Rational Point Count Distributions for del Pezzo Surfaces over Finite Fields

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A Counting Problem

A homogeneous cubic polynomial in w, x, y, z is defined by 20 coefficients:

$$f_3(w,x,y,z) = a_0 w^3 + a_1 w^2 x + \cdots + a_{19} z^3.$$

Question

1. *How many of these q^{20} cubic polynomials define a smooth cubic surface in $\mathbb{P}^3$ with $q^2 + 7q + 1$ $\mathbb{F}_q$-points?*
2. *What about other rational point counts?*

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A homogeneous quartic polynomial in x, y, z is defined by 15 coefficients:

$$f_4(x, y, z) = a_0 x^4 + a_1 x^3 y + \cdots + a_{14} z^4.$$

Question

1. *How many of these q^{15} quartic polynomials define a smooth quartic curve such that $w^2 = f_4(x, y, z)$ has $q^2 + 8q + 1$ $\mathbb{F}_q$-points?*
2. *What about other rational point counts?*

del Pezzo Surface Definitions

Definition

A del Pezzo surface X is a smooth surface such that $-K_X$ is ample. The degree of X is $K_X^2 = d$.

- $1 \leq d \leq 9$.

Example

Let X_d be a degree d del Pezzo surface defined over an arbitrary field k.

1. X_4 is isomorphic to the intersection of two quadrics in $\mathbb{P}^4$.
2. X_3 is isomorphic to a cubic surface in $\mathbb{P}^3$.
3. X_2 is isomorphic to a hypersurface of degree four in the weighted projective space $\mathbb{P}(2, 1, 1, 1)$.
4. X_1 is isomorphic to a hypersurface of degree six in the weighted projective space $\mathbb{P}(3, 2, 1, 1)$.

del Pezzo Surfaces as Blow-ups

Theorem

Let X_d be a del Pezzo surface of degree d over an algebraically closed field. If $1 \leq d \leq 7$, X_d is the blow-up of $\mathbb{P}^2$ at $9 - d$ points in general position:

1. *No three points on a line.*
2. *No six lie on a conic.*
3. *No eight lie on a cubic with a singularity at one of the points.*

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Example

- A del Pezzo surface of degree 3 defined over $\mathbb{F}_q$ is the blow-up of $\mathbb{P}^2(\overline{\mathbb{F}_q})$ at 6 points in general position.
- These points **do not** have to be in $\mathbb{P}^2(\mathbb{F}_q)$.
- The anti-canonical embedding takes this blow-up to a smooth cubic surface in $\mathbb{P}^3$.

Exceptional Curves and the Picard Group

Definition

An exceptional curve of X_d is an irreducible genus zero curve with self-intersection -1.

X_3 has 27 exceptional curves:

- $\binom{6}{1}$ from the points you blew up,
- $\binom{6}{2}$ from the lines connecting these points,
- $\binom{6}{5}$ from conics passing through five of the six points.

The anti-canonical embedding takes the 27 exceptional curves to the 27 lines of the cubic surface.

Let $\overline{X} = X \otimes \overline{\mathbb{F}_q}$. Then $\mathrm{Pic}(\overline{X}) = \langle L, E_1, \ldots, E_{9-d} \rangle$.

del Pezzo Surfaces of degree 2

A del Pezzo surface of degree 2 defined over $\mathbb{F}_q$ is the blow-up of $\mathbb{P}^2(\overline{\mathbb{F}_q})$ at 7 points in general position.

The anti-canonical linear system gives a $2:1$ map to $\mathbb{P}^2$ branched over a plane quartic curve.

X_2 has 56 exceptional curves:

- $\binom{7}{1}$ from the points you blew up,
- $\binom{7}{2}$ from the lines connecting these points,
- $\binom{7}{5}$ from conics passing through five of the six points.
- $\binom{7}{6}$ from cubics through these seven points and singular at one.

The anti-canonical linear system maps pairs of these 56 exceptional curves to the 28 bitangents of the plane quartic.

The action of $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on $\text{Pic}(\overline{X})$

$\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ acts on $\text{Pic}(\overline{X})$.
This action preserves $K_{\overline{X}}$ and the intersection form.

We can identify $K_S^{\perp}$ with the lattice E_{9-d}
(where $E_3 = A_2 \times A_1,\ E_4 = A_4,\ E_5 = D_5$).

The image Γ of $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ in $\text{Aut}(\text{Pic}(\overline{X}))$ gives a cyclic subgroup of the corresponding Weyl group $W(E_{9-d})$.

Question (Andrey's Talk)

For each d, each cyclic subgroup $\Gamma \subset W(E_{9-d})$, and each finite field $\mathbb{F}_q$ can we find X_d such that the image of $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is conjugate to Γ?

Cubic surfaces: See [Rybakov-Trepalin, 16].
del Pezzo surfaces of degree 2: See [Trepalin, 16].

$\mathbb{F}_q$-points on del Pezzo surfaces

Theorem

Let X_d be a del Pezzo surface of degree d over $\mathbb{F}_q$. Then

$$\#X_d(\mathbb{F}_q) = q^2+q+1+aq, \quad \text{where } a \in \begin{cases} \{-3,-2,\dots,4,6\} & \text{if } d=3 \\ \{-7,-5,-4,\dots,5,7\} & \text{if } d=2 \end{cases}.$$

Definition

A del Pezzo surface is split if and only if all the exceptional curves are defined over $\mathbb{F}_q$.

X_d is split if and only if:

- $\#X_d(\mathbb{F}_q)$ is as large as possible.
- The [lines/bitangents] of the [cubic surface/plane quartic] are all defined over $\mathbb{F}_q$.
- The points you blew up were all in $\mathbb{P}^2(\mathbb{F}_q)$.

Which rational point counts arise?

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Which rational point counts actually arise?

Banawit-Fité-Loughran answer this question for each q.

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Example

$\#X_2(\mathbb{F}_q) = q^2+8q+1$ if and only if there exist 7 points in $\mathbb{P}^2(\mathbb{F}_q)$ in general position. [Exists for $q \geq 9$.]

$\#X_2(\mathbb{F}_q) = q^2+6q+1$ if and only if there exist 5 points in $\mathbb{P}^2(\mathbb{F}_q)$ and a pair of Galois conjugate points defined over $\mathbb{F}_{q^2}$ in general position. [Exists for $q \geq 5$.]

Counting Points in General Position

Theorem

1. *The number of collections of 6 points in $\mathbb{P}^2(\mathbb{F}_q)$ in general position is*

$$|\mathrm{PGL}_3(\mathbb{F}_q)| \cdot (q-2)(q-3)(q-5)^2.$$

2. *The number of collections of 7 points in $\mathbb{P}^2(\mathbb{F}_q)$ in general position is*

$$|\mathrm{PGL}_3(\mathbb{F}_q)| \cdot (q-7)(q-5)(q-3)(q^3-20q^2+119q-175).$$

Theorem (Elkies, K.)

1. *The number of cubics $f_3(w,x,y,z)$ that define a split cubic surface over $\mathbb{F}_q$ is*

$$\frac{|\mathrm{GL}_4(\mathbb{F}_q)|(q-2)(q-3)(q-5)^2}{|W(E_6)|}.$$

2. *The number of quartics $f_4(x,y,z)$ with $w^2 = f_4(x,y,z)$ split is*

$$\frac{(q-1)|\mathrm{PGL}_3(\mathbb{F}_q)| \cdot (q-7)(q-5)(q-3)(q^3-20q^2+119q-175)}{|W(E_7)|}.$$

Arcs in the Projective Plane

Definition

An n-arc in $\mathbb{P}^2(\mathbb{F}_q)$ *is an ordered collection of n-distinct points, no three on a line.*

Let $C_n(q)$ *denote the number of n-arcs in* $\mathbb{P}^2(\mathbb{F}_q)$*.*

Example

$$C_4(q) = (q^2+q+1)(q^2+q)q^2(q-1)^2 = |\mathrm{PGL}_3(\mathbb{F}_q)|.$$

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Theorem (Glynn)

For q odd,

$$C_7(q) = |\mathrm{PGL}_3(\mathbb{F}_q)|(q - 3)(q - 5)(q^4 - 20q^3 + 148q^2 - 468q + 498).$$

For q even, subtract an additional term for the number of copies of the Fano plane in $\mathbb{P}^2(\mathbb{F}_q)$.

Counting collections of 7-points in general position

1. Start with formula for $C_7(q)$.
2. Count collections of 7 points on a conic.
3. Count 7-arcs with 6 points on a conic.

When q is odd, every point in $\mathbb{P}^2(\mathbb{F}_q)$ not on a conic lies on either 0 or 2 rational tangent lines.

When q is even...

Definition. A curve X in $\mathbf{P}^n$ is *strange* if there is a point A which lies on all the tangent lines of X.

Example 3.8.2. A conic in $\mathbf{P}^2$ over a field of characteristic 2 is strange. For example, consider the conic $y = x^2$. Then $dy/dx \equiv 0$, so all the tangent lines are horizontal, so they all pass through the point at infinity on the x-axis.

Points in General Position to $w^2 = f_4(x, y, z)$

Question

Given $p_1, \ldots, p_7$ *in general position, how do we find an equation of the form* $w^2 = f_4(x, y, z)$ *isomorphic to their blow-up?*

Points in General Position to $w^2 = f_4(x, y, z)$

Question

Given $p_1, \ldots, p_7$ in general position, how do we find an equation of the form $w^2 = f_4(x, y, z)$ isomorphic to their blow-up?

1. Let x, y, z be a basis for the 3-dimensional space of cubics vanishing at $p_1, \ldots, p_7$.
2. There is a 7-dimensional space of sextics vanishing to order at least 2 at each of these 7.
3. $\langle x^2, xy, xz, y^2, yz, z^2 \rangle$ is a 6-dimensional subspace.
4. Choose w not in this subspace.
5. w satisfies an equation $w^2 + w \cdot f_2(x, y, z) + f_4(x, y, z) = 0$.
6. Complete the square. [q odd]

See Dolgachev, *Classical Algebraic Geometry: A Modern View*.

How many times does each curve arise?

Proposition

$$\sum_{C} \frac{1}{\# \operatorname{Aut}(C)} = \frac{2(q-7)(q-5)(q-3)(q^3-20q^2+119q-175)}{|W(E_7)|},$$

where the sum is over non-isomorphic smooth plane quartics with 28 $\mathbb{F}_q$-rational bitangents.

Classification Strategy:

1. Find all collections of 7 points in $\mathbb{P}^2(\mathbb{F}_q)$ in general position.
2. Blow them up to find equations $w^2 = f_4(x,y,z)$.
3. Compute $\# \operatorname{Aut}(C)$ for $\{f_4 = 0\}$.
4. Stop when you've found 'enough'.

Classifying split del Pezzo surfaces of degree 2 over $\mathbb{F}_q$

Example

- For $q = 9$,

$$\frac{2(q-7)(q-5)(q-3)(q^3-20q^2+119q-175)}{|W(E_7)|} = \frac{1}{6048}.$$

  Blow up any tuple of 7 points in general position and get $w^2 = f_4(x,y,z)$ where $\{f_4 = 0\}$ is isomorphic to the Fermat quartic, which has 6048 automorphisms.

- Similar story for $\mathbb{F}_{11}$ and the Klein quartic.
- Over $\mathbb{F}_{13}$, two isomorphism classes.
  Over $\mathbb{F}_{17}$ there are seven.
  Over $\mathbb{F}_{19}$ there are fourteen.
  ...

Full del Pezzo Surfaces

Definition

A del Pezzo surface is full if it is split and every $\mathbb{F}_q$-point lies on an exceptional curve.

Example

1. del Pezzo surfaces of degree 6 or greater are never full.
2. A split del Pezzo surface of degree 5 over $\mathbb{F}_q$ is full if and only if $q \in \{2, 3\}$.
3. A split del Pezzo surface of degree 4 over $\mathbb{F}_q$ is full if and only if $q = 5$.
4. Hirschfeld classified full cubic surfaces:
   They exist only when $q \in \{4, 7, 8, 9, 11, 13, 16\}$.

Full del Pezzo surfaces of degree 2 can only exist for small q.
Must have $56(q + 1) \geq q^2 + 8q + 1$.

Other Rational Point Counts

Theorem (Elkies, K.)

1. *The number of cubics $f_3(w,x,y,z)$ that define a cubic surface S over $\mathbb{F}_q$ with $\#S(\mathbb{F}_q) = q^2+3q+1$ is*

$$\frac{|\operatorname{GL}_4(\mathbb{F}_q)| \cdot 120 \cdot (2q^4+9q^3-27q^2+182q-270)}{|W(E_6)|}.$$

2. *The number of quartics $f_4(x,y,z)$ such that $w^2 = f_4(x,y,z)$ is a degree 2 del Pezzo surface S with $\#S(\mathbb{F}_q) = q^2+5q+1$ is*

$$\frac{|\operatorname{GL}_3(\mathbb{F}_q)| \cdot 63(q-3)(q^5-12q^4+146q^3-1235q^2+4461q-5185)}{|W(E_7)|}.$$

Strategy:

1. Equivalent to finding individual coefficients of the Hamming weight enumerator of a certain evaluation code.
2. Compute all but a few coefficients by analyzing singular varieties.
3. Compute the lowest weight coefficients of the dual code.
4. Use the MacWilliams theorem to solve for the few unknown coefficients.