Local densities compute isogeny classes

without the analytic class number formula

Jeff Achter

j.achter@colostate.edu
Department of Mathematics
Colorado State University

June 2017
Arithmetic, Geometry, Cryptography and Coding Theory
CIRM, Luminy

Prologue

Motivating question

How big is an isogeny class of elliptic curves over a finite field?

*

* Joint work with Julia Gordon (UBC) and S. Ali Altuğ (MIT)

Isogenous elliptic curves

If $E_1, E_2/\mathbb{F}_q$, the following are equivalent:

- E_1 and E_2 are isogenous;
- $\#E_1(\mathbb{F}_q) = \#E_2(\mathbb{F}_q)$;
- $a(E_1) = a(E_2)$, where characteristic polynomial of Frobenius is

$$f_{E_i/\mathbb{F}_q}(T) = T^2 - a(E_i)T + q.$$

Let

$$I(a, \mathbb{F}_q) = \left\{ E/\mathbb{F}_q : a(E) = a \right\}.$$

Motivating question

What is $\#I(a, \mathbb{F}_q)$?

Or $\widetilde{\#}I(a, \mathbb{F}_q)$, *where* E *has weight* $1/\#\operatorname{Aut}(E)$.

First guess: uniform

- $a \in [-2\sqrt{q}, 2\sqrt{q}]$ *(Hasse)*
- $\asymp q$ elliptic curves over $\mathbb{F}_q$. *(Exact, if we weight by automorphism.)*
- Suppose $a(E)$ uniformly distributed on $[-2\sqrt{q}, 2\sqrt{q}]$.

Heuristic

$$\#I(a, \mathbb{F}_q) \asymp q/\sqrt{q} = \sqrt{q}.$$

First guess: uniform

- $a \in [-2\sqrt{q}, 2\sqrt{q}]$ *(Hasse)*
- $\asymp q$ elliptic curves over $\mathbb{F}_q$. *(Exact, if we weight by automorphism.)*
- Suppose $a(E)$ uniformly distributed on $[-2\sqrt{q}, 2\sqrt{q}]$.

Heuristic

$$\#I(a, \mathbb{F}_q) \asymp q/\sqrt{q} = \sqrt{q}.$$

This can't be exactly right. The distribution is *not* uniform.

Second Guess: Sato–Tate

- Frobenius angles:
  $f_E(T) = T^2 - a_E T + q = (T - \sqrt{q}\exp(i\theta_E))(T - \sqrt{q}\exp(-i\theta_E))$.
- Sato-Tate: Distributed like $\sin^2(\theta)$:

$$\Pr(\theta^- \le \theta_E \le \theta^+) \approx \int_{\theta^-}^{\theta^+} \sin^2(\theta)d\theta \text{ and so } \frac{a}{2\sqrt{q}} \sim \sqrt{1 - \frac{a^2}{4q}}.$$

Second Guess: Sato–Tate

- Frobenius angles:
  $f_E(T) = T^2 - a_E T + q = (T - \sqrt{q}\exp(i\theta_E))(T - \sqrt{q}\exp(-i\theta_E))$.
- Sato-Tate: Distributed like $\sin^2(\theta)$:

$$\Pr(\theta^- \le \theta_E \le \theta^+) \approx \int_{\theta^-}^{\theta^+} \sin^2(\theta) d\theta \text{ and so } \frac{a}{2\sqrt{q}} \sim \sqrt{1 - \frac{a^2}{4q}}.$$

Where did $\sin^2(\theta)$ come from?

$$\mathfrak{c} : \mathrm{SU}(2) \longrightarrow \mathbb{R}[T] \longrightarrow [0, \pi)$$

$$\gamma \longmapsto f_\gamma(T) = (T - \exp(i\theta))(T - \exp(-i\theta)) \longmapsto \theta$$

$$\mu_{\mathrm{ST}} = \mathfrak{c}_* \mu_{\mathrm{Haar}}$$

Second Guess: Sato–Tate

- Frobenius angles:
  $f_E(T) = T^2 - a_E T + q = (T - \sqrt{q}\exp(i\theta_E))(T - \sqrt{q}\exp(-i\theta_E))$.
- Sato-Tate: Distributed like $\sin^2(\theta)$:

$$\Pr(\theta^- \leq \theta_E \leq \theta^+) \approx \int_{\theta^-}^{\theta^+} \sin^2(\theta)d\theta \text{ and so } \frac{a}{2\sqrt{q}} \sim \sqrt{1 - \frac{a^2}{4q}}.$$

Heuristic

$$\#I(a, \mathbb{F}_q) \approx \#\left\{E/\mathbb{F}_q\right\} \cdot Pr(a_E = a) = \sqrt{4q - a^2} \asymp \sqrt{q}.$$

Second Guess: Sato–Tate

- Frobenius angles:
  $f_E(T) = T^2 - a_E T + q = (T - \sqrt{q}\exp(i\theta_E))(T - \sqrt{q}\exp(-i\theta_E))$.
- Sato-Tate: Distributed like $\sin^2(\theta)$:

$$\mathrm{Pr}(\theta^- \leq \theta_E \leq \theta^+) \approx \int_{\theta^-}^{\theta^+} \sin^2(\theta) d\theta \text{ and so } \frac{a}{2\sqrt{q}} \sim \sqrt{1 - \frac{a^2}{4q}}.$$

Heuristic

$$\#I(a, \mathbb{F}_q) \approx \#\left\{E/\mathbb{F}_q\right\} \cdot Pr(a_E = a) = \sqrt{4q - a^2} \asymp \sqrt{q}.$$

This can't be exactly right:

- Katz-Sarnak isn't this strong.
- $\sqrt{4q - a^2}$ has no arithmetic.

Third guess: equidistribution at ℓ

Frobenius elements of $E/\mathbb{F}_q$ are:

- Equidistributed in $\mathrm{GL}_2(\mathbb{Z}/\ell)$ and $\mathrm{GL}_2(\mathbb{Z}_\ell)$;
- Independent: equidistributed in $\mathrm{GL}_2(\mathbb{Z}/\ell_1) \times \mathrm{GL}_2(\mathbb{Z}/\ell_2)$.

Set

$$\nu_\ell(a,q) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,q) \bmod \ell^n\}}{\#\mathrm{SL}_2(\mathbb{Z}/\ell^n)/\ell^n}$$

Third guess: equidistribution at ℓ

Frobenius elements of $E/\mathbb{F}_q$ are:

- Equidistributed in $\mathrm{GL}_2(\mathbb{Z}/\ell)$ and $\mathrm{GL}_2(\mathbb{Z}_\ell)$;
- Independent: equidistributed in $\mathrm{GL}_2(\mathbb{Z}/\ell_1) \times \mathrm{GL}_2(\mathbb{Z}/\ell_2)$.

Set

$$\nu_\ell(a,q) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,q) \bmod \ell^n\}}{\#\mathrm{SL}_2(\mathbb{Z}/\ell^n)/\ell^n}$$

Why ν_ℓ?

- Denominator is average number of elements with given charpoly.
- Equivalently, ν_ℓ comes from pushforward of Haar:

$$\mathrm{GL}_2(\mathbb{Z}_\ell) \xrightarrow{\mathfrak{c}} \mathbb{A}^1 \times \mathbb{G}_m$$

$$\gamma \longmapsto (\mathrm{tr}(\gamma), \det(\gamma))$$

Third guess: equidistribution at ℓ

Frobenius elements of $E/\mathbb{F}_q$ are:

- Equidistributed in $\mathrm{GL}_2(\mathbb{Z}/\ell)$ and $\mathrm{GL}_2(\mathbb{Z}_\ell)$;
- Independent: equidistributed in $\mathrm{GL}_2(\mathbb{Z}/\ell_1) \times \mathrm{GL}_2(\mathbb{Z}/\ell_2)$.

Set

$$\nu_\ell(a,q) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,q) \bmod \ell^n\}}{\#\mathrm{SL}_2(\mathbb{Z}/\ell^n)/\ell^n}$$

Heuristic

$$\#I(a,\mathbb{F}_q) \approx \sqrt{q}(\text{Sato-Tate term}) \cdot \prod_\ell \nu_\ell(a,q).$$

Third guess: equidistribution at ℓ

Frobenius elements of $E/\mathbb{F}_q$ are:

- Equidistributed in $\mathrm{GL}_2(\mathbb{Z}/\ell)$ and $\mathrm{GL}_2(\mathbb{Z}_\ell)$;
- Independent: equidistributed in $\mathrm{GL}_2(\mathbb{Z}/\ell_1) \times \mathrm{GL}_2(\mathbb{Z}/\ell_2)$.

Set

$$\nu_\ell(a,q) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,q) \bmod \ell^n\}}{\#\mathrm{SL}_2(\mathbb{Z}/\ell^n)/\ell^n}$$

Heuristic

$$\#I(a,\mathbb{F}_q) \approx \sqrt{q}(\text{Sato-Tate term}) \cdot \prod_\ell \nu_\ell(a,q).$$

This can't be right. Equidistribution only holds for $\ell \ll q$.

Gekeler's Theorem

Set

$$\nu_\ell(a,p) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,p) \bmod \ell^n\}}{\#\mathrm{SL}_2(\mathbb{Z}/\ell^n)/\ell^n}$$

$$\nu_p(a,p) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{Mat}_2(\mathbb{Z}/p^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,p) \bmod p^n\}}{\#\mathrm{SL}_2(\mathbb{Z}/p^n)/p^n}$$

$$\nu_\infty(a,p) = \frac{2}{\pi}\sqrt{1-\frac{a^2}{4p}}$$

Theorem (Gekeler)

If $|a| < 2\sqrt{p}$ *and* $a \neq 0$, *then*

$$\#I(a,\mathbb{F}_p) = \sqrt{p}\,\nu_\infty(a,p)\prod_\ell \nu_\ell(a,p).$$

Gekeler's Theorem

Set

$$\nu_\ell(a,p) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,p) \bmod \ell^n\}}{\#\mathrm{SL}_2(\mathbb{Z}/\ell^n)/\ell^n}$$

$$\nu_p(a,p) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{Mat}_2(\mathbb{Z}/p^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,p) \bmod p^n\}}{\#\mathrm{SL}_2(\mathbb{Z}/p^n)/p^n}$$

$$\nu_\infty(a,p) = \frac{2}{\pi}\sqrt{1 - \frac{a^2}{4p}}$$

Theorem (Gekeler)

If $|a| < 2\sqrt{p}$ *and* $a \neq 0$, *then*

$$\#I(a,\mathbb{F}_p) = \sqrt{p}\,\nu_\infty(a,p)\prod_\ell \nu_\ell(a,p).$$

Counterfactual equidistribution predicts the right answer!

Why does it work?

~~Why~~ How does it work?

~~Why~~ How does it work?

- Let $\Delta = \Delta_{a,q} = a^2 - 4q$.
- Let $K = \mathbb{Q}(\sqrt{\Delta})$.
- Suppose $\mathcal{O}_K = \mathbb{Z}[\sqrt{a^2 - 4q}]$, $\mathcal{O}_K^\times = \pm 1$.

A class number counts the isogeny class:

$$\widetilde{\#} I(a,p) = \frac{1}{2} h(K).$$

Term by term

- Analytic class number formula:

$$\frac{1}{2}h(K) = \frac{\#\mathcal{O}_K^\times \sqrt{|\Delta_K|}}{2\pi} L(1,\chi_K) = \frac{\#\mathcal{O}_K^\times \sqrt{|\Delta_K|}}{2\pi} \prod_\ell \frac{1}{1-\chi_K(\ell)/\ell}$$

where $\chi_K(\ell) = \left(\frac{\Delta_K}{\ell}\right)$ is the quadratic character associated to $K/\mathbb{Q}$.

- Direct calculation:

$$\lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,q) \bmod \ell^n\}}{\#\mathrm{SL}_2(\mathbb{Z}/\ell^n)/\ell^n}$$

$$= \frac{1}{1-\chi_K(\ell)/\ell}.$$

Two questions

Can we find...

- a pure-thought proof of Gekeler's theorem?
- an analogue for isogeny classes of principally polarized abelian varieties?

Weil polynomials and isogeny classes

- $(X, \lambda)/\mathbb{F}_q$ a g-dimensional principally polarized abelian variety.
- Frobenius $\varpi_{X/\mathbb{F}_q,\ell}$ acts on $V_\ell X = T_\ell X \otimes \mathbb{Q}_\ell$.
- Tate: (unpolarized) isogeny class of X determined by

$$f_{X/\mathbb{F}_q}(T) \in \mathbb{Z}[T],$$

  the characteristic polynomial of $\varpi_{X/\mathbb{F}_q,\ell}$.
- λ induces $\langle\cdot,\cdot\rangle_\lambda : V_\ell X \times V_\ell X \to \mathbb{Q}_\ell^\times$.
- (X, λ) determines $\gamma_0 = \gamma_{X/\mathbb{F}_q,\ell} \in \mathrm{GSp}(V_\ell, \langle\cdot,\cdot\rangle_\lambda) \cong \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$, up to conjugacy.

Main result

Theorem (A.–Altuğ–Gordon)

Let $(X, \lambda)/\mathbb{F}_q$ be a simple, ordinary, principally polarized abelian variety of dimension g. Then

$$\widetilde{\#}I(X,\lambda) = \frac{2}{(2\pi)^g}\sqrt{\left|D_{\mathrm{GSp}}(\gamma)\right|}\prod_{\ell<\infty}\nu_\ell(X,\lambda).$$

Still writing, but for $g = 1$ see *Pacific J Math*, 2017.

Main result

Theorem (A.–Altuğ–Gordon)

Let $(X,\lambda)/\mathbb{F}_q$ be a simple, ordinary, principally polarized abelian variety of dimension g. Then

$$\widetilde{\#}I(X,\lambda) = \frac{2}{(2\pi)^g}\sqrt{|D_{\mathrm{GSp}}(\gamma)|}\prod_{\ell<\infty}\nu_\ell(X,\lambda).$$

For all but finitely many ℓ,

$$\nu_\ell(X,\lambda) = \nu_\ell(X,\lambda)^{\text{naïve}} := \frac{\#\left\{\gamma\in \mathrm{GSp}_{2g}(\mathbb{Z}/\ell) : \gamma\sim\gamma_{X/\mathbb{F}_q} \bmod \ell\right\}}{\#\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)/\ell^g}.$$

Main result

Theorem (A.–Altuğ–Gordon)

Let $(X, \lambda)/\mathbb{F}_q$ be a simple, ordinary, principally polarized abelian variety of dimension g. Then

$$\widetilde{\#}I(X,\lambda) = \frac{2}{(2\pi)^g}\sqrt{|D_{\mathrm{GSp}}(\gamma)|}\prod_{\ell<\infty}\nu_\ell(X,\lambda).$$

$D_{\mathrm{GSp}}(\gamma)$ is Weyl discriminant $\det(1 - \mathrm{Ad}(\gamma_0)|\mathfrak{g}/\mathfrak{g}_{\gamma_0})$:

$$D_{\mathrm{GSp}}(\gamma) = q^{-\frac{g(3g-1)}{2}}\operatorname{disc}(f_{X/\mathbb{F}_q}(T))/\operatorname{disc}(f^+_{X/\mathbb{F}_q}(T))$$

$$f_{X/\mathbb{F}_q}(T) = \prod_{1\le j\le g}(T-\alpha_j)(T-\overline{\alpha}_j)$$

$$f^+_{X/\mathbb{F}_q}(T) = \prod_j (T-(\alpha_j+\overline{\alpha}_j)).$$

Langlands-Kottwitz

- Cohomology groups:

$$H^p(X) := \varprojlim_{p \nmid n} H^1(X_{\overline{\mathbb{F}_q}}, \mathbb{Z}/n) = H^1(X_{\overline{\mathbb{F}_q}}, \mathbb{A}_f^p)$$

  has linear operator $\varpi_{X/\mathbb{F}_q}$;

$$H_p(X) = H^1_{\mathrm{cris}}(X) \otimes_{\mathbb{Z}_q} \mathbb{Q}_q$$

  has $\sigma^{\pm 1}$-linear operators F and V.
- Y^p the set of lattices in $H^p(X)$.
- Y_p the set of F, V-stable lattices in $H_p(X)$.
- Automorphisms of (X, λ): $T_{(X,\lambda)}/\mathbb{Q}$ represents

$$T_{(X,\lambda)}(R) = \left\{ \alpha \in (\mathrm{End}(X) \otimes R)^\times : \alpha\alpha^{(\dagger)} \in R^\times \right\}.$$

Isogeny classes and lattices

Lemma

There is a bijection between

$$I((X,\lambda),\mathbb{F}_q)$$

and

$$T_{(X,\lambda)}(\mathbb{Q})\backslash Y^p \times Y_p.$$

Idea

Isogeny $\alpha : X \to X'$ gives $\alpha^* H(X') \subset H(X)$.

From lattices to GSp_{2g}

- Set $G = \mathrm{GSp}_{2g}$.
- Choose $H^p(X) \cong (\mathbb{A}_f^p)^{2g}$, $H_p(X) \cong \mathbb{Q}_q^{2g}$.
- $$\gamma_0 = \gamma_{(X,\lambda)/\mathbb{F}_q} \in G(\mathbb{A}_f^p) \quad \textit{q-Frobenius}$$
  $$\delta_0 = \delta_{(X,\lambda)/\mathbb{F}_q} \in G(\mathbb{Q}_q) \quad \textit{p-Frobenius}$$
- $G_{\gamma_0} \subset G_{\mathbb{A}_f^p}$ centralizer; $T_{(X,\lambda)} \times \mathbb{A}_f^p \cong G_{\gamma_0}$
- $G_{\delta_0\sigma} \subset G_{\mathbb{Q}_p}$ twisted centralizer; $T_{(X,\lambda)} \times \mathbb{Q}_p \cong G_{\delta_0\sigma}$.
- Compatibilities:

$$\mathrm{charpoly}_{\gamma_0}(T) = f_{X/\mathbb{F}_q}(T)$$
$$\mathrm{N}_{\mathbb{Q}_q/\mathbb{Q}_p}(\delta_0) = \delta_0\delta_0^{\sigma}\cdots\delta_0^{\sigma^{[\mathbb{F}_q:\mathbb{F}_p]-1}} \sim \gamma_0$$

Orbital integrals

Theorem (Langlands-Kottwitz)

We have

$$\widetilde{\#}I(X,\lambda) = \mathrm{vol}(T_{(X,\lambda)}(\mathbb{Q})\backslash T_{(X,\lambda)}(\mathbb{A}_f)) \times \int_{G_{\gamma_0}(\mathbb{A}_f^p)\backslash G(\mathbb{A}_f^p)} \mathbb{1}_{G(\hat{\mathbb{Z}}^p)}(g^{-1}\gamma_0 g)\, d\mu^{\mathrm{can}}(g) \times \int_{G_{\delta_0\sigma}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_q)} \mathbb{1}_{G(\mathbb{Z}_q)\left(\begin{smallmatrix} I_g & 0 \\ 0 & pI_g \end{smallmatrix}\right)G(\mathbb{Z}_q)}(h^{-1}\delta_0 h^{\sigma})\, d\mu^{\mathrm{can}}(h)$$

where the measure on the orbit is the canonical measure.

Idea

$\alpha^* H^p(E') = gH^p(E)$; $\mathbb{1}_{G(\hat{\mathbb{Z}}^p)}(g^{-1}\gamma_0 g)$ enforces *Frobenius is ℓ-adic unit.*

Towards local terms

Want natural local factors $\nu_\ell(X,\lambda)$ which compute

$$\int_{G_{\gamma_0}(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \mathbb{1}_{G(\mathbb{Z}_\ell)}(g^{-1}\gamma_0 g)\, d\mu^{\mathrm{can}}(g).$$

$G(\mathbb{Q}_\ell)$ vs. $G(\overline{\mathbb{Q}}_\ell)$ Can't use $f_{X/\mathbb{F}_q}(T)$; conjugacy and stable conjugacy are different.

$G(\mathbb{Q}_\ell)$ vs. $G(\mathbb{Z}_\ell)$ Can't use $G(\mathbb{Z}_\ell)$-conjugacy;

$$\left\{g^{-1}\gamma_0 g : g \in G(\mathbb{Z}_\ell)\right\} \subsetneq \left\{g^{-1}\gamma_0 g : g \in G(\mathbb{Q}_\ell)\right\} \cap G(\mathbb{Z}_\ell).$$

Integral matrices

$G = \mathrm{GSp}(V) \cong \mathrm{GSp}_{2g,\mathbb{Z}}$.

- Integral matrices:

$$M(\mathbb{Z}_\ell) := \mathrm{GSp}(V \otimes \mathbb{Q}_\ell) \cap \mathrm{End}(V \otimes \mathbb{Z}_\ell) \cong \mathrm{GSp}_{2g}(\mathbb{Q}_\ell) \cap \mathrm{Mat}_{2g}(\mathbb{Z}_\ell).$$

$$M(\mathbb{Z}_\ell)_d := \{A \in M(\mathbb{Z}_\ell) : \mathrm{ord}_\ell \det(A) \leq d\}.$$

  Then $M(\mathbb{Z}_\ell)_0 = G(\mathbb{Z}_\ell)$.
- Truncation: Given $Z/\mathbb{Z}_\ell$, we have $\pi_n = \pi_n^Z : Z(\mathbb{Z}_\ell) \to Z(\mathbb{Z}_\ell/\ell^n)$.
- Conjugation without inversion: If $\overline{\gamma} \in G(\mathbb{Z}_\ell/\ell^n)$, write

$$\overline{\gamma} \sim_{M(\mathbb{Z}_\ell/\ell^n)_d} \gamma_0$$

  if there exists $A \in M(\mathbb{Z}_\ell)_d$ such that

$$\pi_n(A)\overline{\gamma} = \pi_n(\gamma_0)\pi_n(A).$$

Local terms

- Generalized conjugacy class:

$$C_{(d,n,\ell)}(\gamma_0) = \left\{\gamma \in G(\mathbb{Z}_\ell/\ell^n) : \gamma \sim_{M(\mathbb{Z}_\ell/\ell^n)_d} \gamma_0\right\}.$$

- Space of characteristic polynomials: $A = \mathbb{A}^g \times \mathbb{G}_m$.
- Local factor:

$$\nu_\ell([X,\lambda]) = \lim_{d\to\infty} \lim_{n\to\infty} \frac{\# C_{(d,n,\ell)}(\gamma_0)}{\# G(\mathbb{Z}_\ell/\ell^n)/\# A(\mathbb{Z}_\ell/\ell^n)}.$$

Local terms

- Generalized conjugacy class:

$$C_{(d,n,\ell)}(\gamma_0) = \left\{\gamma \in G(\mathbb{Z}_\ell/\ell^n) : \gamma \sim_{M(\mathbb{Z}_\ell/\ell^n)_d} \gamma_0\right\}.$$

- Space of characteristic polynomials: $A = \mathbb{A}^g \times \mathbb{G}_m$.
- Local factor:

$$\nu_\ell([X,\lambda]) = \lim_{d\to\infty}\lim_{n\to\infty} \frac{\#C_{(d,n,\ell)}(\gamma_0)}{\#G(\mathbb{Z}_\ell/\ell^n)/\#A(\mathbb{Z}_\ell/\ell^n)}.$$

Lemma

If $\ell \nmid \operatorname{disc}(f(T))$, *then*

$$\nu_\ell([X,\lambda]) = \frac{\#\{\gamma \in G(\mathbb{Z}_\ell/\ell) : \gamma \sim \gamma_0\}}{\#G(\mathbb{Z}_\ell/\ell)/\#A(\mathbb{Z}_\ell/\ell)}.$$

Strategy of proof

- Kottwitz formula integrates against canonical measure. ($G(\mathbb{Z}_\ell)$ gets volume one.)
- $\nu_\ell([X,\lambda])$ is an integral against geometric measure.

$$G \xrightarrow{\mathfrak{c}} A$$

$$\omega_G = \omega^{\text{geom}}_{\mathfrak{c}(\gamma)} \wedge \omega_A.$$

- Theorem reduces to careful comparison of measures on orbit of γ_0.

Also, use fundamental lemma to replace twisted orbital integral on $G(\mathbb{Q}_q)$ with usual integral on $G(\mathbb{Q}_p)$.

Thanks!