Toeplitz operators for spin systems

Alix Deleporte
Advisor : Nalini Anantharaman

Institut de Recherche Mathématique Avancée
Université de Strasbourg

March 14, 2017

- Low-energy states of pseudodifferential operators $a(x, \hbar D)$ **concentrate** microlocally near the minimal set of the symbol a.
- Subprincipal effects make the situation more precise (Helffer-Sjöstrand): **quantum selection**.

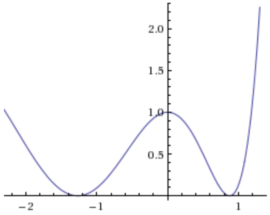

Figure: Under this potential, the first eigenvalue will only concentrate on the left part.

Toeplitz operators

- In Toeplitz quantization, the phase space is not a cotangent space T^*X but a compact Kähler manifold M.
- If L is a convenient complex line bundle over M, let $H_N = H(M, L^{\otimes N})$ be the space of holomorphic sections of the tensor product $L^{\otimes N}$.
- It is a finite-dimensional subspace of $L^2(M, L^{\otimes N})$, with an orthogonal projector S_N, the *Szegő projector*.
- The Toeplitz quantization of a symbol h is then the operator

$$\begin{aligned} T_N(h) \quad : H_N & \mapsto \quad H_N \\ u & \mapsto \quad S_N(hu). \end{aligned}$$

Spin systems

- An important example is $M = \mathbb{CP}^1 = \mathbb{S}^2$, in this case $H_N \simeq \mathbb{C}_N[X]$.
- The quantization of the three coordinate functions x, y, z are the spin operators S_x, S_y, S_z with spin $N/2$.
- Spin systems: given a graph (E, V), on a tensor product $H_N^{\otimes |E|}$, consider operators of the form

$$\sum_{(i,j)\in V} S_x^i S_x^j + S_y^i S_y^j + S_z^i S_z^j.$$

- Toeplitz operators help to analyse the behaviour of such systems as the spin becomes large.

Asymptotics for the Szegő kernel

The Szegő kernel S_N can be seen as the N-th Fourier mode of a Fourier Integral Operator on a circle bundle over M (Boutet-Sjöstrand).

Theorem (Zelditch 00, Charles 03, Ma-Marinescu 06)

The Szegő kernel decreases exponentially fast far from the diagonal. Near the diagonal, in a convenient chart, one has an asymptotical expansion

$$S_N(\rho(z,w)) \simeq \frac{\pi^n}{N^n} e^{-N\frac{|z-w|^2}{2}+iN\,\mathrm{Im}(z\cdot\overline{w})}\left(1+\sum_{j=1}^{+\infty} N^{-j/2} b_j(z,w)\right).$$

Results

The subprincipal criterion for localization is given in terms of the hessian quadratic form q of h at the minimal points, in terms of a real-valued function $q \mapsto \mu(q)$.

Theorem (In publication)

minimal set of a symbol = **non-degenerate critical points (wells)** $\Rightarrow$ *first eigenvector of the Toeplitz operator localizes only where* μ *is minimal.*

Theorem (In preparation)

minimal set of a symbol = **union of submanifolds with nondegenerate crossings (miniwells)** $\Rightarrow$ *first eigenvector of the Toeplitz operator localizes only where* μ *is minimal.*

Perspectives

- Where is μ minimal for spin systems? Numerical evidence that μ is minimal only on planar configurations.
- For now we only have $O(N^{-\infty})$ estimates for localisation. Can we hope for $O(\exp(-cN))$ estimates ?
- Instead of considering a fixed manifold M, we look at a particular symbol on M^n, and we let $n \to +\infty$. What is the behaviour vis-à-vis the semiclassical limit?

The two last problems should be linked together.