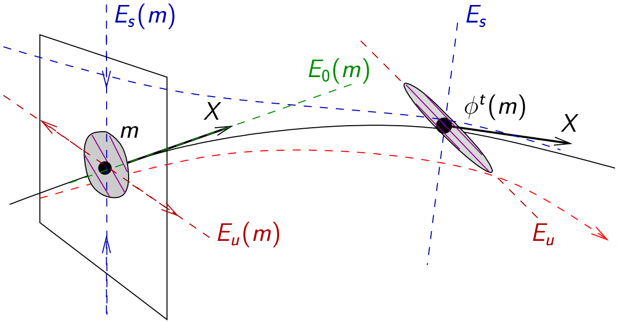
Fractal upper bound for the density of Ruelle Pollicott spectrum of Anosov flows

F. Faure (Grenoble) with M. Tsujii (Kyushu).

March 16,2017, Luminy.

Anosov flow

Let X be a C^∞ **Anosov vector field** on a closed manifold M, $n = \dim M - 1$:



The maps $m \to E_u(m)$, $m \to E_s(m)$ and $m \to E_u(m) \oplus E_s(m)$ are **Hölder continuous** with exponents

$$\beta_u, \beta_s, \quad \beta_0 \geq \min(\beta_u, \beta_s) \in]0, 1], \tag{1}$$

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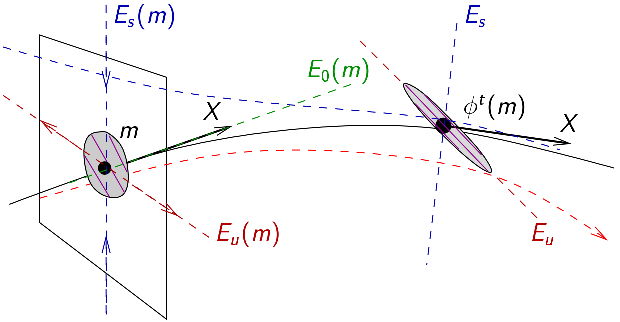

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Special example

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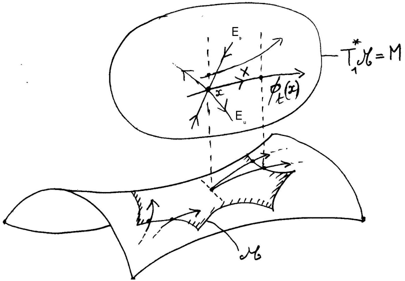
Geodesic flow on $M = T_1^*\mathcal{M}$ is **Anosov**.



It is "**special**": $E_u \oplus E_s = \mathrm{Ker}\left(\underbrace{\sum_j \xi_j dx^j}_{\mathrm{Liouville 1} form}\right)$ is C^∞ hence $\beta_0 = 1$.

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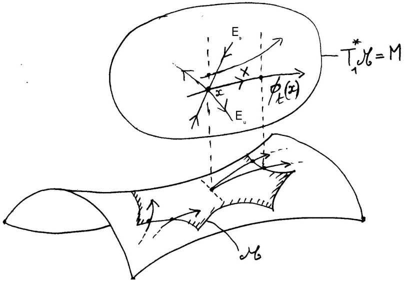

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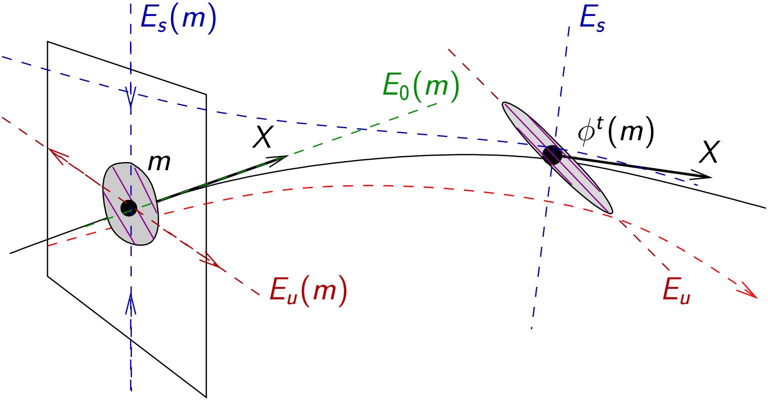
Observations

Let $(m, \eta) \in T^*M$ (lines) and $\tilde{\phi}^t : T^*M \to T^*M$ the lifted flow.



- In $S^*M = T^*M/\mathbb{R}^+$, for $t \to +\infty$, $\left[\tilde{\phi}^t(m,\eta)\right]$ approaches $E_u^* := (E_u \oplus E_0)^\perp$.
  For $t \to -\infty$, $\left[\tilde{\phi}^t(m,\eta)\right]$ approaches $E_s^* := (E_s \oplus E_0)^\perp$.
- The **trapped set** (non wandering set) is $E_0^* := (E_u \oplus E_s)^\perp \subset T^*M$.

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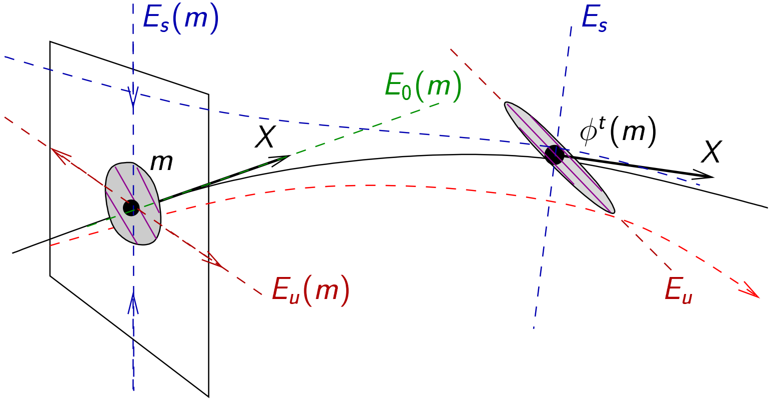

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Transfer operators

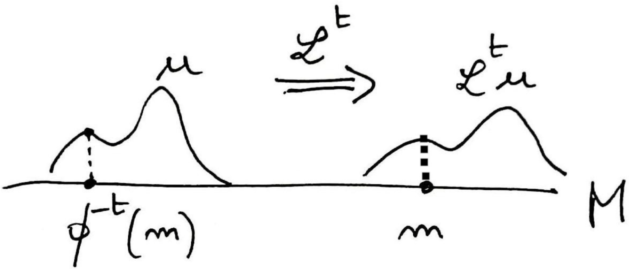
Definition

Let $V \in C^\infty(M;\mathbb{R})$ ("potential"). The first order differential operator on $C^\infty(M)$

$$A := -X + V \tag{2}$$

generates the one-parameter group of **transfer operators,** $t \in \mathbb{R}$**,**

$$\mathcal{L}^t : \begin{cases} C^\infty(M) & \to C^\infty(M) \\ u & \to e^{tA}u = \underbrace{e^{\int_{-t}^{0} V(\phi^s(m))ds}}_{amplitude} \cdot \underbrace{\left(u \circ \phi^{-t}\right)}_{transport} \end{cases} \tag{3}$$



Ruelle Pollicott discrete spectrum of resonances

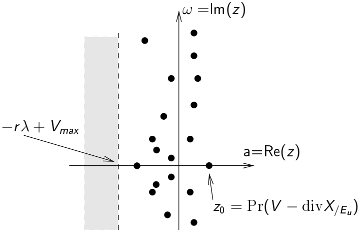
Theorem

[Butterley-Liverani 07, F.-Sjöstrand 11]. For any $r>0$ there exists some ***anisotropic Sobolev space*** *$C^{\infty}(M)\subset\mathcal{H}_W(M)\subset H^{-r}(M)$ such that*

$$\mathcal{L}^t=e^{tA}:\mathcal{H}_W(M)\to\mathcal{H}_W(M),\,t\geq 0$$

extends to a ***strongly continuous semi-group*** *and $A=-X+V$ has* ***discrete spectrum*** *$\sigma(A)$ on $\{z\in\mathbb{C},\quad \mathrm{Re}(z)>\max(V)-r\lambda\}$ independent on the choice of $\mathcal{H}_W(M)$.*

For any generalized eigenfunction u, $WF(u)\subset E_u^*=(E_u\oplus E_0)^{\perp}$ (explained later).



Rem: if $Au=(a+i\omega)u$ then $\mathcal{L}^tu=e^{at}e^{i\omega t}u$

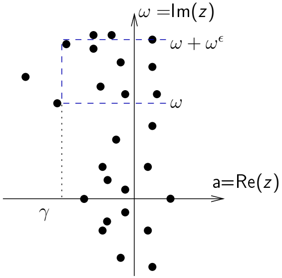
Fractal Weyl law

Theorem

*[F.-Tsujii 17].”**Upper bound for the density of eigenvalues**”. For a general Anosov flow,* $\forall\gamma\in\mathbb{R},\forall 0\leq\epsilon<1,\exists C>0,\forall\omega\geq 1,$

$$\frac{1}{\omega^{\epsilon}}\sharp\left\{\sigma\left(A\right)\cap\left\{z\in\mathbb{C},\mathrm{Re}\left(z\right)>\gamma,\mathrm{Im}\left(z\right)\in\left[\omega,\omega+\omega^{\epsilon}\right]\right\}\right\}\leq C\omega^{\frac{n}{1+\beta_{0}}}.$$

with $\beta_0\in]0,1]$ *is Hölder exponent of* $E_u\oplus E_s$.



Question

Is it sharp generically?

Remarks about the result

$$\frac{1}{\omega^{\epsilon}}\sharp\left\{\sigma(A)\cap\left\{z\in\mathbb{C},\mathrm{Re}(z)>\gamma,\mathrm{Im}(z)\in[\omega,\omega+\omega^{\epsilon}]\right\}\right\}\leq C\omega^{\frac{n}{1+\beta_0}},\qquad \beta_0\in]0,1]. \quad (4)$$

- **Fractal Weyl law** in semiclassical analysis has been discovered by J. Sjöstrand 1990, and many works by Zworski, Nonnenmacher, Weich,..
- Previous results concerning Anosov flows:
  - [F.Sjöstrand 11] Upper bound is $o(\omega^n)$ with $\epsilon=\frac{1}{2}$.
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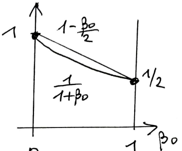
- "Usually" the upper bound is:

$$O\left(\omega^{\frac{\dim_B\left([E_u\oplus E_s]^{\perp}\right)-1}{2}}\right)=O\left(\omega^{n\left(1-\frac{\beta_0}{2}\right)}\right)$$

because $\dim_B\left([E_u\oplus E_s]^{\perp}\right)=(n+1)+n(1-\beta_0)$.
Here $\frac{n}{1+\beta_0}<n\left(1-\frac{\beta_0}{2}\right)$ for $\beta_0\in]0,1[$.



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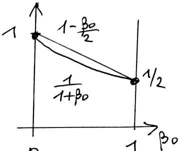

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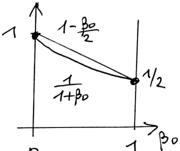
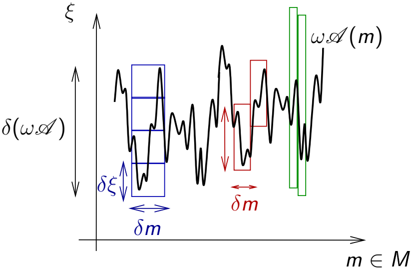
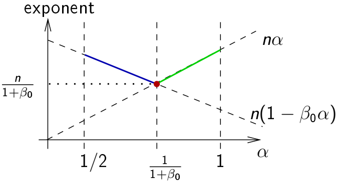


Fast explanation of the fractal exponent $\frac{n}{1+\beta_0}$

Recall **trapped set** is $E_0^* = (E_u \oplus E_s)^\perp = \{\omega \mathscr{A}(m)\,;\ \omega \in \mathbb{R}, m \in M\} \subset T^*M$ with the one form $\mathscr{A}$ s.t. $\mathscr{A}(X) = 1 \quad \mathrm{Ker}(\mathscr{A}) = E_u \oplus E_s$.

Let frequency $\omega \geq 1$. Cover the graph $\omega \mathscr{A}(m)$ by $\mathcal{N}(\omega)$ **symplectic boxes** with size $\delta m \sim \omega^{-\alpha}$ and $\delta\xi \sim \delta m^{-1} \sim \omega^{\alpha}$ (uncertainty principle) with $\frac{1}{2} \leq \alpha < 1$ that we will **optimize**.





$\mathscr{A}(m)$ is β_0-Hölder hence fluctuations have size $\delta(\omega\mathscr{A}) \sim \omega(\delta m)^{\beta_0} = \omega^{1-\alpha\beta_0}$.

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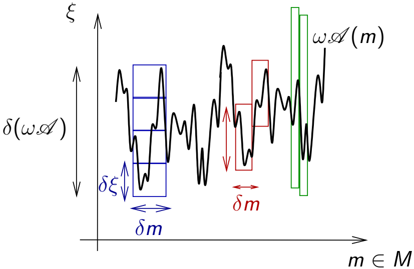

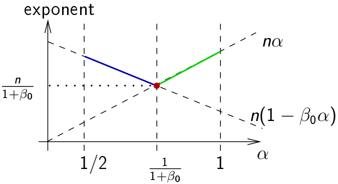

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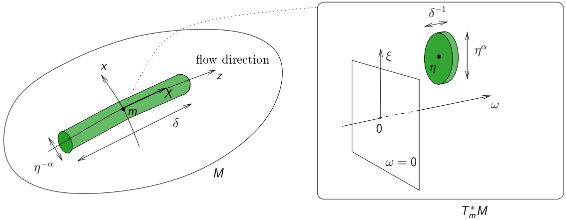
Semiclassical analysis with **wave packets** (or FBI)

- Local flow box coordinates on M: $y = (x, z) \in \mathbb{R}^n \times \mathbb{R}$ s.t. $X = \frac{\partial}{\partial z}$ and dual coordinates $\eta = (\xi, \omega) \in \mathbb{R}^n \times \mathbb{R}$ on $T_y^* M$. Abuse of notations that forget partitions of unity.



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$$\varphi_{(y,\eta)}(y') \underset{|\eta|\gg 1}{\approx} a \exp\left(i\eta.y' - \left| \frac{x' - x}{\langle \eta \rangle^{-\alpha}} \right|^2 - \left| \frac{z' - z}{\delta} \right|^2 \right), \qquad \|\varphi_{(y,\eta)}\|_{L^2(M)} \underset{|\eta|\gg 1}{\approx} 1$$

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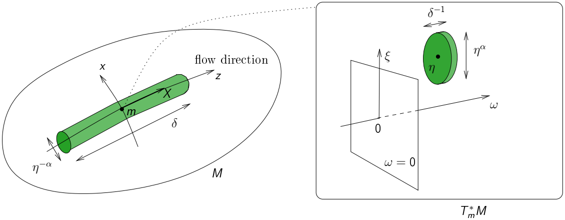

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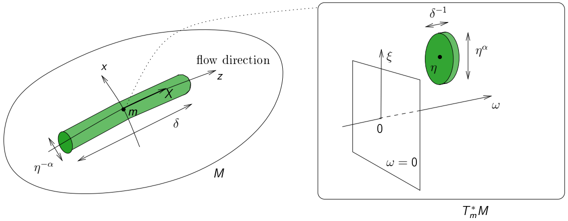

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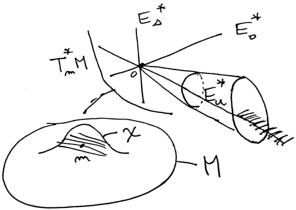
About the **wave front set** of resonances

Theorem

[F.-Sjöstrand 11]. For any (generalized) Ruelle eigenfunction u, i.e. $u \in \mathrm{Im}\Pi(z)$, $WF(u) \subset E_u^* = (E_u \oplus E_0)^\perp$.

It means that $\forall m \in M, \forall \mathrm{Cone}\, C \supset E_u^*(m)$, $\exists \chi \in C^\infty(M)$ with $\chi(m) = 1$, $\forall N > 1, \exists C_N > 0$, $\forall \eta \notin C$,

$$|(\mathcal{F}(\chi u))(\eta)| \leq \frac{C_N}{|\eta|^N}.$$



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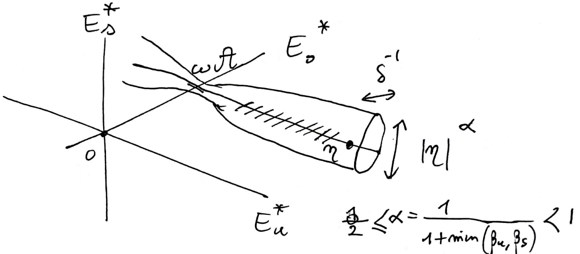
[F.-Tsujii 17] $\forall C>0$, $\forall N, \exists C_N$, *for any (generalized) Ruelle eigenfunction* u, $u \in \mathrm{Im}\Pi(z)$, *with* $\mathrm{Re}(z) > -C$ *then* $\forall \rho = (y,\eta) \in T^*M$,

$$\left|\langle \varphi_\rho, u\rangle_{L^2}\right| \leq \frac{C_N}{\langle \mathrm{dist}_g(\rho, E_u^* + \omega \mathscr{A})\rangle^N}$$

with $\omega = \mathrm{Im} z$.

We choose $\alpha = \frac{1}{1+\min(\beta_u,\beta_s)}$ (but expect $\alpha = \frac{1}{1+\beta_u}$) so that **uncertainty principle absorbs Hölder fluctuations**.



Wave packet transform (or FBI transform)

$$\mathcal{B}_g : \begin{cases} C^\infty(M) & \to \mathcal{S}(T^*M) \\ u(y') & \to (\mathcal{B}_g u)(y,\eta) = \langle \varphi_{y,\eta}, u \rangle_{L^2(M)} \end{cases}$$

Lemma (fundamental 1)

“"*Resolution of identity*”:

$$\mathcal{B}_g^* \circ \mathcal{B}_g = \mathrm{Id} \Leftrightarrow \forall u \in C^\infty(M), \quad u(y') = \int_{T^*M} \varphi_{y,\eta}(y') \langle \varphi_{y,\eta}, u \rangle \frac{dy d\eta}{(2\pi)^{n+1}}.$$

Lemma (fundamental 2)

“"*Micro-locality of the transfer operator*”: $\forall t \geq 0, \forall N > 0, \exists C_{N,t} > 0, \forall \rho, \rho' \in T^*M$,

$$\left| \langle \varphi_{\rho'}, \mathcal{L}^t \varphi_\rho \rangle_{L^2(M)} \right| \leq C_{N,t} \left\langle \mathrm{dist}_g \left(\rho', \tilde{\phi}^t(\rho) \right) \right\rangle^{-N}$$

with $\tilde{\phi}^t : T^*M \to T^*M$ *canonical lift of* ϕ^t.

Rem: particular case $t = 0$.

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Lemma (fundamental 2)

“”*Micro-locality of the transfer operator*”: $\forall t \geq 0, \forall N > 0, \exists C_{N,t} > 0, \forall \rho, \rho' \in T^*M$,

$$\left| \langle \varphi_{\rho'}, \mathcal{L}^t \varphi_\rho \rangle_{L^2(M)} \right| \leq C_{N,t} \left\langle \mathrm{dist}_g \left(\rho', \tilde{\phi}^t(\rho) \right) \right\rangle^{-N}$$

with $\tilde{\phi}^t : T^*M \to T^*M$ *canonical lift of* ϕ^t.

Rem: particular case $t = 0$.

Anti-Wick or Toeplitz quantization

Definition

For a "symbol" $a \in \mathcal{S}(T^*M;\mathbb{R})$

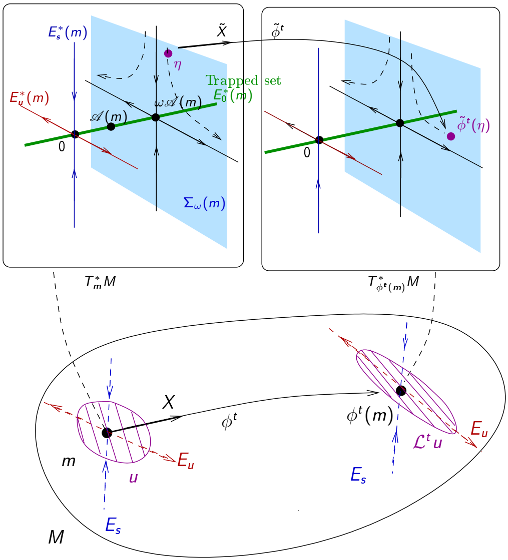
$$\mathrm{Op}(a) := \mathcal{B}_g^* \circ a \circ \mathcal{B}_g = \int_{T^*M} a(\rho)\,\varphi_\rho \langle \varphi_\rho, .\rangle \frac{d\rho}{(2\pi)^{n+1}}.$$

extends to $a \in \mathcal{S}'(T^*M;\mathbb{R})$. We have

$$\|\mathrm{Op}(a)\|_{L^2(M)} \leq |a|_{L^\infty} \tag{5}$$

$$\mathrm{Tr}(\mathrm{Op}(a)) = \int \underbrace{\|\varphi_{y,\eta}\|^2}_{\sim 1} a(\rho) \frac{d\rho}{(2\pi)^{n+1}}.$$

Flow $\tilde{\phi}^t$ in T^*M

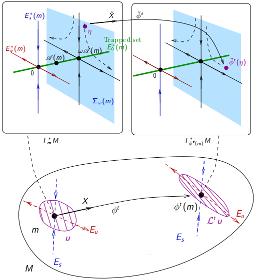


Escape function W in T^*M

Escape function at $\rho = (m, \eta) \in T^*M$:

$$W(\rho) = \frac{\left\langle h_\gamma(\rho) \|\eta_s\|_g \right\rangle^{R_s}}{\left\langle h_\gamma(\rho) \|\eta_u\|_g \right\rangle^{R_s}} \left\langle \|\eta_0\|_g \right\rangle^{R_0}$$

with $R_s, R_u, R_0 > 0$ and $h_\gamma(\rho) = \left\langle \|\eta_u + \eta_s\|_g \right\rangle^{-\gamma}$, $1 - \frac{\alpha^\perp \beta_*}{1-\alpha^\perp} \leq \gamma < 1$, $\frac{1}{1+\beta_0} \leq \alpha^\perp \leq \frac{1}{1+\beta_*}$.



Properties of the escape function W

Escape function at $\rho = (m, \eta) \in T^*M$:

$$W(\rho) = \frac{\left\langle h_\gamma(\rho) \|\eta_s\|_g \right\rangle^{R_s}}{\left\langle h_\gamma(\rho) \|\eta_u\|_g \right\rangle^{R_s}} \left\langle \|\eta_0\|_g \right\rangle^{R_0}$$

with $R_s, R_u, R_0 > 0$ and $h_\gamma(\rho) = \left\langle \|\eta_u + \eta_s\|_g \right\rangle^{-\gamma}$, $1 - \frac{\alpha^\perp \beta_*}{1-\alpha^\perp} \leq \gamma < 1$, $\frac{1}{1+\beta_0} \leq \alpha^\perp \leq \frac{1}{1+\beta_*}$.

Proposition

*W is **temperate**:*

$$\frac{W(\rho')}{W(\rho)} \leq C \left\langle h_\gamma(\rho) \operatorname{dist}_g(\rho', \rho) \right\rangle^{N_0}$$

*and **decays outside a parabolic neighborhood of the trapped set**:*

$$\frac{W\left(\tilde{\phi}^t(\rho)\right)}{W(\rho)} \leq \begin{cases} C & \\ Ce^{-\Lambda t} & \text{if } \|\eta_u + \eta_s\|_{g(\rho)} > C_t \end{cases}$$

(similar to "Weyl-Hörmander calculus 79")

Anisotropic Sobolev space $\mathcal{H}_W(M)$

For $u, v \in C^\infty(M)$

$$\langle u, v \rangle_{\mathcal{H}_W} := \langle W\mathcal{B}_g u, W\mathcal{B}_g v \rangle_{L^2(T^*M)} = \langle u, \mathrm{Op}\left(W^2\right) v \rangle_{L^2(M)}.$$

$$\mathcal{H}_W(M) := \overline{\{u \in C^\infty(M)\}}^{\|.\|_{\mathcal{H}_W}}.$$

Strongly continuous semi-group

Lemma

*Then for any $t \geq 0$, the transfer operator $\mathcal{L}^t = e^{tA} : \mathcal{H}_W \to \mathcal{H}_W$ is **bounded** and they form a **strongly continuous semi-group.***

Proof:

$$\begin{aligned}
\left|\left\langle \delta_{\varrho'}, \left(W\mathcal{B}\mathcal{L}^t\mathcal{B}^* W^{-1}\right)\delta_{\varrho}\right\rangle\right| &= \frac{W(\varrho')}{W(\varrho)} \left|\langle \varphi_{\varrho'}, \mathcal{L}^t \varphi_{\varrho}\rangle\right| \\
&\underset{\text{microlocal of } \mathcal{L}^t}{\leq} \frac{W(\varrho')}{W(\varrho)} C_{N,t} \left\langle \mathrm{dist}_g\left(\rho', \tilde{\phi}^t(\rho)\right)\right\rangle^{-N} \\
&= C_{N,t} \left(\frac{W\left(\tilde{\phi}^t(\rho)\right)}{W(\rho)}\right)\left(\frac{W(\rho')}{W\left(\tilde{\phi}^t(\rho)\right)}\right)\left\langle \mathrm{dist}_g\left(\rho', \tilde{\phi}^t(\rho)\right)\right\rangle^{-N} \\
&\underset{\text{decay\&temperate}}{\leq} C_{N,t}\left\langle \mathrm{dist}_g\left(\rho', \tilde{\phi}^t(\rho)\right)\right\rangle^{N_0-N}
\end{aligned}$$

i.e. decay outside the graph of $\tilde{\phi}^t$. Apply Schur test.
Rem: it also decays outside the trapped set $\to$ **discrete Ruelle Pollicott spectrum**.

Appendix: Hölder exponents

Theorem

[Hasselblatt 94] For any β_s, β_u s.t.

$$\beta_u < \inf_{m \in M} \frac{\lambda^-_{\min}(m) + \lambda^+_{\min}(m)}{\lambda^-_{\max}(m)}, \qquad \beta_s < \inf_{m \in M} \frac{\lambda^-_{\min}(m) + \lambda^+_{\min}(m)}{\lambda^+_{\max}(m)},$$

then $m \to E_u(m)$ (respect. E_s) is β_u-Hölder.