Picard curves with small conductor

joint work with Michel Börner and Stefan Wewers

Irene Bouw

Ulm University

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Motivation

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Smallest cases known to me (curves):

$$g = 1: \quad X_1(11) \qquad N = 11,$$
$$g = 2: \quad X_1(13) \qquad N = 13^2.$$
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The conductor is a product of local factors:

$$N = \prod_p p^{f_p}, \qquad (K = \mathbb{Q}).$$

This talk: For Picard curves over $\mathbb{Q}$ we find restrictions on the conductor exponent f_p, which can be computed from the stable reduction of Y at p.

Models

Let p prime, $K = \mathbb{Q}_p^{\mathrm{nr}}$, $\mathcal{O} \subset K$ ring of integers, $k = \mathcal{O}/(p)$. A **model** $\mathcal{Y}$ of Y is a normal proper flat $\mathcal{O}$-scheme with $\mathcal{Y} \otimes_{\mathcal{O}} K \simeq Y$.

Y has **good reduction** if there exists a model with $\overline{Y} := \mathcal{Y} \otimes_{\mathcal{O}} k$ smooth. Otherwise, **bad reduction**. (This includes potentially good but not good reduction.)

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A result of Deligne–Mumford

Assume $g(Y) \geq 2$. There exists a Galois extension L/K and a unique minimal semistable model $\mathcal{Y}$ over $\mathcal{O}_L$, such that $\Gamma := \mathrm{Gal}(L, K)$ acts on $\mathcal{Y}$, hence on the special fiber $\overline{Y}$.

Picard curve over K, char$(K) \neq 3$

$$Y: \qquad y^3 = f(x), \qquad f \in K[x] \text{ separable of degree } 4$$

The cover

$$Y \to \mathbb{P}^1, \qquad (x, y) \mapsto x.$$

is Galois over $K(\zeta_3)$ with group generated by $\sigma(x, y) = (x, \zeta_3 y)$.

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$$\mathcal{Y}_f \text{ minimal } \Leftrightarrow \quad 0 \leq \nu_p(\Delta(f)) < 36.$$

Assume $\mathcal{Y}_f$ minimal and $p \neq 3$, then

$$Y \text{ good reduction } \Leftrightarrow p \nmid \Delta(f).$$

Hence if $p \mid \Delta(f)$ there is no **other** model with $\overline{Y}$ smooth.

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$$\exists \tau \in \Gamma = \mathrm{Gal}(L/K) \text{ with } \tau(\zeta_3) = \zeta_3^2.$$

τ acts k-linearly on $\overline{Y}$ and

$$\tau^{-1}\sigma\tau = \sigma^2 \neq \sigma \qquad \in \mathrm{Aut}_k(\overline{Y}).$$

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Remark: Assume that Y acquires good reduction over $L \ni \zeta_3$. Analyzing the action of σ and $\tau \in \mathrm{Gal}(L/K)$ on $\overline{Y}$ shows that

$$g(\overline{Y}/\langle\tau\rangle) = 0.$$

The conductor exponent

Let $\mathcal{Y}/\mathcal{O}_L$ stable model, $\Gamma = \mathrm{Gal}(L/K)$, $\overline{Y}^0 = \overline{Y}/\Gamma$.

Proposition

$f_p = \delta + \epsilon$ with

$$\epsilon = 2g(Y) - \dim H^1_{\mathrm{et}}(\overline{Y}^0, \mathbb{Q}_\ell),$$
$$\delta = 0 \quad \Leftrightarrow \quad L/K \quad \text{tame}.$$

We have

$$\dim H^1_{\mathrm{et}}(\overline{Y}^0, \mathbb{Q}_\ell) = \sum_{\overline{W}} 2g(\overline{W}) + \gamma(\overline{Y}^0),$$

where the sum runs over the irreducible components of $\overline{Y}^0$ and $\gamma(\overline{Y}^0)$ is the number of loops in the graph of components of $\overline{Y}^0$.

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Corollary

Assume that $Y/\mathbb{Q}_3$ has potentially good reduction at $p = 3$. Then $\epsilon = 6$, hence $f_p \geq 6$.

The case $p \neq 3$

A similar analysis for $p \neq 3$ yields:

Proposition

- $f_2 \neq 1$,
- $f_p \in \{0, 2, 4, 6\}$. (In particular, $\delta = 0$.)

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Computing the stable reduction is much easier if $p \neq 3$. We know the field L over which Y acquires stable reduction explicitly. $\overline{Y}$ is completely determined by the configuration of the branch points.

Brumer–Kramer prove an upper bound for f_p. This yields $f_2 \leq 28$.

Example

The curve

$$Y : \qquad y^3 = x^4 - 1 =: f(x), \qquad \Delta(f) = -2^8$$

has good reduction for $p \neq 2, 3$. Fact: $|\operatorname{Aut}_{\mathbb{C}}(Y)| = 48$.
The conductor is $N = 2^6 \cdot 3^6$.
Y has potentially good reduction at $p = 3$ over a tame extension.
Y reduces to a chain of 3 elliptic curves over $\mathbb{Q}_2^{\mathrm{nr}}(\sqrt[3]{2}, i)$.

The twist $y^3 = x^4 + 1$ has $N = 2^{16} \cdot 3^6$.

Searching for curves with small conductor

Faltings

Let K be a number field, S a finite set of places, and $g \geq 2$.

- The number of curves over K with good reduction outside S (up to isomorphism) if finite.
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Malmskog–Rasmussen have determines all Picard curves over $\mathbb{Q}$ with good reduction outside $S = \{3\}$. These curves have $10 \leq f_3 \leq 21$, hence $N > 2^6 \cdot 3^6$.

The method generalizes in principle to other sets S.

The exceptional primes

Example Let

$$Y: \qquad y^3 = f(x) = x^4 + 14x^2 + 72x - 41, \qquad \Delta(f) = -2^{10} \cdot 3^4 \cdot 5^6.$$

Hence Y has bad reduction at $p = 2, 3, 5$.

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However, we have

$$N = 2^{19} \cdot 3^{13}.$$

Necessary conditions for p to be an exceptional prime:

- $6 \mid \nu_p(\Delta(f))$,
- f splits over $K = \mathbb{Q}_p^{\mathrm{nr}}$. This implies that Y acquires stable reduction over K,
- the Jacobian of Y has good reduction over K.