Rank two root systems and maximal curves

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June 19, 2017

Hermitian, Suzuki and Ree curves

The curves are the Deligne-Lusztig varieties of dimension one.

- Lie groups and Lie algebras
  (˜1900: Lie, Engel, Cartan, Dickson)
  (˜1960: Chevalley, Steinberg, Suzuki, Ree, Tits)
- Deligne-Lusztig theory
- Function fields
  (Henn, Hansen and Stichtenoth, Hansen, Pedersen)
- Class field theory (Lauter)
- Elementary construction of the groups (Wilson)
- Plucker embeddings (Kane, Eid and D)
- Maximal covers (Giulietti and Korchmaros, Skabelund)

We give a description of the last three aspects in terms of the root systems A_2, B_2 and G_2.

Their function fields

In parentheses are the field sizes for which the function field is Hasse-Weil maximal.

$$H/\mathbb{F}_q \;:\; y^{q_0} - y = x^{q_0+1} \qquad q = q_0^2 \qquad (q, q^3, q^5, \ldots)$$

$$S/\mathbb{F}_q \;:\; y^q - y = x^{q_0}(x^q - x) \qquad q = 2q_0^2 \qquad (q^4, q^{12}, q^{20}, \ldots)$$

$$R/\mathbb{F}_q \;:\; \begin{cases} {y_2}^q - y_2 = x^{2q_0}(x^q - x) \\ {y_1}^q - y_1 = x^{q_0}(x^q - x) \end{cases} \qquad q = 3q_0^2 \qquad (q^6, q^{18}, q^{30}, \ldots)$$

The following covers $\tilde{X}/X$ are Hasse-Weil maximal.

$$\tilde{H}/\mathbb{F}_{q^3} \;:\; t^{q-q_0+1} = x^q - x \qquad \text{(Giulietti and Korchmaros 2008)}$$

$$\tilde{S}/\mathbb{F}_{q^4} \;:\; t^{q-2q_0+1} = x^q - x \qquad \text{(Skabelund 2016)}$$

$$\tilde{R}/\mathbb{F}_{q^6} \;:\; t^{q-3q_0+1} = x^q - x \qquad \text{(Skabelund 2016)}$$

Their smooth embeddings and automorphism groups

Projective line $\xrightarrow{(1\,:\,u)} \mathbb{P}^1$ $\quad A_1(q)$

Hermitian curve $\xrightarrow{(1\,:\,u\,:\,v)} \mathbb{P}^2$ $\quad {}^2A_2(q) \subset A_2(q)$

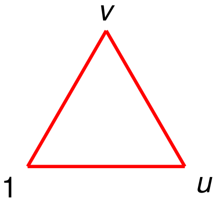
Suzuki curve $\xrightarrow{(1\,:\,x\,:\,-\,:\,z\,:\,w)} \mathbb{P}^4$ $\quad {}^2B_2(q) \subset B_2(q)$

Ree curve $\xrightarrow{(1\,:\,x\,:\,-\,:\,-\,:\,y\,:\,-\,:\,z\,:\,-\,:\,-\,:\,u\,:\,-\,:\,-\,:\,v\,:\,w)} \mathbb{P}^{13}$

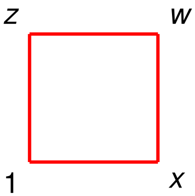
${}^2G_2(q) \subset G_2(q)$

Rotational symmetries of order 2, 3, 4 and 6

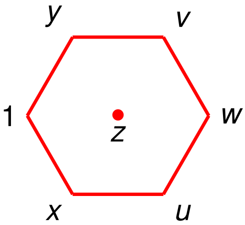
Projective line



Hermitian curve



Suzuki curve



Ree curve

From the root system A_1 to the group $SL(2, F)$

The single edge

has two directions, α and $-\alpha$.

Define

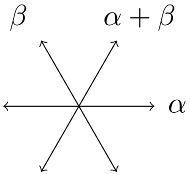
$$X_\alpha(t) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \quad \text{(directed edge } u \to 1 \text{ with weight } t\text{)}$$

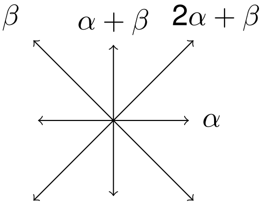
$$X_{-\alpha}(t) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \quad \text{(directed edge } 1 \to u \text{ with weight } t\text{)}$$

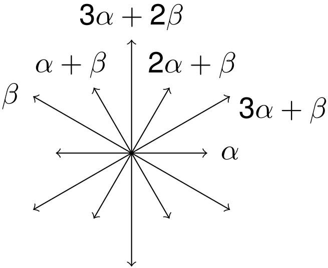
Then

$$SL(2, F) = \langle \exp X_\alpha(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \exp X_{-\alpha}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in F \rangle.$$

Root systems A_2, B_2 and G_2

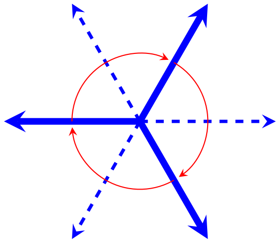






(6, 8 and 12 directions in the triangle, square, hexagon, resp.)

Lie Algebra A_2 of dimension 8



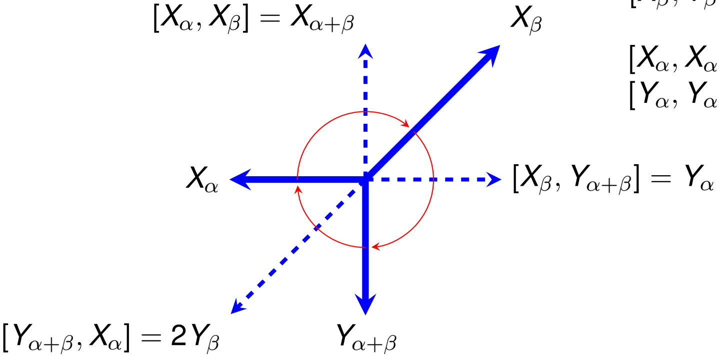
$$[X_\alpha, Y_\alpha] = H_\alpha$$
$$[X_\beta, Y_\beta] = H_\beta$$

The span $\langle X_\alpha, X_\beta, X_{\alpha+\beta}, Y_\alpha, Y_\beta, Y_{\alpha+\beta}, H_\alpha, H_\beta \rangle$ is closed under the operation $[A, B] = AB - BA$.

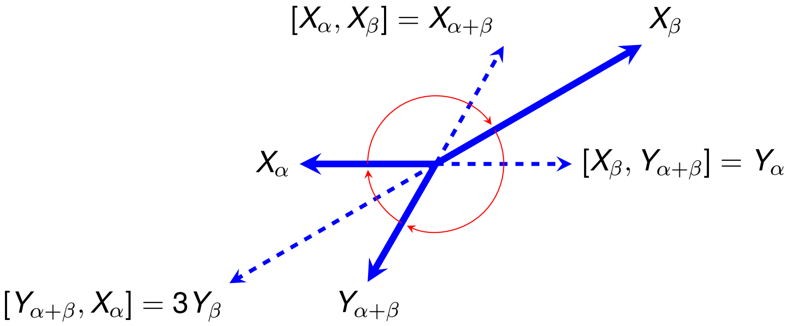
Lie Algebra B_2 of dimension 10

$$[X_\alpha, Y_\alpha] = H_\alpha$$
$$[X_\beta, Y_\beta] = H_\beta$$

$$[X_\alpha, X_{\alpha+\beta}] = 2X_{2\alpha+\beta}$$
$$[Y_\alpha, Y_{\alpha+\beta}] = 2Y_{2\alpha+\beta}$$



Lie Algebra G_2 of dimension 14



$$[X_\alpha, Y_\alpha] = H_\alpha \qquad [X_\alpha, X_{2\alpha+\beta}] = 3X_{3\alpha+\beta}$$
$$[X_\beta, Y_\beta] = H_\beta \qquad [Y_\alpha, Y_{2\alpha+\beta}] = 3Y_{3\alpha+\beta}$$

$$[X_\alpha, X_{\alpha+\beta}] = 2X_{2\alpha+\beta} \qquad [X_{3\alpha+\beta}, X_\beta] = -X_{3\alpha+2\beta}$$
$$[Y_\alpha, Y_{\alpha+\beta}] = 2Y_{2\alpha+\beta} \qquad [Y_{3\alpha+\beta}, Y_\beta] = -Y_{3\alpha+2\beta}$$

Hermitian curve

Equation $v^{q_0} + v + u^{q_0+1} = 0$ over $\mathbb{F}_q$, for $q = q_0^2$. Clearly

$$\begin{pmatrix} 1 & u & v \\ 1^q & u^q & v^q \end{pmatrix} \begin{pmatrix} v^{q_0} \\ u^{q_0} \\ 1^{q_0} \end{pmatrix} = 0$$

And thus, with $[a, b] = ab^q - a^q b$,

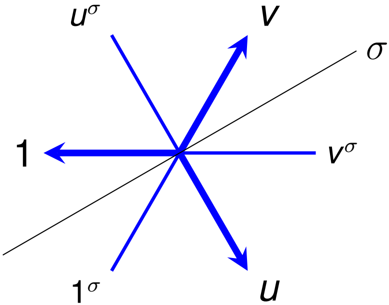
$$([1, u] : [v, 1] : [u, v]) = (1 : u : v)^{(q_0)}$$

The equation

$$([1, u], [v, 1], [u, v]) = t^{q-q_0+1}(1, u, v)^{(q_0)}$$

defines the Giulietti-Korchmaros cover $\tilde{H} = H(t)$.

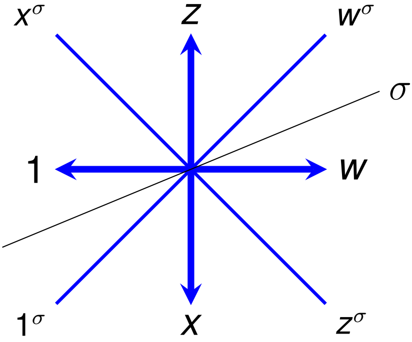
From Root system A_2 to Hermitian cover $\tilde{H}$



$$\begin{pmatrix} [1,u] \\ [u,v] \\ [v,1] \end{pmatrix} = t^{q+1-\sigma} \begin{pmatrix} 1^\sigma \\ v^\sigma \\ u^\sigma \end{pmatrix}$$

Root system A_2 with polarization σ defining the subgroup ${}^2A_2$
($\sigma : x \to x^{q_0}$)

From Root system B_2 to Suzuki cover $\tilde{S}$

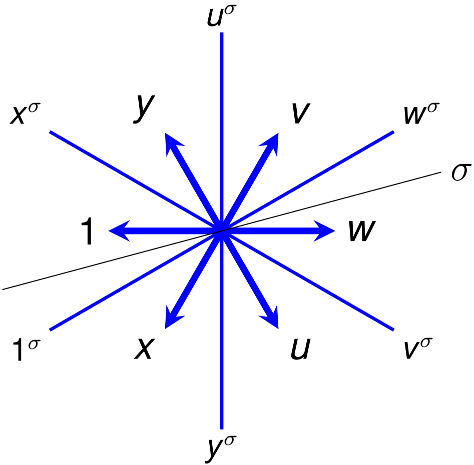


$$\begin{pmatrix} [1,x] \\ [w,z] \end{pmatrix} = t^{q+1-\sigma} \begin{pmatrix} 1^\sigma \\ w^\sigma \end{pmatrix}$$

$$\begin{pmatrix} [1,z] \\ [w,x] \end{pmatrix} = t^{q+1-\sigma} \begin{pmatrix} x^\sigma \\ z^\sigma \end{pmatrix}$$

Root system B_2 with polarization σ defining the subgroup ${}^2B_2$ ($\sigma : x \to x^{2q_0}$)

From Root system G_2 to Ree cover $\tilde{R}$



$$\begin{pmatrix} [1,x] \\ [u,w] \\ [v,y] \end{pmatrix} = t^{q+1-\sigma} \begin{pmatrix} 1^\sigma \\ v^\sigma \\ u^\sigma \end{pmatrix}$$

$$\begin{pmatrix} [1,y] \\ [v,w] \\ [u,x] \end{pmatrix} = t^{q+1-\sigma} \begin{pmatrix} x^\sigma \\ w^\sigma \\ y^\sigma \end{pmatrix}$$

Root system G_2 with polarization σ defining the subgroup ${}^2G_2$
$(\sigma : x \to x^{3q_0})$

Hermitian hypersurface for $\tilde{S}$

Lemma

For the cover $\tilde{S}$ of the Suzuki curve,

$$w^{q^2} + w + xz^{q^2} + x^{q^2}z = t^{q^2+1}.$$

Proof. Let

$$\Delta(1, x, z) = \begin{vmatrix} 1 & 1 & 1 \\ x & x^q & x^{q^2} \\ z & z^q & z^{q^2} \end{vmatrix}$$

Using $[x, z] = [1, w]$,

$$\Delta = w^{q^2} + w + xz^{q^2} + x^{q^2}z.$$

Using $[1, z] = x^\sigma [1, x]$,

$$\Delta = [1, x]^{q+\sigma+1} = t^{q^2+1}.$$

Dickson's description of $G_2(q)$

The full group $G_2(q)$ is defined in Dickson (1901) as a linear group acting on the variety defined by the quadric

$$z^2 + w + vx + yu = 0$$

and the relations

$$\begin{pmatrix} [1,u] \\ [u,v] \\ [v,1] \end{pmatrix} = \begin{pmatrix} [z,x] \\ [z,w] \\ [z,y] \end{pmatrix}, \quad \begin{pmatrix} [w,y] \\ [y,x] \\ [x,w] \end{pmatrix} = \begin{pmatrix} [z,v] \\ [z,1] \\ [z,u] \end{pmatrix}.$$

Where $[u,v] = \det((u,v),(u,v)^{(q)}) = uv^q - u^q v$.

The previous equations imply

$$[1,w] + [v,x] + [u,y] = 0$$

${}^2G_2(q)$

The subgroup ${}^2G_2(q) \subset G_2(q)$ acts on

$$z^2 + w + vx + uy = 0$$

$$\begin{pmatrix} [1,u] \\ [u,v] \\ [v,1] \end{pmatrix} = \begin{pmatrix} [z,x] \\ [z,w] \\ [z,y] \end{pmatrix}. \quad \begin{pmatrix} [w,y] \\ [y,x] \\ [x,w] \end{pmatrix} = \begin{pmatrix} [z,v] \\ [z,1] \\ [z,u] \end{pmatrix}$$

$$[1,w] + [v,x] + [u,y] = 0.$$

$$\begin{pmatrix} [u,w] \\ [v,y] \\ [1,x] \end{pmatrix} = t^{q+1-\sigma} \begin{pmatrix} v^\sigma \\ u^\sigma \\ 1^\sigma \end{pmatrix} \quad \begin{pmatrix} [1,y] \\ [u,x] \\ [v,w] \end{pmatrix} = t^{q+1-\sigma} \begin{pmatrix} x^\sigma \\ y^\sigma \\ w^\sigma \end{pmatrix}$$

Hermitian hypersurface for $\tilde{R}$

Lemma

For the cover $\tilde{R}$ of the Ree curve,

$$w^{q^3} + w + vx^{q^3} + x^{q^3}v + uy^{q^3} + u^{q^3}y - z^{q^3+1} = t^{q^3+1}.$$

Proof. Similar to the previous, let

$$\Delta(1, x, y, z) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & x^q & x^{q^2} & x^{q^3} \\ y & y^q & y^{q^2} & y^{q^3} \\ z & z^q & z^{q^2} & z^{q^3} \end{vmatrix}$$

Using Dickson's equations for $G_2(q)$,

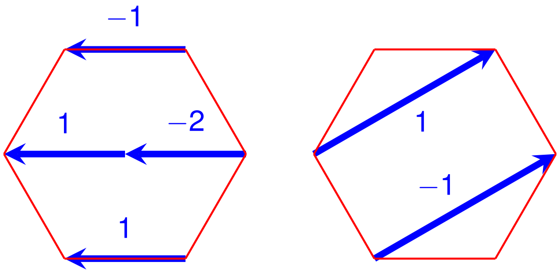
$$\Delta = w^{q^3} + w + vx^{q^3} + x^{q^3}v + uy^{q^3} + u^{q^3}y - z^{q^3+1}$$

Using the polarization relations for ${}^2G_2$,

$$\Delta = [1, x]^{(q+1)(q+\sigma+1)} = t^{q^3+1}.$$

The Lie Algebra G_2

Operators X_α (short root) and X_β (long root). Other operators by conjugation (rotation).



The corresponding action of $x_\alpha(t) = \exp X_\alpha(t)$ and $x_\beta(t) = \exp X_\beta(t)$ is given by the automorphisms (of Dickson's G_2 invariant variety)

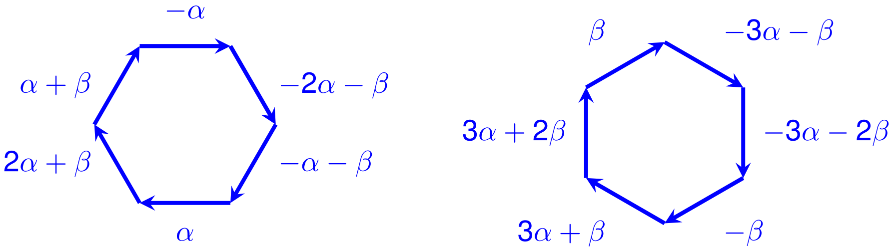
$$(1, x, y, z, u, v, w) \to (1, x, y, z + t, u + tx, v - ty, w - 2tz - t^2)$$

and

$$(1, x, y, z, u, v, w) \to (1 + tv, x - tw, y, z, u, v, w)$$

Root system G_2 and the Steinberg automorphism σ

The twelve directions in the hexagon divide into six short roots (left) and six long roots (right).



σ maps short roots $\alpha(t) \to -\beta(t^{3q_0})$ and long roots $\beta(t) \to -\alpha(t^{q_0})$.
So that $\sigma^2(t) = t^{3q_0^2} = t^q$.

The remaining coordinate functions

$$\text{Ree curve} \xrightarrow{(1\,:\,x\,:\,-\,:\,-\,:\,y\,:\,-\,:\,z\,:\,-\,:\,-\,:\,u\,:\,-\,:\,-\,:\,v\,:\,w)} \mathbb{P}^{13}$$

Lemma

The functions of type $[1, z]$ *satisfy*

$$[1, z]^3 = \begin{pmatrix} [1, y] & [1, x] \end{pmatrix} \begin{pmatrix} [u, x] & [y, u] \\ [v, x] & [y, v] \end{pmatrix} \begin{pmatrix} [1, y] \\ [1, x] \end{pmatrix}$$
$$= [1, x]^3(-u + x^2 y + xz)^\sigma.$$

Thank you.