Counting quartic extensions of $\mathbb{F}_q(t)$

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(joint ongoing work with Yongqiang Zhao)

AGC^{2}T-16 conference
June 23, 2017

Counting number fields of a given degree

Theorem (Hermite): *For all $X \in \mathbf{R}$ there are only finitely many number fields K such that $|\Delta_K| \leq X$.*

Thus the quantity

$$N_d(X) = \#\,\{\text{ number fields } K \text{ such that } [K:\mathbf{Q}] = d \text{ and } |\Delta_K| \leq X \,\}\,/\cong$$

is well-defined.

Natural question: what are the asymptotics as $X \to \infty$?

Counting number fields of a given degree

Conjecture ("folklore", Linnik, Narkiewicz, Bhargava, …): *If $d \geq 2$ then for some $c_d > 0$ we have*

$$N_d(X) = c_d X + o(X).$$

Known cases: $N_2(X) = \frac{1}{\zeta(2)} X + o(X) = \frac{6}{\pi^2} X + o(X)$ (**Gauss, 1801**)

$N_3(X) = \frac{1}{3\zeta(3)} X + o(X)$ (**Davenport–Heilbronn, 1971**)

$N_4(X) = c_{\text{very ugly}} X + o(X)$ (**Baily, Cohen–Diaz y Diaz–Olivier, Wong, Bhargava, 2005**)

$N_5(X) = c_{\text{ugly}} X + o(X)$ (**Bhargava, 2010**)

Best known result for $d > 5$: $N_d(X) = O\left(X^{\exp(c\sqrt{\log d})}\right)$ (**Ellenberg-Venkatesh, 2007**)

Several refined statements available for $N_d(X, G)$ where $G = \mathrm{Gal}(K, \mathbf{Q}) \subseteq S_d$.

Counting number fields of a given degree

In $d = 2$ number fields are of the form

$$K = \mathbf{Q}(\sqrt{a})$$

for some squarefree integer a and

$$\Delta_K = \begin{cases} a & \text{if } a \equiv 1 \bmod 4, \\ 4a & \text{if not.} \end{cases}$$

So roughly $N_2(X)$ counts squarefree integers a for which $|a| \leq X$. Heuristic:

$$\frac{N_2(X)}{X} \sim \prod_p (1 - p^{-2}) = \frac{1}{\zeta(2)}.$$

Not hard to make argument precise (inclusion-exclusion sieve), yielding $N_2(X) = \frac{1}{\zeta(2)} X + \boldsymbol{O}(\sqrt{\boldsymbol{X}})$.

Counting number fields of a given degree

In the much harder case $d = 3$ the Davenport-Heilbronn proof does not come along with $O(\sqrt{X})$:

$$N_3(X) = \frac{1}{3\zeta(3)} X \; + \; \textbf{?}$$

Fung, Williams, 1990: Experiments show significantly fewer cubic fields K with $|\Delta_K| \leq X$.

"If it weren't a theorem, you might doubt it was true!" (quote **Ellenberg**)

Roberts, 2000: Heuristic predicting a large negative second term:

$$N_3(X) = \frac{1}{3\zeta(3)} X + \frac{4\left(1+\sqrt{3}\right)\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} + o\left(X^{5/6}\right) \approx 0.277X - 0.403X^{5/6} + o(X^{5/6})$$

based on Shintani zeta function $\xi_3(s)$ which counts cubic rings (and has poles at $s = 1$ and $s = 5/6$).

Now a theorem by **Taniguchi, Thorne, 2013** and **Bhargava, Shankar, Tsimerman, 2013**.

Counting number fields of a given degree

In case $d = 4$ we ask the same question:

$$N_4(X) = c_{\text{very ugly}}X \ + \ \textbf{?}$$

No theorems or precise conjectures publicly available, but the quartic Shintani zeta function $\xi_4(s)$ was shown to have poles at $s = 1$, $s = 5/6$ and $s = 3/4$ by **Yukie, 1993**.

From an analysis of $\mathrm{Res}_{5/6}(\xi_4)$ it is believed that this should imply a second term

$$\sim X^{5/6}$$

The role of $s = 3/4$ might be explained by tertiary terms of the form $\sim X^{3/4}$ and/or $\sim X^{3/4}\log X$, but this evidence appears to be less convincing.

For $d > 4$: no serious attempts yet.

The function field case

We switch to extensions of $\mathbf{F}_q(t)$:

$$N_d(X) = \#\left\{\text{field extensions } K \text{ of } \mathbf{F}_q(t) \text{ such that } \left[K\colon \mathbf{F}_q(t)\right] = d \text{ and } |\Delta_K| \leq X\right\} / \cong$$

Assumption: char $\mathbf{F}_q > d$ to avoid inseperable extensions, wild ramification, ...

We rewrite:

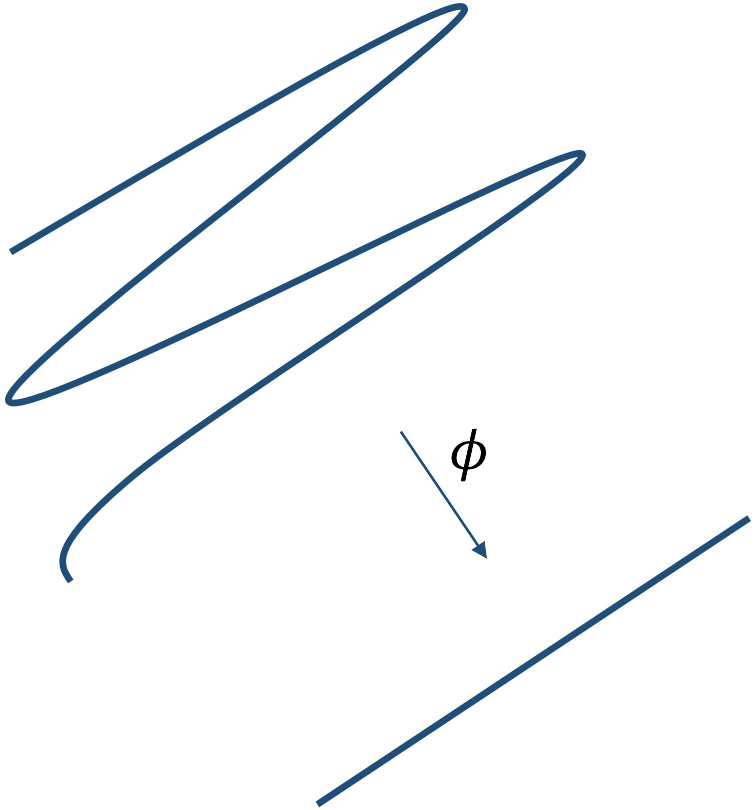
$$N_d(X) = \#\left\{\text{field extensions } K \text{ of } \mathbf{F}_q(t) \text{ such that } \left[K\colon \mathbf{F}_q(t)\right] = d \text{ and } q^{\deg \Delta_K} \leq X\right\} / \cong$$

Assuming $K \cap \mathbf{F}_q^{\text{alg.cl.}} = \mathbf{F}_q$, plus a sloppy use of Riemann-Hurwitz, we replace this by

$$\#\left\{\text{smooth proj. genus } g \text{ curves } C/\mathbf{F}_q \text{ together with a morphism } C \xrightarrow{d:1} \mathbf{P}^1 \text{ and } q^{2g} \leq X\right\} / \cong_{\mathbf{P}^1}$$

which could affect the leading constants, but should leave the asymptotics unharmed.

The function field case



Instead of $N_d(X)$ we will consider

$$T_d(q^{2g}) = \#\left\{ \text{genus } g \text{ curves } C/\mathbf{F}_q \text{ together with a morphism } \phi\colon C \xrightarrow{d:1} \mathbf{P}^1 \right\} / \cong_{\mathbf{P}^1}$$

which again should exhibit the same asymptotics.

Because of our transformations and future sloppinesses, we will not put effort in specifying leading constants.

The function field case

In $d = 2$ we count **hyperelliptic** curves $y^2 = f(t)$ with $f(t)$ squarefree of degree $2g + 1$ or $2g + 2$.

This corresponds to the fields $K = \mathbf{F}_q(\sqrt{f(t)})$.

Thus $T_2(q^{2g})$ roughly counts squarefree polynomials of a given degree. The same proof as in the number field case gives

$$T_2(q^{2g}) = \mathrm{c}_{2,q} q^{2g} + O(q^g)$$

for some constant $c_{2,q} > 0$.

This can be made much more precise.

The function field case

In $d = 3$ we are counting **trigonal** curves.

Theorem (Datkovsky-Wright, 1988): $T_3(q^{2g}) = c_{3,q} q^{2g} + o(q^{2g})$ *for some constant* $c_{3,q} > 0$.

(In fact they deal with any global field of characteristic at least 5.)

What about the secondary term?

Theorem (Zhao, 2013): $T_3(q^{2g}) = c_{3,q} q^{2g} - d_{3,q} q^{5g/3} + o(q^{5g/3})$ *for some constant* $d_{3,q} > 0$.

His proof gives a remarkable geometric interpretation for the second term in $X^{5/6} = q^{5g/3}$!

Overview of the remainder of this talk

We will:

- define the **Maroni invariants** of an algebraic curve,
- explain the idea behind Zhao's proof,
- define the **Schreyer invariants** of an algebraic curve,
- discuss a similar heuristic for the quartic case,
- wonder about number theoretic versions of these invariants.

Maroni invariants (= scrollar invariants)

A **rational normal scroll** of type $(e_1, e_2, \ldots, e_r)$ is an r-dimensional variety in

$$\mathbf{P}^N = \mathbf{P}^{e_1+e_2+\cdots+e_r+r-1}$$

swept out by simultaneously parameterizing rational normal curves $\mathbf{P}^1 \hookrightarrow \mathbf{P}^N$:

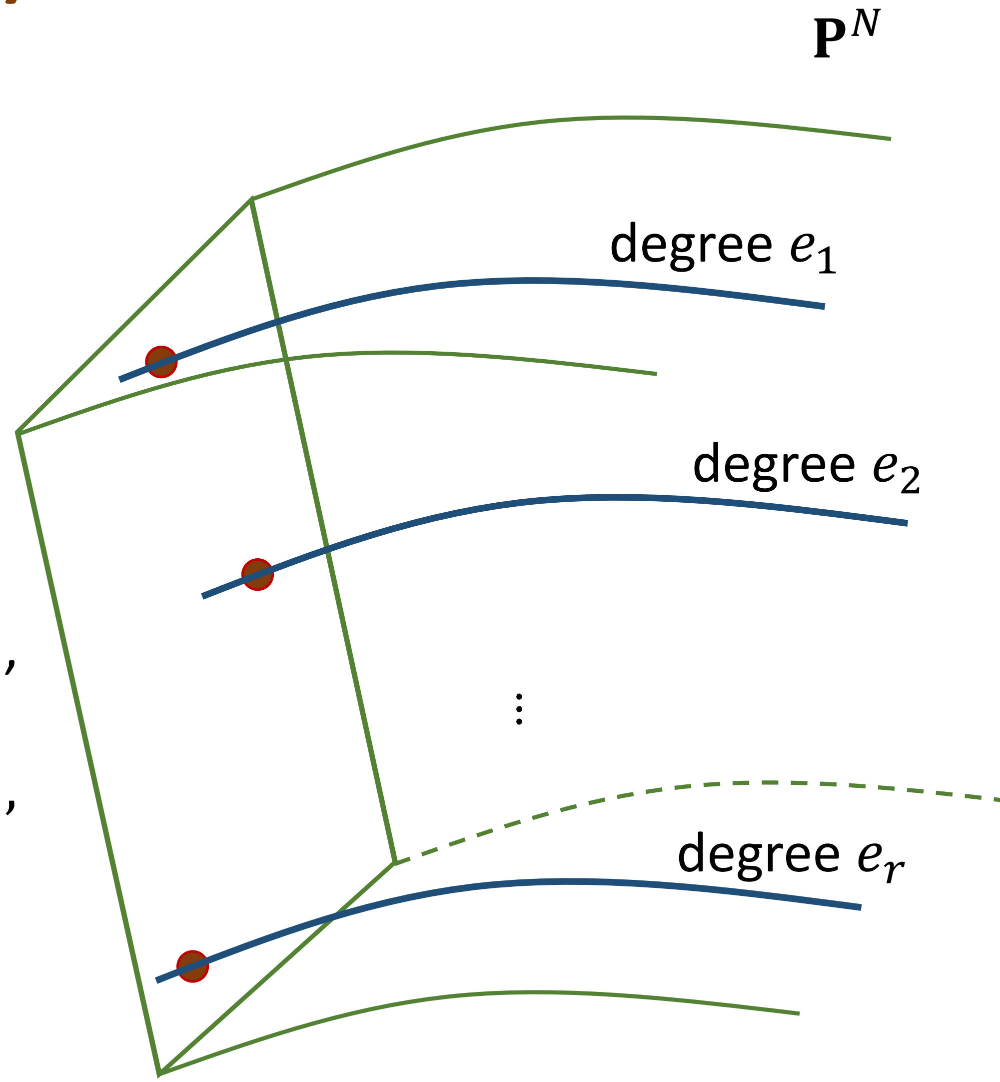
$$(s,t) \mapsto (s^{e_1} : s^{e_1-1}t : s^{e_1-2}t^2 : \ldots : t^{e_1} : 0 : 0 : \ldots : 0 : \ldots : 0 : 0 : \ldots : 0),$$

$$(s,t) \mapsto (0 : 0 : \ldots : 0 : s^{e_2} : s^{e_2-1}t : s^{e_2-2}t^2 : \ldots : t^{e_2} : \ldots : 0 : 0 : \ldots : 0),$$

$$\vdots$$

$$(s,t) \mapsto (0 : 0 : \ldots : 0 : 0 : 0 : \ldots : 0 : \ldots : s^{e_r} : s^{e_r-1}t : s^{e_r-2}t^2 : \ldots : t^{e_r}),$$

each time taking the linear span of the image points.

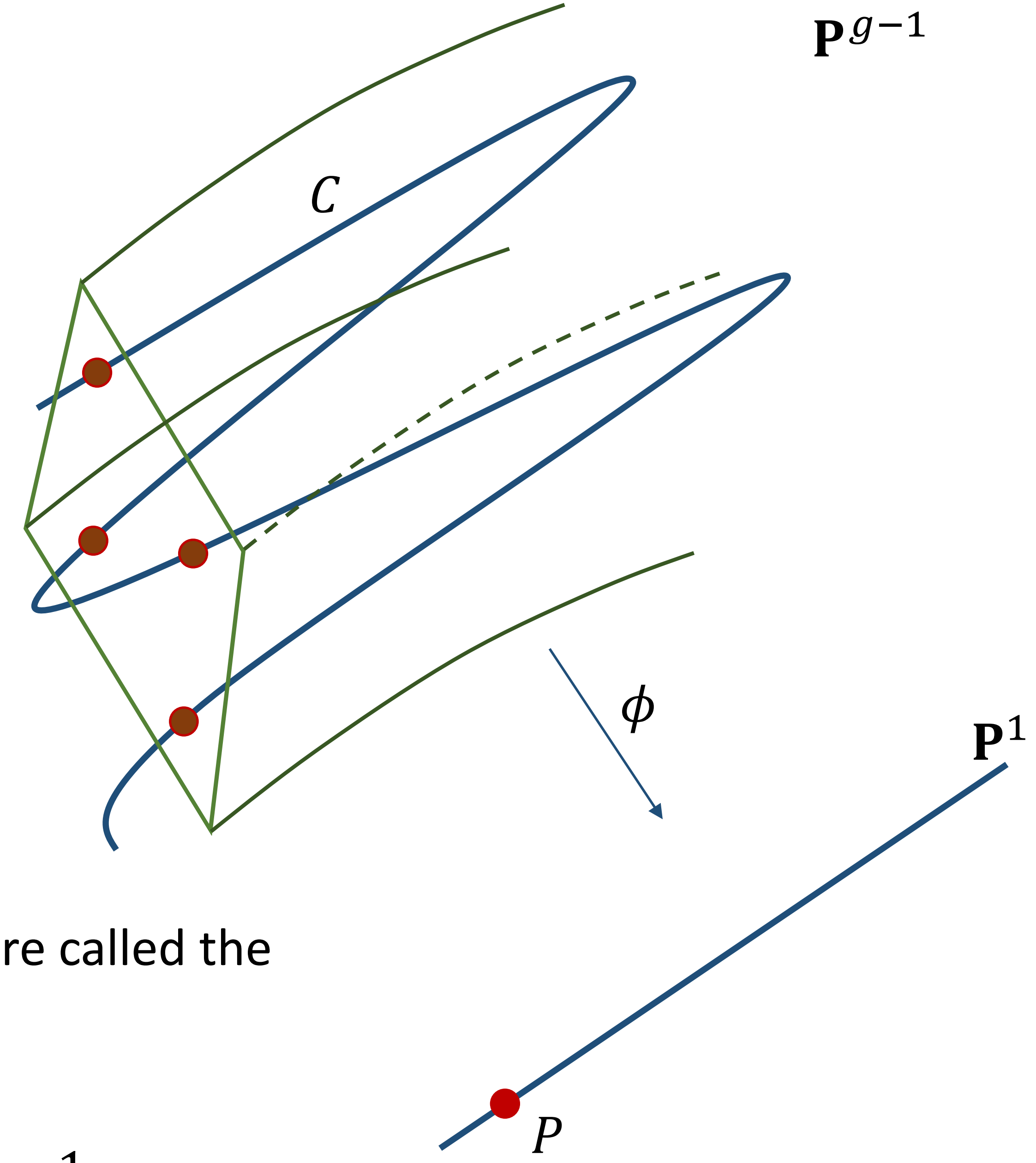


Maroni invariants

Consider a curve C over a field k along with a morphism $\phi: C \to \mathbf{P}^1$ of degree d.

Assume that C is canonically embedded in $\mathbf{P}^{g-1}$.

Take the linear spans of the fibers $\phi^{-1}\{P\}$ as P runs through all points of $\mathbf{P}^1$.



Theorem (Eisenbud-Harris, 1987): *The variety swept out by these linear spans is a rational normal scroll.*

The degrees $e_1, e_2, \dots, e_r$ corresponding to this scroll are called the multiset of **Maroni invariants** of C with respect to ϕ.

We assume that {fibers of ϕ} is complete, then $r = d - 1$.

Maroni invariants

Sum formula:

$$e_1 + e_2 + \cdots + e_{d-1} = g - d + 1$$

This follows from the definition of a rational normal scroll.

Maroni bound:

$$e_i \leq \frac{2g-2}{d}.$$

This follows essentially from the Riemann-Roch theorem.

Zhao's observation

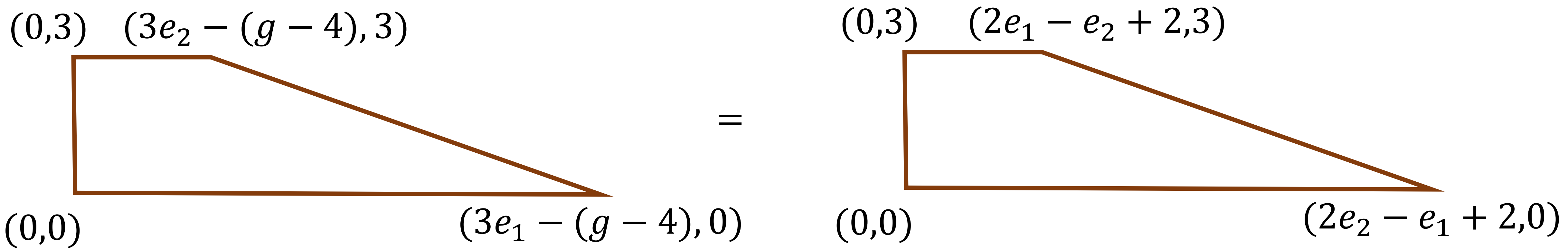
If $d = 3$ then we have two invariants, say $e_1 \leq e_2$ which satisfy

$$e_1 + e_2 = g - 2$$

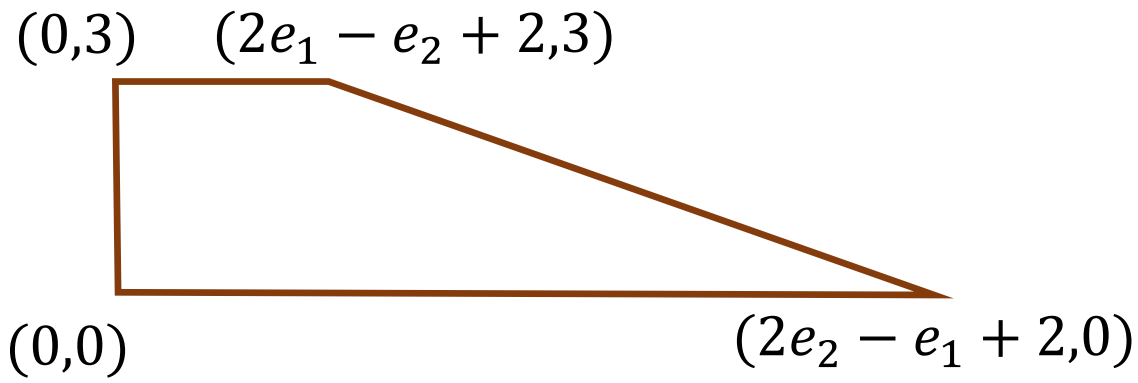
and the Maroni bound

$$\frac{g-4}{3} \leq e_1 \leq e_2 \leq \frac{2g-2}{3}$$

Inside the rational normal scroll our curve is in the linear system $|3H - (g-4)R|$, which on an appropriate chart corresponds to the polynomials $f(x,y)$ supported on



Zhao's observation



So by counting lattice points one sees that $\dim|3H-(g-4)R| = 2g+7$.
Well-known that $\dim \mathrm{Aut}(\mathrm{scroll}) = e_2 - e_1 + 5 + \delta_{e_1,e_2}$.

Assume that proportion of smooth irreducible members is "constant enough" for our purposes.

Then modulo some further self-admitted sloppinesses $T_3(q^{2g})$ is proportional to

$$\sum_{e_2=\frac{g}{2}}^{\frac{2g-2}{3}} q^{2g+7-(e_2-e_1+5)-3} = \sum_{e_2=\frac{g}{2}}^{\frac{2g-2}{3}} q^{2g+7-(e_2-(g-2-e_2)+5)-3} \approx \sum_{e_2=\frac{g}{2}}^{\frac{2g}{3}} q^{3g-2e_2} = \sum_{r=\frac{5g}{3}}^{2g} q^r \approx q^{2g} - q^{5g/3}$$

which gives the desired error term, which is directly related to the Maroni bound!

Tetragonal curves

If $d = 4$ then we have three Maroni invariants, say $e_1 \leq e_2 \leq e_3$ which satisfy

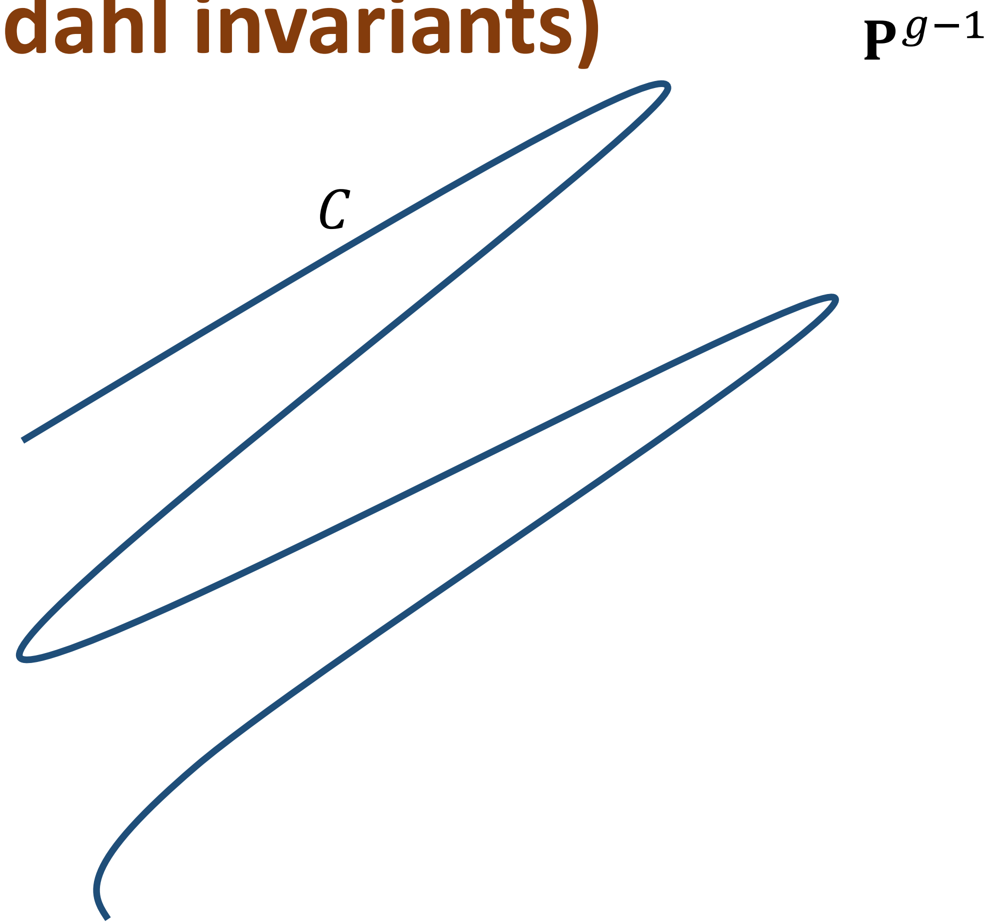
$$e_1 + e_2 + e_3 = g - 3$$

and the Maroni bound

$$0 \leq e_1 \leq e_2 \leq e_3 \leq \frac{2g-2}{4}$$

But according to the Shintani zeta function we expect an exponent $5/6$: this does not seem compatible?

Schreyer invariants (= Casnati-Ekedahl invariants)



Consider a curve C over a field k along with a morphism $\phi: C \to \mathbf{P}^1$ of degree $d \geq 4$.

Assume that C is non-hyperelliptic, non-trigonal, and canonically embedded in $\mathbf{P}^{g-1}$.

Well-known: C arises* as the intersection of

$$\frac{(g-2)(g-3)}{2}$$

quadratic hypersurfaces of $\mathbf{P}^{g-1}$, or if you want, effective divisors in the class $2H$.

* except if C is isomorphic to a smooth plane quintic

Schreyer invariants

Consider a curve C over a field k along with a morphism $\phi: C \to \mathbf{P}^1$ of degree $d \geq 4$.

Assume that C is non-hyperelliptic, non-trigonal, and canonically embedded in $\mathbf{P}^{g-1}$.

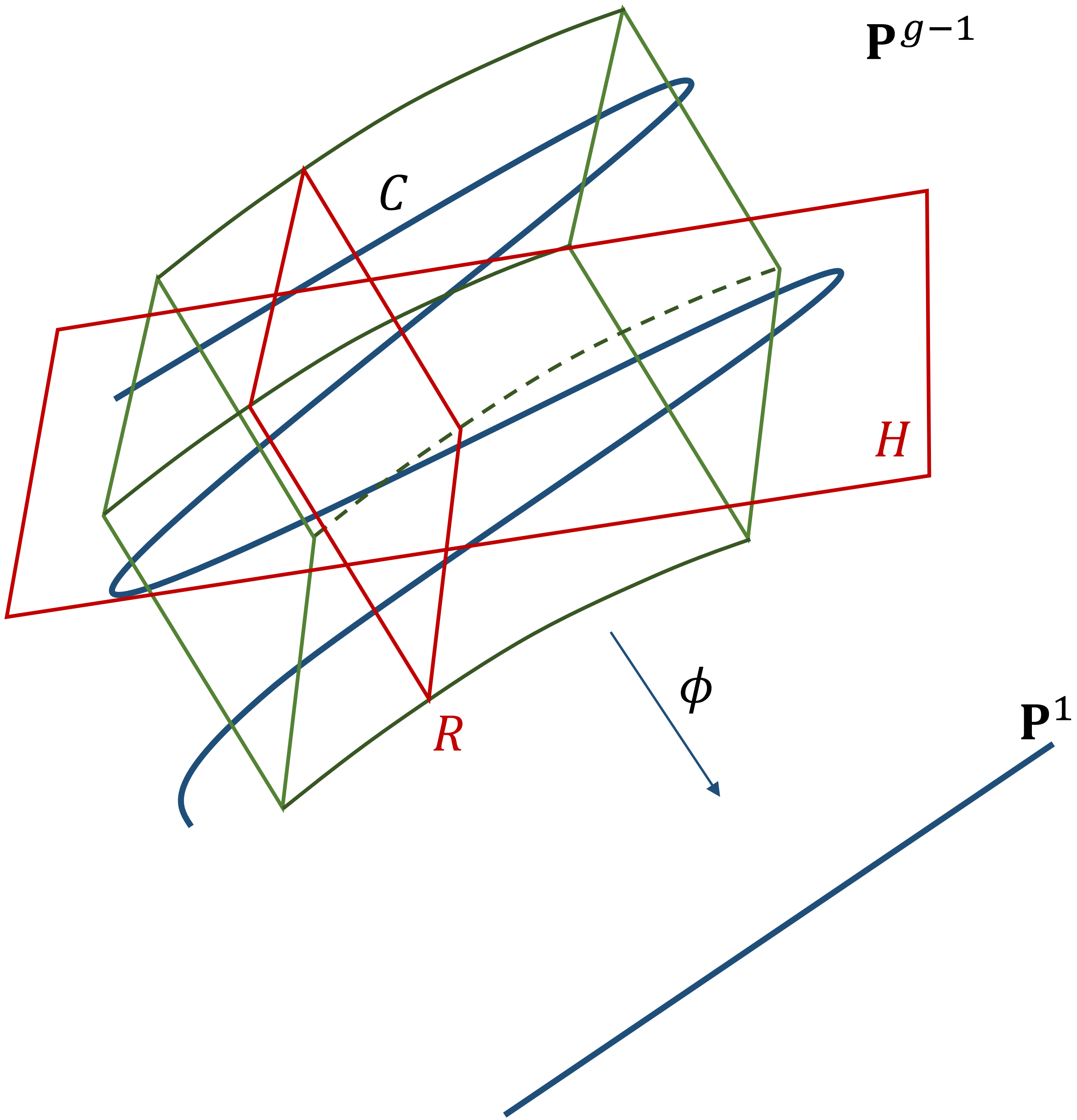
Theorem (Schreyer, 1986): *Inside the scroll associated to ϕ the curve C arises as the intersection of*

$$\frac{d(d-3)}{2}$$

effective divisors in classes of the form

$$2H - b_i R$$

for invariants $b_i \in \mathbf{Z}$ that are unique up to order.*



Schreyer invariants

We call the numbers

$$b_1, b_2, \ldots, b_{\frac{d(d-3)}{2}}$$

the **Schreyer invariants** of C with respect to the map ϕ. They satify

$$b_1 + b_2 + \cdots + b_{\frac{d(d-3)}{2}} = (d-3)(g-d-1)$$

They were introduced as a tool in the study of syzygies of algebraic curves (Green's conjecture).

A generalized treatment was given by **Casnati-Ekedahl, 1996**.

Back to tetragonal curves

If $d = 4$ then we have three Maroni invariants, say $e_1 \leq e_2 \leq e_3$ which satisfy

$$e_1 + e_2 + e_3 = g - 3$$

and the Maroni bound

$$0 \leq e_1 \leq e_2 \leq e_3 \leq \frac{2g - 2}{4}$$

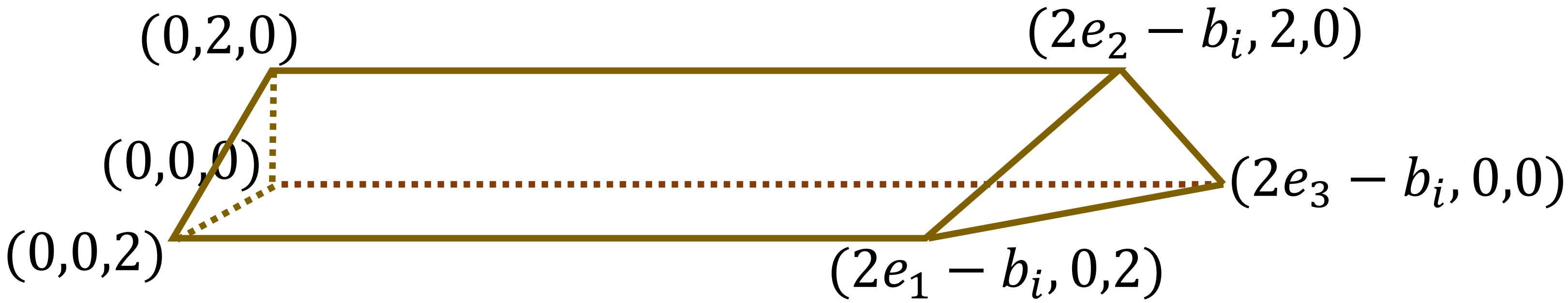
We also have two Schreyer invariants, say $b_1 \leq b_2$ which satisfy

$$b_1 + b_2 = g - 5$$

Inside our three-dimensional scroll C is a complete intersection of two divisors Y and Z, which belong to $|2H - b_1 R|$ and $|2H - b_2 R|$, respectively.

Back to tetragonal curves

On an appropriate chart the members of $|2H - b_i R|$ are defined by polynomials supported on

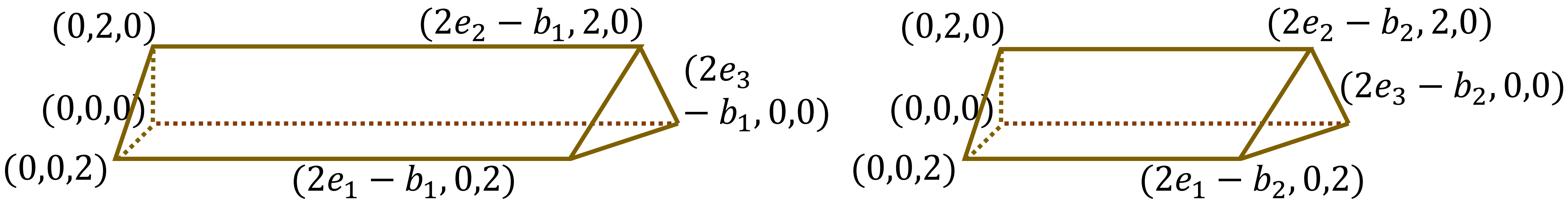


So by counting lattice points one sees that $\dim|2H - b_i R| = 4g - 7 - 6b_i$.

Well-known that $\dim \mathrm{Aut}(\mathrm{scroll}) = 2(e_3 - e_1) + 8 + \delta_{e_1,e_2} + \delta_{e_2,e_3} + \delta_{e_1,e_3}$.

Back to tetragonal curves

We have two polynomials $f_Y(x,y,z)$ and $f_Z(x,y,z)$ supported on:



To $f_Y(x,y,z)$ we can add $g(x)f_Z(x,y,z)$ for $\deg g(x) \leq b_2 - b_1$, without changing the curve.

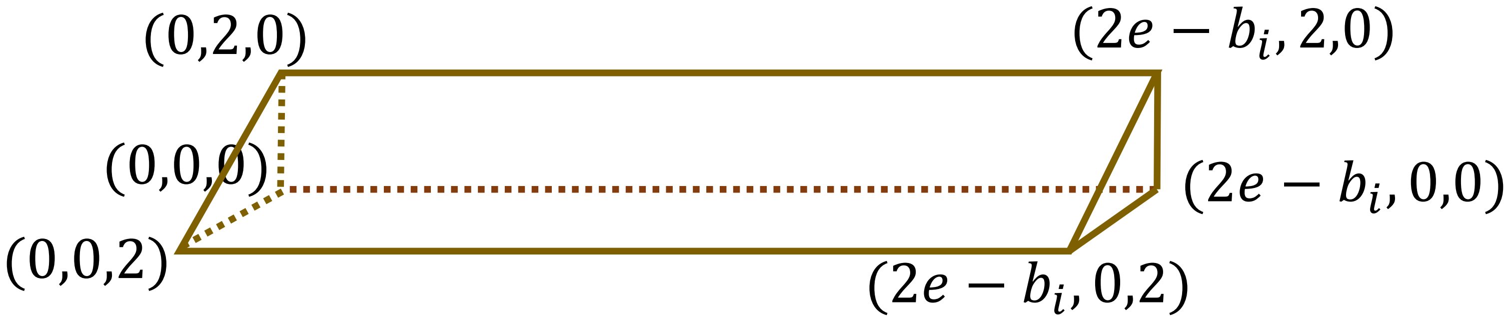
Therefore from the above pair we expect a contribution proportional to:

$$q^{4g-7-6b_1+4g-7-6b_2-(b_2-b_1+1)-8-2(e_3-e_1)} = q^{8g-6(b_1+b_2)-b_2+b_1-15} = q^{2g+15-b_2+b_1}$$

$$= q^{2g+15-b_2+g-5-b_2} \approx q^{3g-2b_1-2(e_3-e_1)}$$

Back to tetragonal curves

This must be fed to a **double** sum running over all triples e_1, e_2, e_3 and all corresponding pairs b_1, b_2. Let us look at the "main" case where $e_1 = e_2 = e_3 = e$.



We see that $b_i \leq 2e = 2(e_1 + e_2 + e_3)/3 = 2(g-3)/3 \leq 2g/3 - 2$.

Assume that the proportion of smooth complete intersections is "constant enough" inside this range, plus some self-admitted sloppiness, we get a contribution proportional to

$$\sum_{b_2 = g/2}^{2g/3} q^{3g - b_2}$$

But this we recognize from the trigonal count! It gives terms in q^{2g} and $q^{5g/3}$ as wanted.

Back to tetragonal curves

The other cases work similarly.

Everything combines nicely (but very heuristically) to the desired secondary term, suggesting that indeed

$$T_4(q^{2g}) = c_{4,q} q^{2g} - d_{4,q} q^{5g/3} + o(q^{5g/3})$$

for some constants $c_{4,q}, d_{4,q} > 0$.

Here the exponent in $q^{5g/3} = X^{5/6}$ is explained by **a bound of Maroni type** on Schreyer's invariants.

Is this a coincidence?

Recillas' trigonal construction

Consider a curve C over $\mathbf{F}_q$ along with a morphism $\phi: C \to \mathbf{P}^1$ of degree 4.

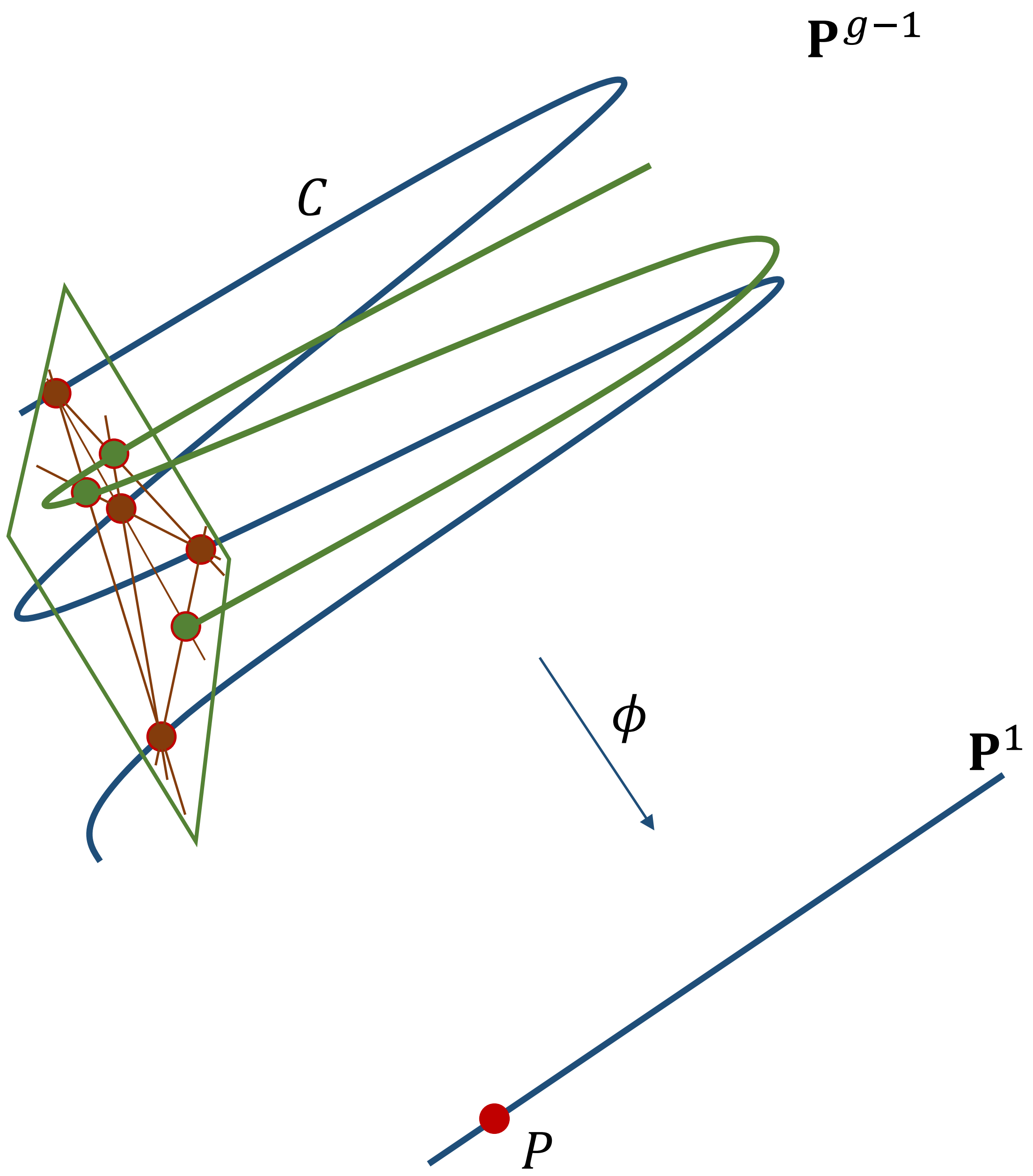
Assume that C is canonically embedded in $\mathbf{P}^{g-1}$.

Take the linear spans of the fibers $\phi^{-1}\{P\}$ as P runs through all points of $\mathbf{P}^1$, these are $\mathbf{P}^2$'s.

In each such $\mathbf{P}^2$, take the three "dual" points.

Theorem (Recillas 1974): *If $\phi: C \to \mathbf{P}^1$ has no ramification of type $4P$ or $2P + 2Q$ then these dual points cut out a smooth trigonal curve of genus $g + 1$.*

This is now known as **Recillas' trigonal construction**.

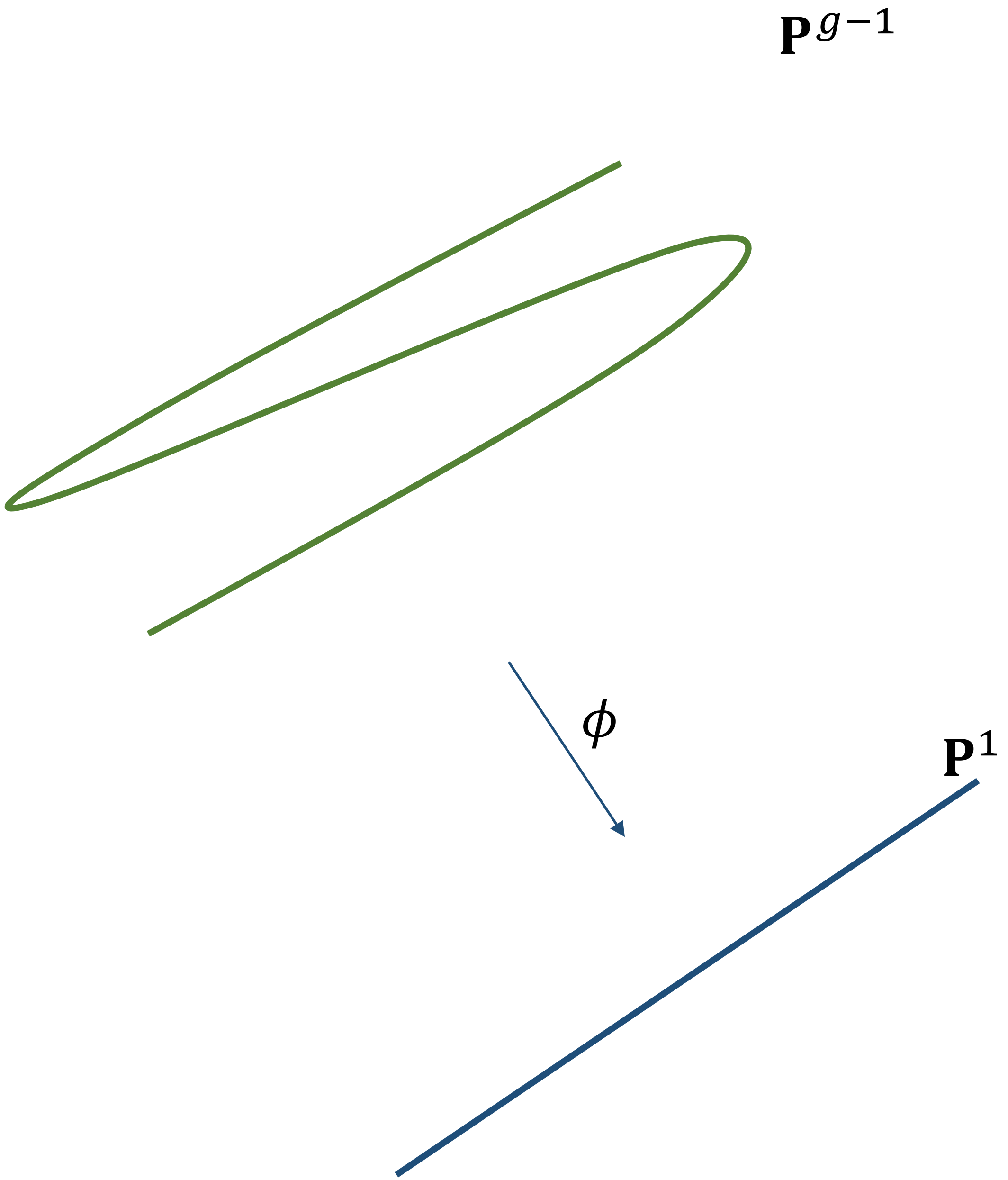


Recillas' trigonal construction

By explicit computation we refound the following striking fact:

Theorem (Casnati, 1995): *Under the same assumptions, the Maroni invariants of Recillas' trigonal construction applied to* C *are* $b_1 + 2$ *and* $b_2 + 2$*, where* b_1*,* b_2 *are the Schreyer invariants of* C*.*

This gives a very satisfactory explanation for the Maroni type bound on the b_i's!



Back to number fields

Is there a similar theory working behind the scenes in the case of number fields K?

It seems so!

Older (unpublished) idea due to Yongqiang: based on the alternative definition

$$\phi_* O_C = O_C \oplus O_C(-e_1 - 2) \oplus O_C(-e_2 - 2) \oplus \cdots \oplus O_C(-e_{d-1} - 2)$$

it is natural to define **the Maroni invariants of K** as

$$\log \|v_1\|, \log \|v_2\|, \ldots, \log \|v_{d-1}\|$$

where $1, v_1, v_2, \ldots, v_{d-1}$ is a Minkowski-reduced basis of the lattice $\sigma(O_K)$, with σ the canonical embedding.

Back to number fields

Compare

Theorem (Minkowski's second theorem): $\|v_1\| \cdot \|v_2\| \cdots \|v_{d-1}\| \sim_d \operatorname{vol}\left(\mathbf{R}^d / \sigma(O_K)\right) = \sqrt{|\Delta_K|}.$

with

$$e_1 + e_2 + \cdots + e_{d-1} = g - d + 1.$$

and

Theorem (Peikert-Rosen, 2007): $\|v_i\| = O_d(\Delta_K^{1/d})$

with the Maroni bound

$$e_i \leq \frac{2g-2}{d}.$$

(Similar bound appears in **Bhargava-Shankar-Taniguchi-Thorne-Tsimerman-Zhao, 2017**)

Back to number fields

What about the **Schreyer invariants of K**?

Fact: Recillas' trigonal construction is the geometric counterpart of the **cubic resolvent**

$$(x-\alpha_1\alpha_2-\alpha_3\alpha_4)(x-\alpha_1\alpha_3-\alpha_2\alpha_4)(x-\alpha_1\alpha_4-\alpha_2\alpha_3)$$

which is a Galois resolvent for the group $D_4 \subseteq S_4$.

Thus: natural to define the Schreyer invariants of a quartic field as the Maroni invariants of its cubic resolvent (ignoring potential reducibility concerns).

What about higher degree fields? Is this part of a richer theory?

Back to number fields

Experiments computing Maroni invariants of Galois resolvents strongly suggest so, although we cannot yet pin down how it works exactly.

Experiments in degree three:

Genus of input curve: g

Maroni invariants of input curve: e_1, e_2 (sum: $g-2$)

subgroup $G \subseteq S_3$	generators	index	generic genus	Maroni invariants of G-resolvent
trivial	id	6	$3g+1$	e_1, e_1, e_2, e_2, g
curve itself → C_2	(12)	3	g	e_1, e_2
$A_3 \cong C_3$	(123)	2	$g+1$	g

Back to number fields

Experiments in degree four:

Genus of input curve: g

Maroni invariants of input curve: e_1, e_2, e_3 (sum: $g-3$)
Schreyer invariants of input curve: b_1, b_2 (sum: $g-5$)

	subgroup $G \subseteq S_4$	generators	index	generic genus	Maroni invariants of G-resolvent
	C_4	(1234)	6	$3g+4$	$g-e_1-1, g-e_2-1, g-e_3-1, g-b_1-3, g-b_2-3$
	V_4	$(12),(34)$	6	$2g+1$	$e_1, e_2, e_3, b_1+2, b_2+2$
curve itself →	S_3	$(12),(123)$	4	g	e_1, e_2, e_3
cubic res. →	D_4	$(1234),(12)(34)$	3	$g+1$	b_1+2, b_2+2
	A_4	even perm.	2	$g+2$	$g+1$

Back to number fields

Experiments in degree five:

Genus of input curve: g

Maroni invariants of input curve: e_1, e_2, e_3, e_4 (sum: $g-4$)
Schreyer invariants of input curve: b_1, b_2, b_3, b_4, b_5 (sum: $2g-12$)

subgroup $G \subseteq S_5$	generators	index	generic genus	Maroni invariants of G-resolvent
S_4 (curve itself)	perm. fixing 5	5	g	e_1, e_2, e_3, e_4
F_{20} (Cayley res.)	$(1234), (12345)$	6	$3g+7$	$g-b_1-2, g-b_2-2,$ $g-b_3-2, g-b_4-2,$ $g-b_5-2$
A_5	even perm.	2	$g+3$	$g+2$

Back to number fields

Experiments in degree six:

Genus of input curve: g

Maroni invariants of input curve: e_1, e_2, e_3, e_4, e_5 (sum: $g - 5$)
Schreyer invariants of input curve: $b_1, b_2, \dots, b_9$ (sum: $3g - 21$)

subgroup $G \subseteq S_5$	generators	index	generic genus	Maroni invariants of G-resolvent
curve itself → S_5	perm. fixing 6	6	g	e_1, e_2, e_3, e_4, e_5
$S_3 \wr C_2$	$(12), (123)$ $(45), (456)$ $(14)(25)(36)$	10	$3g + 6$	$b_1 + 2, \dots, b_9 + 2$
A_6	even perm.	2	$g + 4$	$g + 3$

Questions?

Thanks for your attention!