Correlation spectrum of Morse-Smale flows

(Resonances: Geometric Scattering and Dynamics, CIRM)

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Introduction

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Denote by $d^\nabla : \Omega^k(M, \mathcal{E}) \to \Omega^{k+1}(M, \mathcal{E})$ the induced coboundary operator ($d^\nabla \circ d^\nabla = 0$).

Denote by $\Phi_k^{-t*}(\psi_1)$ the solution of

$$\boxed{\partial_t \psi = -\mathcal{L}_{V,\nabla}^{(k)} \psi, \quad \psi(t=0) = \psi_1,}$$

with

$$\mathcal{L}_{V,\nabla}^{(k)} = (d^{\nabla} + \iota_V)^2 : \Omega^k(M, \mathcal{E}) \to \Omega^k(M, \mathcal{E}).$$

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A question in dynamical systems. Fix $0 \leq k \leq n$. Under which condition

$$\Phi_k^{-t*}(\psi_1)$$

has a limit as $t \to +\infty$ for every ψ_1 in $\Omega^k(M, \mathcal{E})$?

It is convenient to introduce the "correlation function" :

$$\forall t \geq 0, \quad C_{\psi_1,\psi_2}(t) := \int_M \psi_2 \wedge \Phi_k^{-t*}(\psi_1),$$

where $\psi_1 \in \Omega^k(M, \mathcal{E})$ and $\psi_2 \in \Omega^{n-k}(M, \mathcal{E}')$.

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Define also its Laplace transform, for $\text{Re}(z) > 0$ large enough,

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Meromorphic continuation ? Pollicott (1985), Ruelle (1987) : case of Axiom A flows.

This problem can be solved by determining a Banach space $\mathcal{H}_k^m(M, \mathcal{E})$ such that

$$\mathcal{L}_{V,\nabla}^{(k)} : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E})$$

has good spectral properties (e.g. discrete spectrum).

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- Anosov flows (**microlocal approach**) : Tsujii (2010-12), Faure-Sjöstrand (2011), Faure-Tsujii (2013), Dyatlov-Zworski (2013), etc.

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- Axiom A flows : Dyatlov-Guillarmou (2014). Also, in the case of diffeomorphisms : Baladi-Tsujii (2007), Gouëzel-Liverani (2008).

Morse-Smale flows

A point x is said to be wandering if there exist some open neighborhood U of x and some $t_0 > 0$ such that

$$U \cap \left(\cup_{|t| \geq t_0} \varphi^t(U) \right) = \emptyset.$$

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Define the unstable (resp.) stable manifolds :

$$W^{u/s}(\Lambda) := \left\{ x \in M : \lim_{t \to -/+\infty} d(\varphi^t(x), \Lambda) = 0 \right\}.$$

One can prove that, for every x in M, there exists an **unique** (i, j) such that

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If we suppose in addition that

$$\forall x \in M, \quad T_x M = T_x W^u(\Lambda_i) + T_x W^s(\Lambda_j) \quad \text{(transversality)},$$

then we say that φ^t is a **Morse-Smale flow.**

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Hypothesis. In the following, we will always assume that the Lyapunov exponents of the **Morse-Smale flow** verify some (generic) **non-resonance** assumptions related to the Sternberg-Chen Theorem.

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Csq. The flow can be linearized near every Λ_j in a smooth chart.

Pollicott-Ruelle spectrum

Theorem (Dang-R. 2017)

Morse-Smale flow + nonresonance assumption. Let $0 \leq k \leq n$.
Then, there exists a (minimal) discrete subset $\mathcal{R}_k(V, \nabla) \subset \mathbb{C}$ *such that, given any* (ψ_1, ψ_2) *in* $\Omega^k(M, \mathcal{E}) \times \Omega^{n-k}(M, \mathcal{E}')$,

$$\hat{C}_{\psi_1,\psi_2}(z) := \int_0^{+\infty} e^{-tz} \left(\int_M \psi_2 \wedge \Phi_k^{-t*}(\psi_1) \right) dt$$

has a meromorphic extension to $\mathbb{C}$ *whose poles are contained inside* $\mathcal{R}_k(V, \nabla)$.

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has a meromorphic extension to $\mathbb{C}$ *whose poles are contained inside* $\mathcal{R}_k(V, \nabla)$.

$$\mathcal{R}_k(V, \nabla) := \{\text{Pollicott-Ruelle resonances}\}.$$

Elements inside $\mathcal{R}_k(V, \nabla) \subset \mathbb{C}$ correspond to the **discrete spectrum** of

$$\mathcal{L}_{V,\nabla}^{(k)} : \mathcal{H}_k^m(M, \mathcal{E}) \to \mathcal{H}_k^m(M, \mathcal{E})$$

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acting on an appropriate Sobolev space.

Each eigenvalue is associated with a spectral projector $\pi_{z_0}^{(k)}$ and we set

$$\boxed{C_{V,\nabla}^k(z_0) := \text{Ran}\left(\pi_{z_0}^{(k)}\right) \quad \text{(Pollicott-Ruelle resonant states).}}$$

Some comments.

- Compared with previous result on the Axiom A case (Pollicott, Ruelle, Baladi-Tsujii, Gouëzel-Liverani, Dyatlov-Guillarmou), no assumptions on the supports of ψ_1 and ψ_2 (i.e. no cutoff function near the Λ_j).

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  This is **a global result on the dynamics**.

- **Goal.** Computation of this dynamical spectrum + links with topology (global results).

Explicit description of the spectrum

We need to fix some conventions in order to compute the spectrum. For simplicity, we will now suppose that ∇ **preserves an hermitian structure**.

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For a **fixed point** Λ, we define

$$\boxed{\sigma_\Lambda = \{0\},}$$

and the multiplicity

$$\mu_\Lambda(0) = N,$$

where N is the rank of the complex vector bundle.

For a **closed orbit** Λ, we set $\mathcal{P}_\Lambda$ for the minimal period and

$$\varepsilon_\Lambda = 0 \text{ if } W^u(\Lambda) \text{ is orientable, and } \varepsilon_\Lambda = \frac{1}{2} \text{ otherwise.}$$

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We denote by $(e^{2i\pi\gamma_j^\Lambda})_{j=1,\ldots,N}$ the eigenvalues of the monodromy for the parallel transport around Λ.

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We denote by $(e^{2i\pi\gamma_j^\Lambda})_{j=1,\ldots,N}$ the eigenvalues of the monodromy for the parallel transport around Λ. Finally, we set

$$\boxed{\sigma_\Lambda = \left\{ -\frac{2i\pi(\gamma_j^\Lambda + m + \varepsilon_\Lambda)}{\mathcal{P}_\Lambda} : 1 \leq j \leq N,\ m \in \mathbb{Z} \right\},}$$

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and the multiplicity of $z_0 \in \sigma_\Lambda$

$$\mu_\Lambda(z_0) = \left| \left\{ (j, m) : z_0 = -\frac{2i\pi(\gamma_j^\Lambda + m + \varepsilon_\Lambda)}{\mathcal{P}_\Lambda} \right\} \right|,$$

where N is the rank of the complex vector bundle.

Theorem (Dang-R. 2017)

Morse-Smale flow + nonresonance assumption + ∇ preserves an hermitian structure. Let $0 \leq k \leq n$.
Then, one has

$$\mathcal{R}_k(V, \nabla) \subset \{z : Re(z) \leq 0\},$$

and

$$\mathcal{R}_k(V, \nabla) \cap i\mathbb{R} = \bigcup_{\Lambda \text{ fixed point: } \dim W^s(\Lambda) = k} \sigma_\Lambda \cup \bigcup_{\Lambda \text{ closed orbit: } \dim W^s(\Lambda) \in \{k, k+1\}} \sigma_\Lambda .$$

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Moreover, the multiplicity of $z_0 \in \mathcal{R}_k(V, \nabla) \cap i\mathbb{R}$ is

$$\mu_k(z_0) = \sum_{\Lambda \text{ fixed point: } \dim W^s(\Lambda) = k} \mu_\Lambda(z_0) + \sum_{\Lambda \text{ closed orbit: } \dim W^s(\Lambda) \in \{k, k+1\}} \mu_\Lambda(z_0).$$

The rest of the spectrum is described by the following theorem :

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Then, for every critical element Λ, there exists a sequence $(z_{\Lambda,k}(j))_{j\geq 1}$ such that

$$\text{Re}(z_{\Lambda,k}(j)) \leq 0, \quad \lim_{j\to+\infty} \text{Re}(z_{\Lambda,k}(j)) = -\infty,$$

and

$$\boxed{\mathcal{R}_k(V,\nabla) = \bigcup_{\Lambda,j\geq 1} \left(z_{\Lambda,k}(j) + \sigma_\Lambda\right).}$$

Some comments.

- **Closed orbits generate vertical bands of resonances**. We recover, in the context of Morse-Smale flows, the band structure exhibited by Faure and Tsujii in the case of Anosov geodesic flows (2013).

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- The $z_{\Lambda,k}(j)$ are explicit (linear combination of eigenvalues of the linearized system near Λ).

We can already observe that

$$\boxed{\dim\ C^k_{V,\nabla}(0) = \sum_{\Lambda \text{ fixed point: } \dim W^s(\Lambda)=k} N + \sum_{\Lambda \text{ closed orbit: } \dim W^s(\Lambda)\in\{k,k+1\}} m_\Lambda,}$$

where m_Λ is the multiplicity of $e^{2i\pi\varepsilon_\Lambda}$ as an eigenvalue of the monodromy around Λ.

In particular, if the flow has[1] **no fixed point and if** $e^{2i\pi\varepsilon_\Lambda}$ **is never an eigenvalue of** $M_{\mathcal{E}}(\Lambda)$, then

$$\boxed{\forall 0 \leq k \leq n, \quad C^k_{V,\nabla}(0) = \{0\}.}$$

1. These "topological" assumptions appear in the works of Fried on Reidemeister torsion.

Links with topology

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- Twisted De Rham complex :

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- Spectral Morse-Smale complex :

$$0 \xrightarrow{d^{\nabla}} C^0_{V,\nabla}(0) \xrightarrow{d^{\nabla}} C^1_{V,\nabla}(0) \xrightarrow{d^{\nabla}} \dots \xrightarrow{d^{\nabla}} C^n_{V,\nabla}(0) \xrightarrow{d^{\nabla}} 0.$$

Theorem (Dang-R. 2017)

Morse-Smale flow + nonresonance assumption.
Then, the maps

$$\pi_0^{(k)} : \Omega^k(M, \mathcal{E}) \to C^k_{V,\nabla}(0)$$

induce **isomorphisms between the cohomology of the twisted De Rham complex and the cohomology of the spectral Morse-Smale complex**.

Recall that $\pi_0^{(k)}$ is the spectral projector appearing in the residue (at $z = 0$) of the meromorphic extension of

$$\hat{C}_{\psi_1,\psi_2}(z) := \int_0^{+\infty} e^{-tz} \left(\int_M \psi_2 \wedge \Phi_k^{-t*}(\psi_1) \right) dt$$

Some comments.

- In order to prove this Theorem, we use the formal analogy between our problem and Hodge theory :

$$\mathcal{L}_{V,\nabla} = (d^{\nabla} + \iota_V)^2 \quad \text{and} \quad \Delta_{\nabla} = (d^{\nabla} + (d^{\nabla})^*)^2.$$

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- In the case of gradient flows and of the trivial bundle $M \times \mathbb{C}$, we already obtained this result (2016, see Viet's talk).

- In the case of geodesic flows on negatively curved surfaces, Dyatlov and Zworski computed the dimension of

$$C_V^k(0) \cap \text{Ker}(\iota_V)$$

in terms of the Betti numbers of the underlying surface (2016).

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$$\boxed{\sum_{\Lambda:\text{dim } W^s(\Lambda)=k+1} m_\Lambda + N\sum_{j=0}^{k}(-1)^{k-j}c_j(V) \geq \sum_{j=0}^{k}(-1)^{k-j}b_j(M,\mathcal{E}),}$$

with equality in the case $k = n$ and with $c_j(V)$ the **number of fixed points** such that $\text{dim } W^s(\Lambda) = j$.
Recall that m_Λ is the **multiplicity** of $e^{2i\pi\varepsilon_\Lambda}$ as an eigenvalue of the monodromy around Λ.

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Recall that m_Λ is the **multiplicity** of $e^{2i\pi\varepsilon_\Lambda}$ as an eigenvalue of the monodromy around Λ.

In the case of the trivial bundle $M \times \mathbb{C}$, we recover the results of Smale (1959) and Franks (1982).

Torsion

Suppose now that

$$\boxed{\forall 0 \leq k \leq n,\ 0 \notin \mathcal{R}_k(V, \nabla).}$$

Recall that it is equivalent to say that the flow has no fixed points and that $e^{2i\pi\varepsilon_\Lambda}$ is not an eigenvalue of the monodromy (**Fried's assumptions**). This also implies that the **twisted De Rham complex is acyclic.**

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In analogy with Ray-Singer definition of analytic torsion (=Reidmeister torsion, Cheeger and Muller 1978-79), we set :

$$\zeta_{V,\nabla}(s) := \sum_{k=0}^{n} (-1)^k k \sum_{z_0 \in \mathcal{R}_k(V,\nabla) \cap i\mathbb{R}} \frac{\dim \left(C^k_{V,\nabla}(z_0) \right)}{|z_0|^s}.$$

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- The spectral zeta function $\zeta_{V,\nabla}(s)$ has a meromorphic extension to $\mathbb{C}$ with (at most) one pole at $s=1$ which is simple.

- Moreover,

$$\boxed{e^{-\zeta'_{V,\nabla}(0)} = \text{Reidemeister torsion of } (\mathcal{E},\nabla).}$$

Some comments.

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- Proof follows from our explicit description of the spectrum + Fried's dynamical formula for the Reidemeister torsion.

Strategy of the proof

- Construction of anisotropic Sobolev spaces of currents à la Faure-Sjöstrand. It requires to understand the **global** dynamical properties of the Hamiltonian flow induced by :

$$\forall (x,\xi) \in T^*M, \quad H_V(x,\xi) = \xi(V(x)).$$

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$$\forall (x,\xi) \in T^*M, \quad H_V(x,\xi) = \xi(V(x)).$$

- Explicit construction of generalized eigenmodes using Sobolev regularity and the Morse-Smale dynamics.
- Show that these eigenmodes generate all the Pollicott-Ruelle spectrum.

Thank you for your attention.