_n^\vee)$ are the same.

We give the second definition of relatively unipotent de Rham fundamental group $\pi_1(X/S, \iota)$ by using $W = \{(W_n, e_n)\}_n$.
We have that for any E

$$f_{\mathrm{dR}*}\mathcal{H}om(W, E) \xrightarrow{\cong} \iota^*_{\mathrm{dR}} E$$

Then we apply this to $E = W$ and to $E = W \widehat{\otimes} W$ (the projective system $\{W_m \otimes W_n\}_{m,n}$) to have a product and coproduct structure

We see that, with these structures, $(\iota^*_{\mathrm{dR}} W)^\vee := \varinjlim_n (\iota^*_{\mathrm{dR}} W_n)^\vee$ forms a commutative Hopf algebra object in the ind-category of $\mathrm{MIC}^{\mathrm{n}}(S/k)$.

Definition

Let the notations be as above. We define the relatively unipotent de Rham fundamental group $\pi_1^{\mathrm{dR}}(X/S,\iota)$ *by*

$$\pi_1^{\mathrm{dR}}(X/S,\iota) := \mathrm{Spec}\,(\iota_{\mathrm{dR}}^{*}W)^{\vee},$$

which is a group scheme in $\mathrm{MIC}^{\mathrm{n}}(S/k)$.

Theorem

The first and the second definitions of $\pi_1^{\mathrm{dR}}(X/S, \iota)$ *are canonically isomorphic.*

How to prove it?
Proof. We apply Hadian's method. For any E in $\mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k)$ (old fund. group). We have

$$f_{\mathrm{dR}*}\mathcal{H}om(W,W)\otimes f_{\mathrm{dR}*}\mathcal{H}om(W,E^\vee)\longrightarrow f_{\mathrm{dR}*}\mathcal{H}om(W,E^\vee)$$

and this corresponds to the map

$$\iota_{\mathrm{dR}}^*W\otimes\iota_{\mathrm{dR}}^*E^\vee\longrightarrow\iota_{\mathrm{dR}}^*E^\vee$$

This map defines the representation of $\pi_1^{\mathrm{dR}}(X/S,\iota)'$ (the new one) on ι_{dR}^*E in $\mathrm{MIC}^{\mathrm{n}}(S/k)$. Hence we have defined the functor

$$\mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k)\longrightarrow\mathrm{Rep}_{\mathrm{MIC}^{\mathrm{n}}(S/k)}(\pi_1^{\mathrm{dR}}(X/S,\iota)').....$$

Relative Minimal Model of Navarro-Aznar. Third definition.

We start with definition in the absolute case (Sullivan 1-minimal model), then the sheafication (Navarro-Aznar). We consider a **dgca**: it is a graded algebra $A = \bigoplus_{i=0}^{\infty} A^i$ endowed with a differential $d : A^i \longrightarrow A^{i+1}$ ($i \geq 0$) with $d \circ d = 0$ ..p, q.........

A Hirsch extension of an (augmented) R-dgca A is an (augmented) inclusion $A \hookrightarrow A \otimes \bigwedge(E)$ of (augmented) R-dgca's, where E is a free R-module homogeneous of degree 1, $\bigwedge(E)$ is the free graded commutative algebra generated by E

(in general it will be indicated as Hirsch extension of degree 1: but we will omit the term 'of degree 1' because we only need 1-degree)

An (augmented) R-dgca A is called $(1, q)$-minimal if there exists a sequence of Hirsch extensions

$$R = A(0) \subseteq A(1) \subseteq \cdots \subseteq A(q) = A. \tag{11}$$

An (augmented) R-dgca A is called 1-minimal if A is the union of $(1, q)$-minimal (augmented) R-dgca's ($q \in \mathbb{N}$) with respect to Hirsch extensions.

A $(1, q)$-minimal model of an (augmented) R-dgca B is an augmented morphism

$$\rho_q : M(q) \longrightarrow B$$

from a $(1, q)$-minimal (augmented) R-dgca $M(q)$ (endowed with filtration $\{M(q')\}_{q'=0}^{q}$ (as slides before), such that the induced maps $H^i(M(q')) \longrightarrow H^i(B)$ $(i = 0, 1)$ are isomorphisms for $1 \leq q' \leq q$ and the induced map $H^2(M(q' - 1), B) \longrightarrow H^2(M(q'), B)$ is zero for $2 \leq q' \leq q$. A 1-minimal model of an (augmented) R-dgca B is an augmented morphism

$$\rho : M \longrightarrow B$$

which is the union of $(1, q)$-minimal models $(q \in \mathbb{N})$ as above with respect to Hirsch extensions $M(q - 1) \subseteq M(q)$.

The main result is the following

Proposition

Let A, A' be augmented R-dgca's and assume given the following augmented diagram of augmented R-dgca's $A \to A'$ the $(1, q)$-minimal models $\rho : M(q) \to A$ and $\rho' : M(q)' \to A'$. Then there exists a unique morphism $\varphi : M(q) \longrightarrow M'(q)$ such that there exists a unique **Sullivan homotopy** $h : f \circ \rho \simeq_{\mathrm{Su}} \rho' \circ \varphi$. *Moreover, if f is a quasi-isomorphism, φ is an isomorphism.*

This unicity in the absolute case will allow us to contruct a sheaf version.But we will have to respect the Sullivan homotopy obstruction. We will need the notion of **ho-morphism**: morphisms which are compatible with the Sullivan homotopy.

We prove the sheaf (and stratifications) existence of $(1, q)$-minimal model and 1-minimal model along the line of Navarro-Aznar (Sullivan). Let $\mathcal{A}$ be a quasi-coherent sheaf of augmented $\mathcal{O}_S$-dgca's and with a structure of stratification on $H^i(\mathcal{A})$. **We assume moreover that $H^i(\mathcal{A})$ belongs to $\mathrm{MIC^n}(S/k)$.**

Theorem

Let the situation be as above. Then there exist unique (ho-morphism) $(1, q)$-minimal models

$$\rho : \mathcal{M}(q) \longrightarrow \mathcal{A},$$

Moreover there is structure of stratification on the degree i part $\mathcal{M}(q)^i$ of $\mathcal{M}(q)$. With respect to the structure of stratification defined before $\mathcal{M}(q)^i$ belongs to $\mathrm{MIC^n}(S/k)$.

We can put all together our previous $(1, q)$-minimal models and to get **a 1-minimal model**

$$\mathcal{M} = \bigcup_q \mathcal{M}(q)$$

on each $\mathcal{M}^i$ a structure of stratification and **it belongs to the ind-category of** $\mathrm{MIC}^{\mathrm{n}}(S/k)$**.**

Now we apply the previous corollary to our geometric situation getting the third definition. Let $X = \bigcup_{i \in I} U_i$ be a finite affine open covering of X. From the strict simplicial scheme $X_\bullet$, we have a morphism $\pi : X_\bullet \longrightarrow X$. Let $\pi_S : S_\bullet \longrightarrow S$ be the pull-back of π by ι. Then the section ι induces a surjective morphism of strict cosimplicial sheaves of $\mathcal{O}_S$-dgca's

$$\mathcal{A}^{\bullet,*} := f_* \pi_* \Omega^*_{X_\bullet/S} \xrightarrow{\iota^*} \pi_{S*} \mathcal{O}_{S_\bullet}.$$

By applying the **Thom-Whitney functor** $\mathbf{s}_{\mathrm{TW}}$ (it associates a dcga to a co-simplicial complex...) , we obtain the surjective morphism of sheaves of $\mathcal{O}_S$-dgca's $\mathbf{s}_{\mathrm{TW}}(\iota^*) : \mathbf{s}_{\mathrm{TW}}(\mathcal{A}^{\bullet,*}) \longrightarrow \mathbf{s}_{\mathrm{TW}}(\pi_{S*}\mathcal{O}_{S_\bullet})$ such that $H^0(\mathbf{s}_{\mathrm{TW}}(\pi_{S*}\mathcal{O}_{S_\bullet})) = \mathcal{O}_S$.

The sheaf before is not augmented.So:

Definition

We define the sheaf of $\mathcal{O}_S$-dgca's $\mathcal{A}_{X/S} \subseteq \mathbf{s}_{\mathrm{TW}}(\mathcal{A}^{\bullet,*})$ as the inverse image of $\mathcal{O}_S = H^0(\mathbf{s}_{\mathrm{TW}}(\pi_{S*}\mathcal{O}_{S_\bullet})) \hookrightarrow \mathbf{s}_{\mathrm{TW}}(\pi_{S*}\mathcal{O}_{S_\bullet})$ by $\mathbf{s}_{\mathrm{TW}}(\iota^*)$. Then, $\mathcal{A}_{X/S}$ is in fact a **sheaf of augmented $\mathcal{O}_S$-dgca's**.

We have functorial quasi-isomorphisms

$$\mathcal{A}_{X/S} \longrightarrow \mathbf{s}_{\mathrm{TW}}(\mathcal{A}^{\bullet,*}) \longrightarrow \mathbf{s}(\mathcal{A}^{\bullet,*})$$

But $H^i(\mathbf{s}(\mathcal{A}^{\bullet,*}))$ is nothing but $R^i f_{\mathrm{dR}*}(\mathcal{O}_X, d)$, we have the functorial isomorphism

$$H^i(\mathcal{A}_{X/S}) \xrightarrow{\cong} R^i f_{\mathrm{dR}*}(\mathcal{O}_X, d). \tag{12}$$

Using this construction one can define a stratification on $H^i(\mathcal{A}_{X/S})$: it belongs to $\mathrm{MIC}^{\mathrm{n}}(S/k)$.

Therefore there exists a unique 1-minimal model $M_{X/S}$ of $\mathcal{A}_{X/S}$ endowed with the stratification with respect to which $M^i_{X/S}$ ($i \in \mathbb{N}$) belongs to the ind-category of $\mathrm{MIC}^{\mathrm{n}}(S/k)$ (differentials $M^i_{X/S} \longrightarrow M^{i+1}_{X/S}$ are morphisms in the ind-category of $\mathrm{MIC}^{\mathrm{n}}(S/k)$).
In particular, **each $M^i_{X/S}$ is flat over $\mathcal{O}_S$**. Note that $M_{X/S}$ is the union of $(1, q)$-minimal models, which we denote by $M_{X/S}(q)$(and each $M_{X/S}(q)^i$ belongs to $\mathrm{MIC}^{\mathrm{n}}(S/k)$).

Now put $(M^1_{X/S})^\vee := \varprojlim_q (M_{X/S}(q)^1)^\vee$ and denote the dual isomorphism of the stratification by by ϵ^m. Then the pair $((M^1_{X/S})^\vee, \{\epsilon^m\})$ is an object in the pro-category of $\mathrm{MIC}^{\mathrm{n}}(S/k)$. The minus of the differential $-d : M^1_{X/S} \longrightarrow M^2_{X/S} = M^1_{X/S} \wedge M^1_{X/S}$ endows it with a structure of pro-nilpotent Lie algebra in the pro-category of $\mathrm{MIC}^{\mathrm{n}}(S/k)$. The sign "Minus" because we see it at "degree 0" while it was defined at level 1.

Definition (Third definition)

We define the relatively unipotent de Rham fundamental group $\pi_1^{\mathrm{dR}}(X/S, \iota)$ *as the pro-unipotent algebraic group in* $\mathrm{MIC^n}(S/k)$ *corresponding to the pro-nilpotent Lie algebra* $((M^1_{X/S})^\vee, \{\epsilon^m\})$.

Relative Bar construction. Fourth definition

We need a fourth definition: it gives the link between "tannakian" and "minimal model". The **bar construction** will give the link to the minimal model. Moreover it will be connected with a category of “connections” on our dgca $\mathcal{A}_{X/S}$ and then we will be able to introduce Tannakian formalism... ......closing in this way the circle.
These kinds of equivalences are known in the absolute case by the work of Bloch-Kriz/Hain and Terasoma : here we reduce to those cases by showing that we have base change in our constructions.

We start in the absolute case. Let R be a commutative ring and let M be an augmented R-dgca. Assume that, for any $i \in \mathbb{N}$, the degree i part M^i of M and the i-th cohomology $H^i(M)$ are flat R-modules and that $H^0(M) = R$. Let $\overline{M} := \mathrm{Ker}(M \longrightarrow R)$ be the augmentation ideal. For $s, t \in \mathbb{N}$, let $B^{-s,t}(M)$ **be the set of elements in** $\bigotimes_R^s \overline{M}$ **which are homogeneous of degree** t.

Then we define $B(M)$ as the single complex associated to $(B^{-s,t}(M), d_C, d_I)$

This construction has a structure! In particular for the 0-th cohomology

$$
\begin{aligned}
& i_H : R \longrightarrow H^0(B(M)), \quad e_H : H^0(B(M)) \longrightarrow R, \\
& \wedge_H : H^0(B(M)) \otimes_R H^0(B(M)) \longrightarrow H^0(B(M)), \\
& \Delta_H : H^0(B(M)) \longrightarrow H^0(B(M)) \otimes_R H^0(B(M))
\end{aligned} \tag{13}
$$

and we can check that $H^0(B(M)) := (H^0(B(M)), i_H, e_H, \wedge_H, \Delta_H)$ form **a Hopf algebra over R.**

Now: go back to our geometric situation.
We apply the sheafication of the previous to $M_{X/S}$

Definition (Fourth definition of relatively unipotent)

We define the relatively unipotent de Rham fundamental group $\pi_1^{\mathrm{dR}}(X/S, \iota)$ *by*

$$\pi_1^{\mathrm{dR}}(X/S, \iota) := \mathrm{Spec}\, H^0(B(M_{X/S})),$$

which is a pro-group scheme in $\mathrm{MIC}^{\mathrm{n}}(S/k)$*.*

Because we use $M_{X/S}$ we have also base change....flatness

Strategy of the equivalence: **a**) the first and fourth definitions are equivalent. **b**) The fourth and third are equivalent.

Let's see **b**):We prove that the pro-nilpotent Lie algebra in the pro-category of $\mathrm{MIC}^{\mathrm{n}}(S/k)$ associated to $\mathrm{Spec}\, H^0(B(M_{X/S}))$ is naturally isomorphic to $(M^1_{X/S})^\vee$. We put

$$QH^0(B(M_{X/S})) := \mathrm{Ker}(e_H)/\wedge_H (\mathrm{Ker}(e_H) \otimes \mathrm{Ker}(e_H)),$$

which is called the module of indecomposables of $H^0(B(M_{X/S}))$. Then it has a structure of a coLie algebra. On the other hand, there is a structure of coLie algebra on $M^1_{X/S}$. By base change we reduce to the case proven by Bloch-Kriz/Hain.

To prove **a**) **we compare Tannakian formalism and Bar construction on minimal model**. We will need to define the notion of connection on a dcga algebras This will be done for $\mathrm{MIC}(\mathcal{A}_{X/S})$ and $M_{X/S}$. But we will need also a linearization for the differential structure over S/k.

This again the linearization will play a role: a unipotent object will be iterated extensions of $Strat(S/k)$ in the category of connections over $\mathrm{MIC}(\mathcal{A}_{X/S})$ and $\mathrm{MIC}(M_{X/S})$: $\mathcal{E}$ and $\mathcal{E}_M$. Moreover $\mathrm{N}_f\mathcal{E}^{\mathrm{n}}$ (resp. $\mathrm{N}_f\mathcal{E}^{\mathrm{n}}_M$) will be the full subcategory of $\mathcal{E}$ (resp. $\mathcal{E}_M$) whose objects are iterated extensions of objects in $\mathrm{MIC}^{\mathrm{n}}(S/k)$ (in $Strat(S/k)$)

Proposition

$\mathrm{N}_f\mathcal{E}^{\mathrm{n}}$ *(resp.* $\mathrm{N}_f\mathcal{E}^{\mathrm{n}}_M$*) is a neutral Tannakian category.*

We can do again what we did for the tannakian formalism and we have base change. Then the two relative fundamental groups $G(\mathrm{N}_f\mathcal{E}^{\mathrm{n}}, \iota^*)$ and $G(\mathrm{N}_f\mathcal{E}^{\mathrm{n}}_M, \iota^*)$ in $\mathrm{MIC}^{\mathrm{n}}(S/k)$ are isomorphic

$$G(\mathrm{N}_f\mathcal{E}^{\mathrm{n}}, \iota^*) \xrightarrow{\cong} G(\mathrm{N}_f\mathcal{E}^{\mathrm{n}}_M, \iota^*)$$

We have base change:

Proposition

The morphism

$$G(\mathrm{N}_{f_s}\mathrm{MIC}(\mathcal{A}_{X_s/s}), x^*) \longrightarrow s^*_{\mathrm{dR}}G(\mathrm{N}_f\mathcal{E}^{\mathrm{n}}, \iota^*). \tag{14}$$

is an isomorphism.

The same base change holds for the 1-minimal model for the minimal model.

Now we compare the categories $\mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k)$ and $\mathrm{N}_f\mathcal{E}^{\mathrm{n}}$,

Proposition

There exists an equivalence of categories $\mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k) \xrightarrow{\cong} \mathrm{N}_f\mathcal{E}^{\mathrm{n}}$ *over* $\mathrm{MIC}^{\mathrm{n}}(S/k)$*. Equivalently, there exists an isomorphism*

$$G(\mathrm{N}_f\mathcal{E}^{\mathrm{n}}, \iota^*) \xrightarrow{\cong} \pi_1^{\mathrm{dR}}(X/S, \iota)$$

of group schemes in $\mathrm{MIC}^{\mathrm{n}}(S/k)$*, where* $\pi_1^{\mathrm{dR}}(X/S, \iota)$ *denotes the first definition of relatively unipotent de Rham fundamental group.*

Let $E := (E, \{\epsilon^m\})$ be an object in $\mathrm{N}_f\mathrm{MIC}^{\mathrm{n}}(X/k) \cong \mathrm{N}_f\mathrm{StrCrys}^{\mathrm{n}}(X/k)$. We move it to a covering and we apply Thom-Whitney functor $\mathbf{s}_{\mathrm{TW}}$...

Finally the next step of the proof is to compare the category $\mathrm{N}_f\mathcal{E}^{\mathrm{n}}_M$ and the category $\mathrm{Rep}_{\mathrm{MIC^n}(S/k)}(\mathrm{Spec}\, H^0(B(M_{X/S})))$ of representations of $\mathrm{Spec}\, H^0(B(M_{X/S}))$ in $\mathrm{MIC^n}(S/k)$.

Proposition

There exists an equivalence of categories

$$\mathrm{Rep}_{\mathrm{MIC^n}(S/k)}(\mathrm{Spec}\, H^0(B(M_{X/S}))) \xrightarrow{\cong} \mathrm{N}_f\mathcal{E}_M^{\mathrm{n}} \tag{15}$$

over $\mathrm{MIC^n}(S/k)$*. Equivalently, there exists an isomorphism*

$$G(\mathrm{N}_f\mathcal{E}_M^{\mathrm{n}}, \iota^*) \xrightarrow{\cong} \mathrm{Spec}\, H^0(B(M_{X/S})) \tag{16}$$

of group schemes in $\mathrm{MIC^n}(S/k)$*.*

The proof is inspired by the argument of Terasoma: he proved the theorem in the absolute case. In our case we use base change.

It is clear that for curves over the punctured point S it would be using the minimal model that we will have our "algebraic" method to handle monodromy. In fact in that case we will deal with

$$X \xrightarrow{f} S \xrightarrow{g} \operatorname{Spec} k \qquad X \xleftarrow{\iota} S \tag{17}$$

S is just a point and we are in the absolute case. The 1-minimal model will be linked to the log-cohomology vector spaces $H^1(X)$ and we will be in the absolute case i.e. with vector spaces and no sheaves. The monodromy will be linked to this exact sequence

$$1 \longrightarrow \pi_1^{\mathrm{dR}}(X_s, x) \longrightarrow \pi_1^{\mathrm{dR}}(X, x) \underset{\iota_*}{\overset{f_*}{\rightleftarrows}} \pi_1^{\mathrm{dR}}(S, s) \longrightarrow 1$$

by the action by conjugation of $\pi_1^{\mathrm{dR}}(S, s) = \mathbb{G}_a$ on $\pi_1^{\mathrm{dR}}(X_s, x)$.