Spectral properties of the semi-classical scattering matrix

M. Ingremeau

Université Paris-Sud

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Schrödinger operators and generalised eigenfunctions

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such that

$$u(x) = |x|^{-(d-1)/2}\big(e^{-i|x|/h}\phi_{in}(-\omega)+e^{i|x|/h}\phi_{out}(\omega)\big)+O(|x|^{-(d+1)/2}).$$

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We then set

$$S_h\phi_{in} := e^{i\pi(d-1)/2}\phi_{out}.$$

From scattering amplitudes to the scattering matrix

For any $\omega' \in \mathbb{S}^{d-1}$, we may find a function $E_h(x;\omega')$ such that $(P_h - 1)E_h = 0$ and

$$E_h(x;\omega') = e^{\frac{i}{h}\langle\omega',x\rangle} + e^{i|x|/h}|x|^{-\frac{1}{2}(d-1)}\Big(a_h(\omega';\omega) + O\Big(\frac{1}{|x|}\Big)\Big),$$

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We have

$$S_h(\phi_{in})(\omega) = \phi_{in}(\omega) + \int_{\mathbb{S}^{d-1}} a_h(\omega,\omega')\phi_{in}(\omega')\mathrm{d}\omega'.$$

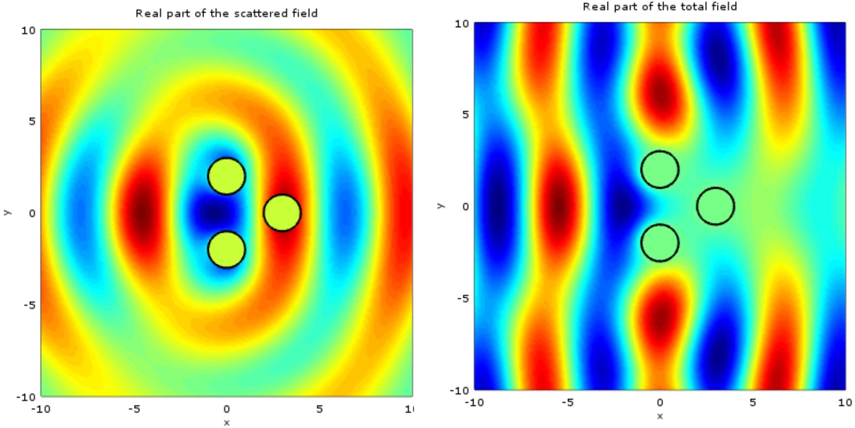

Figure: Here, ω is directed towards the right, and $h = 2\pi$.

Picture made with μ-diff, a program developed by X. Antoine and B. Thierry

Properties of S_h

- S_h is unitary.
- $S_h - Id$ is trace class.
- For every $h > 0$, S_h has discrete spectrum, accumulating only at 1.

These eigenvalues are called *phase shifts*. We will write these eigenvalues as $(e^{i\beta_{h,n}})_{n\in\mathbb{N}}$.

Main theorem

We suppose that *1 is a non-degenerate energy level*, and that *the non-trivial periodic points of the scattering relation have volume zero*.

Theorem (I. 2016)

Let $f \in C^0(\mathbb{S}^1, \mathbb{C})$ such that $1 \notin \mathrm{supp} f$. Then

$$\lim_{h \to 0} \langle \mu_h, f \rangle = \lim_{h \to 0} (2\pi h)^{d-1} \sum_{n \in \mathbb{N}} f(e^{i\beta_{h,n}}) = \frac{\mathrm{Vol}(\mathcal{I})}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

Corollary

Let $0 < \phi_1 < \phi_2 < 2\pi$ be two angles, and let $N_h(\phi_1, \phi_2)$ be the number of eigenvalues $e^{i\beta_{h,n}}$ of S_h with $\phi_1 \leq \beta_{h,n} \leq \phi_2$ modulo 2π. We then have:

$$\lim_{h \to 0} (2\pi h)^{d-1} N_h(\phi_1, \phi_2) = \mathrm{Vol}(\mathcal{I}) \frac{\phi_2 - \phi_1}{2\pi}.$$

Similar results

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- 2015, Gell-Redmann and Hassell: (very) short-range potential, completely different behaviour!

Idea of proof

We have

$$N_h(\phi_1, \phi_2) = \mathrm{Tr}(\mathbf{1}_{[\phi_1,\phi_2]}(S_h)).$$

We approach $\mathbf{1}_{[\phi_1,\phi_2]}$ by polynomials vanishing at 1.

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Proposition

Suppose that the potential V is such that the hypotheses of diversion and weak trapping are satisfied. Let $k \in \mathbb{Z}\backslash\{0\}$. We then have

$$\mathrm{Tr}\big((S_h^k - Id)\big) = -\frac{\mathrm{Vol}(\mathcal{I})}{(2\pi h)^{d-1}} + o(h^{-(d-1)}).$$

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To obtain $\mathrm{Tr}\big((S_h^k - Id)\big) = -\frac{\mathrm{Vol}(\mathcal{I})}{(2\pi h)^{d-1}} + o(h^{-(d-1)})$, we have to show that the trace of S_h^k in this last set is negligible, since the trace of Id in $\mathcal{I}$ is $\frac{\mathrm{Vol}(\mathcal{I})}{(2\pi h)^{d-1}}$.

Link between the scattering matrix and the scattering map

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- 2006 : Alexandrova extended this result to short-range potentials.
- 2008 : Hassell-Wunsch showed an analogous result for non-trapping metric perturbations of asymptotically conical manifolds.

Gaussian states at infinity

Let $(\omega_0, \eta_0) \in T^*\mathbb{S}^{d-1}$, and Γ_0 be a $d \times d$ symmetric matrix, with positive definite real part, and let Q_0 be a polynomial of d variables. We shall write

$$\phi_{\omega_0,\eta_0,\Gamma_0,Q_0}(\omega; h) = Q_0\left(\frac{\omega - \omega_0}{\sqrt{h}}\right) e^{-\frac{i}{h}\eta_0\cdot\omega} e^{-\frac{1}{2h}(\omega-\omega_0)\cdot\Gamma_0(\omega-\omega_0)}.$$

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If A is a trace-class operator, we have

$$\mathrm{Tr}(A) = c_h \int_{T^*\mathbb{S}^{d-1}} d\omega_0 d\eta_0 \langle \phi_{\omega_0,\eta_0,Id,1}, A\phi_{\omega_0,\eta_0,Id,1}\rangle,$$

where $c_h \sim_{h\to 0} (2\pi h)^{-3(d-1)/2}$.

The scattering matrix and Gaussian states

Suppose to simplify that $K = \emptyset$.

Theorem (I. 2017)

Let $(\omega_0, \eta_0) \in T^*\mathbb{S}^{d-1}$, and Γ_0 be a $d \times d$ symmetric matrix, with positive definite real part, and let Q_0 be a polynomial of d variables.
Then there exists $\delta_1 \in \mathbb{R}$, Γ_1 a $d \times d$ symmetric matrix, with positive definite real part, and, for any $N \in \mathbb{N}$, a polynomial of d variables Q_1^N such that

$$S_h \phi_{\omega_0,\eta_0,\Gamma_0,Q_0} = e^{i\frac{\delta_1}{h}} \phi_{\omega_1,\eta_1,\Gamma_1,Q_1^N} + O_{C^0}(h^{(N-1)/2}),$$

with

$$(\omega_1, \eta_1) = \kappa(\omega_0, \eta_0).$$

Corollary

Let $(\omega_0, \eta_0) \in T^*\mathbb{S}^{d-1}$ *be such that* $\kappa^k(\omega_0, \eta_0)$ *is well-defined, and that* $\kappa^k(\omega_0, \eta_0) \neq (\omega_0, \eta_0)$*. Then we have*

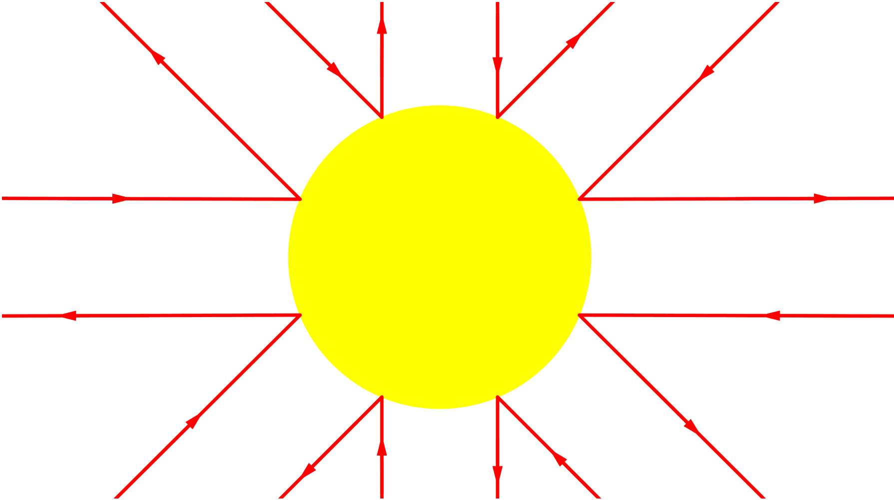
$$\langle S_h^k \phi_{\omega_0,\eta_0,Id,1}, \phi_{\omega_0,\eta_0,Id,1} \rangle = O(h^\infty).$$

Proof.

By iterating the previous theorem, we have $S_h \phi_{\omega_0,\eta_0,Id,1} = \phi_{\omega_k,\eta_k,\Gamma_k,Q_k^N} + O(h^N)$, with $(\omega_k, \eta_k) \neq (\omega_0, \eta_0)$. □

Is it possible to obtain an estimate on the remainder ?

The case of convex obstacles



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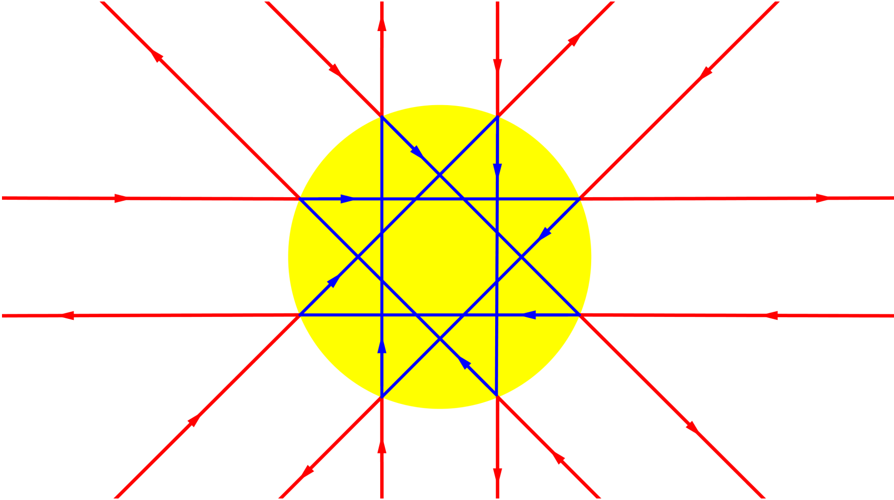

The diversion hypothesis

Fact

For a strictly convex obstacle, the diversion hypothesis is equivalent to Ivrii's conjecture, that the periodic orbits of the interior billiard map have measure zero.

It holds for *analytic* obstacles, as well as for *generic* obstacles (Petkov-Soyanov, 88).

Equidistribution of phase shifts for obstacle scattering

Theorem (Gell-Redman, I., work in progress)

Let Ω be a smooth strictly convex obstacle, which is analytic or generic. Let $k \in \mathbb{Z}\backslash\{0\}$. We then have

$$\mathrm{Tr}\big((S_h^k - Id)\big) = -\frac{\mathrm{Vol}(\mathcal{I})}{(2\pi h)^{d-1}} + O(h^{-(d-1)-1/3}).$$

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Corollary

There exists $\epsilon > 0$ such that

$$(2\pi h)^{d-1} N_h(\phi_1, \phi_2) = \mathrm{Vol}(\mathcal{I}) \frac{\phi_2 - \phi_1}{2\pi} + O\big(|\log h|^{\epsilon}\big).$$

Kirchhoff's approximation

Recall that a_h is the integral kernel of $S_h - Id$.

Theorem (Melrose-Taylor 85)

$$a_h(\omega, \omega') = -\frac{1}{2}(2\pi h)^{1-d} \times \int_{\partial\Omega} e^{\frac{i}{h}(\omega-\omega')\cdot y}(-\nu_y \cdot \omega' + |\nu_y \cdot \omega| + R_h(\omega, y))\mathrm{d}y$$

with

$$R_h \in h^{1/3} S_{1/3}.$$

Here ν_y is the outgoing normal vector at y.

Idea of proof which does not work

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$$\mathrm{Vol}\{(y,\omega) \text{ such that } |\nu_y\cdot\omega| < h^{1/2}\} = O\big(h^{(d-1)/2}\big),$$

which does not compensate the factor $(2\pi h)^{1-d}$ when k becomes large...

Idea of proof which does work

Use Gaussian states !
$\langle \phi_{\omega_0,\eta_0}, (S_h - Id)^k \phi_{\omega_0,\eta_0} \rangle$ can be computed easily, as long as (ω_0, η_0) is far away from the glancing set.
The set of (ω_0, η_0) close to the glancing set has a small volume, and $(S_h - Id)^k$ is bounded by 2^k, so that

$$\int_{(\omega_0,\eta_0) \text{ almost glancing}} \langle \phi_{\omega_0,\eta_0}, (S_h - Id)^k \phi_{\omega_0,\eta_0} \rangle$$

gives a negligible contribution.

Open problems and future projects

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- Is the equidistribution result true for non-convex obstacles ?
- Can we use the Gaussian states construction to describe the properties of the scattering matrix close to the trapped trajectories ?

Thank you for your attention