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Arithmetic aspects of the Burkhardt quartic

Nils Bruin (Simon Fraser University),
joint with Brett Nasserden (University of Waterloo)

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The Burkhardt Quartic

Defining Equation:

$$B : f(y_0, \ldots, y_4) := y_0(y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3) + 3y_1y_2y_3y_4 = 0.$$

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Some properties:

- 45 nodal singularities (maximum possible); over $\mathbb{Q}(\sqrt{-3})$.
- Only quartic with that property (over $\mathbb{C}$)
- Linear action of $\mathrm{PSp}_4(\mathbb{F}_3)$, the simple group of order 25920.
- f is the unique quartic invariant for this action.

Moduli interpretation

B is birational to $\mathcal{A}_2(3)$, the moduli space of principally polarized abelian surfaces A with full level 3 structure, i.e., together with an isomorphism $(\mathbb{Z}/3)^2 \times (\mu_3)^2 \to A[3]$.

Questions

Question 1: It is known that the Burkhardt quartic is rational over $\mathbb{Q}(\sqrt{-3})$. Is it also rational over $\mathbb{Q}$?

Question 2: The Burkhardt quartic has good reduction at primes $p \neq 3$. We know the zeta function of B over $\mathbb{F}_p$ for $p \equiv 1 \pmod 3$ (Hoffman-Weintraub, 2000). Can we determine it for all $p \neq 3$?

Question 3: The moduli space $\mathcal{A}_2(3)$ is *fine*, and an open part of it is formed by Jacobians of genus 2 curves. There should exist a *universal* genus 2 curve C_α over that part. Can we write down a model in term of coordinates α on B?

Question 4: How do we mark the level 3 structure on such a curve C_α?

(**Question 5:** Genus 2 curves with 3-torsion class arise as discriminants of cubic genus 1 covers of $\mathbb{P}^1$. Can we recognize these?)

J-Planes on the Burkhardt

Burkhardt: $f(y_0,\ldots,y_4) = y_0(y_0^3+y_1^3+y_2^3+y_3^3+y_4^3)+3y_1y_2y_3y_4 = 0$

Hessian: $$\mathrm{Hess}(B):\ \det\left(\frac{\partial f}{\partial y_i \partial y_j}\right)_{i,j} = 0$$

J-planes: $$B \cap \mathrm{Hess}(B) = \bigcup_{i=1}^{40} J_i$$

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- Each J_i is a linear 2-space (e.g. $y_0 = y_1 = 0$)
- $\mathrm{PSp}_4(\mathbb{F}_3)$ acts transitively on them
- Each J_i contains 9 of the 45 nodes
- 8 are defined over $\mathbb{Q}$; 16 pairs conjugate over $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$.

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Steiner primes: 40 hyperplanes that intersect B in 4 J-planes.

Moduli interpretation: Points in $B \setminus \mathrm{Hess}(B)$ correspond to genus 2 curves

Question 1: Rationality of the Burkhardt Quartic

Todd (1936): *The Burkhardt quartic is birational to $\mathbb{P}^3$ over $\mathbb{C}$.*

Baker (1942): Produced a paramametrization over $\mathbb{Q}(\sqrt{-3})$.

Question 1: Can we adjust Baker's idea to work over $\mathbb{Q}$?

Approach: Modify Baker's argument to be Galois invariant.

Useful fact: The lines through 3 planes in $\mathbb{P}^4$ form a rational variety.

Idea: Take 3 planes J_1, J_2, J_3 on B and parametrize the lines through them. Parametrize B using the fourth intersection point.

Watch out: Not all choices of J_1, J_2, J_3 produce a dominant map.

Executing the idea

Theorem. $\phi\colon \mathbb{P}^3 \overset{\sim}{\dashrightarrow} B;\quad (1:t_1:t_2:t_3)\mapsto(\xi_0:\xi_1:\xi_2:\xi_3:\xi_4)$

$$\begin{aligned}
\xi_0 &= t_1^3-3t_1^2t_3-3t_1t_2^2-3t_1t_2t_3-t_2^3-1,\\
\xi_1 &= -t_1^3+3t_1^2t_3-3t_1t_3^2+t_2^3+1,\\
\xi_2 &= -t_1^4+t_1^3t_2+3t_1^3t_3-3t_1^2t_2t_3-3t_1^2t_3^2-2t_1t_2^3-3t_1t_2^2t_3+t_1-t_2^4-t_2,\\
\xi_3 &= -t_1^4+4t_1^3t_3+3t_1^2t_2^2+3t_1^2t_2t_3-3t_1^2t_3^2+t_1t_2^3-3t_1t_2^2t_3-3t_1t_2t_3^2+t_1-t_2^3t_3-t_3,\\
\xi_4 &= -t_1^4-t_1^3t_2+2t_1^3t_3+3t_1^2t_2t_3+t_1t_2^3+3t_1t_2^2t_3+t_1+t_2^4+t_2^3t_3+t_2+t_3
\end{aligned}$$

has birational inverse $\psi\colon (y_0:y_1:y_2:y_3:y_4)\mapsto(t_0:t_1:t_2:t_3)$,

$$\begin{aligned}
t_0 &= y_0(y_0^2-y_0y_1+y_1^2),\\
t_1 &= y_0(y_1y_2-y_0y_3-y_0y_4),\\
t_2 &= y_0(y_0y_2-y_1y_2+y_1y_3+y_1y_4),\\
t_3 &= y_0y_1y_2-y_0y_1y_3+y_1^2y_3-y_0^2y_4.
\end{aligned}$$

Question 2: Zeta functions

Zeta function: Let $X/\mathbb{F}_q$ be a variety over a finite field $\mathbb{F}_q$.

$$Z(X/\mathbb{F}_q, T) := \exp\left(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n}) \frac{T^n}{n}\right)$$

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Unions: $Z(X\cup Y, T)Z(X\cap Y, T) = Z(X,T)Z(Y,T)$

Disjoint Conjugate components: $X = Y \cup Y'$

$$Z(X/\mathbb{F}_q, T) = Z(Y/\mathbb{F}_{q^2}, T^2).$$

Birational map induces an isomorphism: $\mathbb{P}^3 \setminus V_\phi \simeq B \setminus V_\psi$

Applied to our problem: $Z(B,T) = Z(V_\psi, T)\dfrac{Z(\mathbb{P}^3, T)}{Z(V_\phi, T)}$

Answer to Question 2

Case $q \equiv 1 \pmod 3$:

$$Z(B/\mathbb{F}_q, T) = \frac{(1-qT)^{29}}{(1-T)(1-q^2T)^{16}(1-q^3T)}$$

Case $q \equiv 2 \pmod 3$:

$$Z(B/\mathbb{F}_q, T) = \frac{(1-qT)^{15}(1+qT)^{14}}{(1-T)(1-q^2T)^{10}(1+q^2T)^6(1-q^3T)}$$

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Further computation:

$$\#(B \setminus \mathrm{Hess}(B))(\mathbb{F}_q) = \begin{cases} (q-4)(q-7)(q-13) & \text{if } q \equiv 1 \pmod 3 \\ (q-2)(q^2-2q-1) & \text{if } q \equiv 2 \pmod 3 \end{cases}$$

Question 3: Moduli interpretation

Moduli space:

$\mathcal{A}_2(3) = \{A :$ abelian surface together with $(\mathbb{Z}/3\mathbb{Z})^2 \times \mu_3^2 \xrightarrow{\sim} A[3]\}$

(abelian surfaces together with a basis for the 3-torsion)

Caveat: Principal polarization; Weil pairing on $A[3]$.

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Classical: $B \setminus \mathrm{Hess}(B) = \{\mathrm{Jac}(C) : C$ genus 2 curve$\}$

Question: Can we make this explicit?

$$\alpha \in B \setminus \mathrm{Hess}(B) \rightsquigarrow \text{genus 2 curve } C_\alpha$$

Example of explicit moduli interpretation for g=1

Modular curve: $\mathcal{A}_1(3) \subset X(3) \simeq \mathbb{P}^1$

Hesse Pencil:

$$E_{(s:t)} : s(X^3 + Y^3 + Z^3) + tXYZ = 0$$

Cubics passing through $(0:1:1), (0:\zeta_3:1), \dots$ (9 points)

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Classical result: Any elliptic curve with $\mathbb{Z}/3 \times \mu_3 \simeq E[3]$ occurs for some values $(s:t)$

The 9 points mark the 3-torsion on these elliptic curves.

The family $E_{(s:t)}$ makes the moduli interpretation of $\mathbb{P}^1$ as $X(3)$ completely explicit.

Explicit moduli problem

Problem: Give a formula of C_α in terms of $\alpha \in B \setminus \mathrm{Hess}(B)$. such that $\mathrm{Jac}(C_\alpha)$ realizes the moduli interpretation.

Known results: Hunt gives a model for $\mathrm{Pic}^1(C_\alpha) \subset \mathbb{P}^8$ in terms of $\alpha \in B \setminus \mathrm{Hess}(B)$.

Model for dual Kummer: $\mathcal{K}_\alpha^\vee := \mathrm{Pic}^1(C_\alpha)/\langle \iota \rangle$.

Dual Kummers: come with a conic through six of the nodes.

We have curve specified as 6 points on a conic.

Field of definition obstruction: We need to have conic isomorphic to $\mathbb{P}^1$.

Quadratic twists: Level 3 structure determines which quadratic twist.

Moduli questions for 6 points

The following are equivalent moduli questions:

- Six points in $\mathbb{P}^1$
- Six points in $\mathbb{P}^3$ in general position
  (embed $\mathbb{P}^1$ a s a rational normal curve)
- 4-dimensional systems

$$\mathcal{Q} = \langle Q_1, Q_2, Q_3, Q_4 \rangle$$

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Derived quartic surfaces:

- Weddle surface: $\mathcal{W}_{\mathcal{Q}} = \bigcup\{\mathrm{sing}(Q) : Q \in \mathcal{Q}\}$
- $\mathcal{K}^{\vee}_{\mathcal{Q}} : \det(y_1 Q_1 + \cdots + y_4 Q_4) = 1$ (*symmetroid* of $\mathcal{W}_{\mathcal{Q}}$)

Conic on $\mathcal{K}^{\vee}_{\mathcal{Q}}$ is image of rational normal curve on $\mathcal{W}_{\mathcal{Q}}$.

Question 3: Answer

Let $(1:\alpha_1:\alpha_2:\alpha_3:\alpha_4)$ be a point on B. Set

$$\begin{aligned}
H &:= \alpha_2\alpha_4X^2 - \alpha_3\alpha_4X - \alpha_1\alpha_4^2 \\
\lambda &:= \alpha_1^3\alpha_4^6 - 3\alpha_1\alpha_2\alpha_3\alpha_4^4 + \alpha_1^3\alpha_4^3 - \alpha_2^3\alpha_4^3 - \alpha_3^3\alpha_4^3 - 3\alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_2^3 - \alpha_3^3 \\
G &:= (3\alpha_1\alpha_2\alpha_3\alpha_4^4 + \alpha_1^3\alpha_4^3 + 2\alpha_2^3\alpha_4^3 + \alpha_3^3\alpha_4^3 + \alpha_2^3)X^3 \\
&\quad + (3\alpha_1^2\alpha_2\alpha_4^5 - 3\alpha_2^2\alpha_3\alpha_4^3 + 3\alpha_1^2\alpha_2\alpha_4^2 - 3\alpha_2^2\alpha_3)X^2 \\
&\quad + (-3\alpha_1^2\alpha_3\alpha_4^5 + 3\alpha_2\alpha_3^2\alpha_4^3 - 3\alpha_1^2\alpha_3\alpha_4^2 + 3\alpha_2\alpha_3^2)X \\
&\quad - 2\alpha_1^3\alpha_4^6 + 3\alpha_1\alpha_2\alpha_3\alpha_4^4 - \alpha_1^3\alpha_4^3 + \alpha_2^3\alpha_4^3 - \alpha_3^3
\end{aligned}$$

Theorem: If $C_\alpha : y^2 + Gy = \lambda H^3$ is a genus 2 curve. Then

$$(\mathbb{Z}/3)^2 \times \mu_3^2 \simeq \mathrm{Jac}(C_\alpha)[3]$$

Warning: This model is bad for $\alpha_4 = 0$.

Question 4: Marking 3-torsion

Suppose: $C\colon y^2 = G(x)^2 + 4\lambda H(x)^3$

Consider divisor:

$$T = \{H(x) = 0, y - G(x) = 0\} - \{x = \infty\}$$

Then T represents a 3-torsion class:

$$\mathrm{div}(y - G(x)) = \pm 3T$$

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Full level structure: $(\mathbb{Z}/3)^2 \times (\mu_3)^2 \subset \operatorname{Jac}(C)$ means two decompositions of each.

3-torsion and J-planes

First polar of B at α:

$$P_\alpha = (4y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3)\alpha_0 + (3y_0y_1^2 + 3y_2y_3y_4)\alpha_1 + (3y_0y_2^2 + 3y_1y_3y_4)\alpha_2 + (3y_0y_3^2 + 3y_1y_2y_4)\alpha_3 + (3y_0y_4^2 + 3y_1y_2y_3)\alpha_4,$$

Construction of dual kummer:

- Take enveloping cone $\mathrm{EC}_\alpha(P(\alpha))$
- Take projection $\pi_\alpha : \mathbb{P}^4 \to \mathbb{P}^3$ from α
- $\mathcal{K}_\alpha^\vee = \pi_\alpha(\mathrm{EC}_\alpha(P(\alpha)))$

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- Take enveloping cone $\mathrm{EC}_\alpha(P(\alpha))$
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- $\mathcal{K}_\alpha^\vee = \pi_\alpha(\mathrm{EC}_\alpha(P(\alpha)))$

Computation:

- Take a J-plane J of B
- Then $\pi_\alpha(J)$ is tangent to $\mathcal{K}_\alpha^\vee$.
- Point on $\mathcal{K}_\alpha$ comes from 3-torsion point
- $2 \cdot 40 = 81 - 1$

Further computation:

Two 3-torsion points have trivial Weil pairing iff their J-planes lie in a common Steiner prime.

Some relevant literature

Classical work by Burkhardt, Coble, Todd, Baker

Bruce Hunt, *The geometry of some special arithmetic quotients*, Springer LNM 1637 (1996)

Noam D. Elkies, *The identification of three moduli spaces*, (1999), arXiv:math/9905195.

J. William Hoffman and Steven H. Weintraub, The Siegel modular variety of degree two and level three, *Trans. Amer. Math. Soc.* 353 (2001), no. 8, 3267–3305

Nils Bruin, Brett Nasserden, *Arithmetic aspects of the Burkhardt quartic threefold*, `https://arxiv.org/abs/1705.09006` (2017)

Brett Nasserden, *Arithmetic aspects of the Burkhardt quartic threefold*, M.Sc. thesis, Simon Fraser University (2016) `https://theses.lib.sfu.ca/thesis/etd9738`