Quasi-shuffle Algebras

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Outline

Introduction

I gave a definition of quasi-shuffle products in my *Journal of Algebraic Combinatorics* paper in 2000. It was inspired by my study of multiple zeta values and their generalizations. In the same year Li Guo and William Keigher gave an essentially equivalent construction of "mixable shuffle products" in *Advances in Mathematics*. It took a few years before this was generally recognized. My original definition had some technical conditions that were needlessly restrictive, and starting in 2012 Kentaro Ihara and I generalized the definition, while also emphasizing some algebraic features neglected by other authors. Our work has recently appeared (*J. Algebra*, July) and I will describe some of its features and applications, particularly to interpolated multiple zeta values.

Quasi-shuffle products

Here is the basic construction. Let A be a countable set, k a field. Suppose we have a commutative product $\diamond$ on kA, so that, for any $a, b \in A$, $a \diamond b$ is a finite sum of elements of A with coefficients in k. Now let $k\langle A \rangle$ be the noncommutative polynomial algebra on A; its elements are sums of monomials in elements of A with coefficients in k. We define a new product $*$ on $k\langle A \rangle$ inductively by setting $w * 1 = 1 * w = w$ for any monomial w, and

$$au * bv = a(u * bv) + b(au * v) + (a \diamond b)(u * v)$$

for $a, b \in A$ and monomials u, v. Then $(k\langle A \rangle, *)$ is commutative and associative.

Quasi-shuffle products cont'd

Another commutative, associative product $\star$ on $k\langle A\rangle$ can be defined similarly: let $w \star 1 = 1 \star w = w$ for any monomial w, and let

$$au \star bv = a(u \star bv) + b(au \star v) - (a \diamond b)(u \star v)$$

for $a, b \in A$ and monomials u, v. If the product $\diamond$ is trivial, i.e., $a \diamond b = 0$ for all $a, b \in A$, then $* = \star$ is just the shuffle product $⧢$ on $k\langle A\rangle$. But if $\diamond$ is nontrivial, the products $*$ and $\star$ are different.

A motivating example

A basic example is provided by multiple zeta values (MZVs) and multiple zeta-star values (MZSVs) , defined by

$$\zeta(i_1,\ldots,i_k)=\sum_{n_1>n_2>\cdots>n_k\geq 1}\frac{1}{n_1^{i_1}n_2^{i_2}\cdots n_k^{i_k}}$$

and

$$\zeta^\star(i_1,\ldots,i_k)=\sum_{n_1\geq n_2\geq\cdots\geq n_k\geq 1}\frac{1}{n_1^{i_1}n_2^{i_2}\cdots n_k^{i_k}}$$

respectively, where $i_1,\ldots,i_k$ are positive integers with $i_1>1$. It is easy to see that the product of two MZVs is a sum of MZVs, and similarly for MZSVs.

A motivating example cont'd

To put this in the framework of quasi-shuffle products, let $A = \{z_1, z_2, \dots\}$, with product $z_i \diamond z_j = z_{i+j}$. Then the $*$ product on the underlying vector space $\mathfrak{H}^1$ of $\mathbb{Q}\langle A\rangle$ is given by

$$z_i * z_j = z_i z_j + z_j z_i + z_{i+j},$$

$$z_i z_j * z_k = z_k z_i z_j + z_i z_k z_j + z_i z_j z_k + z_{i+k} z_j + z_i z_{j+k},$$

and so on. If we let $\mathfrak{H}^0$ be the subspace of $\mathfrak{H}^1$ generated by 1 and monomials that don't start with z_1, then the linear function $\zeta : \mathfrak{H}^0 \to \mathbb{R}$ given by $\zeta(z_{i_1} \cdots z_{i_k}) = \zeta(i_1, \dots, i_k)$ and $\zeta(1) = 1$ is a homomorphism from $(\mathfrak{H}^0, *)$ to the reals with their usual product.

A motivating example cont'd

The $\star$ product on $\mathfrak{H}^1$ looks like

$$z_i \star z_j = z_i z_j + z_j z_i - z_{i+j}$$

$$z_i z_j \star z_k = z_k z_i z_j + z_i z_k z_j + z_i z_j z_k - z_{i+k} z_j - z_i z_{j+k},$$

etc. The linear function $\zeta^\star : \mathfrak{H}^0 \to \mathbb{R}$ sending $z_{i_1} \cdots z_{i_k}$ to $\zeta^\star(i_1, \dots, i_k)$ is a homomorphism from $(\mathfrak{H}^0, \star)$ to $\mathbb{R}$. There is an isomorphism $\Sigma : (\mathfrak{H}^1, \star) \to (\mathfrak{H}^1, *)$ so that $\zeta^\star = \zeta\Sigma$; it is given by $\Sigma(z_i) = z_i$, $\Sigma(z_i z_j) = z_i z_j + z_{i+j}$,

$$\Sigma(z_i z_j z_k) = z_i z_j z_k + z_{i+j} z_k + z_i z_{j+k} + z_{i+j+k},$$

and so forth.

A motivating example cont'd

We'll return to Σ later, but first let's look at just what the algebra $(\mathfrak{H}^1, *)$ is in our example: it turns out to be the rational algebra QSym of quasi-symmetric functions, first defined by Ira Gessel in 1984. The algebra QSym contains the well-known algebra Sym of symmetric functions. Recognizing that MZVs are homomorphic images of quasi-symmetric functions was a key insight of my 1997 *Journal of Algebra* paper: since Sym $\subset$ QSym, this meant that identities of MZVs can be obtained from identities of symmetric functions. Also, by a 1995 theorem of Malvenuto and Reutenauer, QSym is a polynomial algebra: we can think of $(\mathfrak{H}^1, *)$ as the polynomial algebra on "Lyndon words" in the z_i.

Exotic MZVs

Quasi-shuffle algebras apply to other quantities besides MZVs and MZSVs. For example, if we let $a_1 > a_2 > \cdots$ be the zeros of the Airy function $\mathrm{Ai}(z)$ (all real and negative), then the sum

$$\zeta_{\mathrm{Ai}}(i_1, \dots, i_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{a_{n_1}^{i_1} \cdots a_{n_k}^{i_k}}$$

converges when $i_1 > 1$. These "Airy MZVs" are the images of the *same* quasi-shuffle algebra as the MZVs, but they are certainly different in many ways. For example, every $\zeta_{\mathrm{Ai}}(n)$, $n \geq 2$, is a rational polynomial in

$$\kappa = \frac{3^{\frac{5}{6}} \Gamma(\frac{2}{3})^2}{2\pi} \approx 0.729011.$$

Cf. Wakhare and Vignat `arXiv 1702.05534`.

Multiple Polylogarithms

Other sets of numbers are homomorphic images of different quasi-shuffle algebras. For a positive integer r, we can define multiple polylogarithms at rth roots of unity by

$$\mathrm{Li}_{(i_1,\dots,i_k)}(\omega^{j_1},\dots,\omega^{j_k}) = \sum_{n_1>\cdots>n_k\geq 1} \frac{\omega^{n_1 j_1 \cdots + n_k j_k}}{n_1^{i_1}\cdots n_k^{i_k}},$$

where $\omega = e^{\frac{2\pi i}{r}}$, and there is an obvious "starred" version. These are homormorphic images of the algebra $(k\langle A\rangle, *)$ with $A = \{z_{i,j} : \ i \geq 1, 0 \leq j \leq r-1\}$ and product

$$z_{i,j} \diamond z_{p,q} = z_{i+p,j+q},$$

with the second subscript is understood mod r. The case $r = 1$ is MZVs; for $r \geq 2$ one has "colored" MZVs.

q-MZVs and q-MZSVs

Another application is to the q-versions of MZVs and MZSVs. If one defines

$$\zeta_q(i_1,\dots,i_k)=\sum_{n_1>n_2>\cdots>n_k\geq 1}\frac{q^{n_1(i_1-1)+\cdots+n_k(i_k-1)}}{[n_1]^{i_1}[n_2]^{i_2}\cdots[n_k]^{i_k}},$$

where $[n]=(1-q^n)/(1-q)$ (and similarly for $\zeta_q^\star$), then the algebra of these quantities the homomorphic image of the quasi-shuffle algebra $(k\langle A\rangle,*)$ with $A=\{z_1,z_2,\dots\}$ and

$$z_i\diamond z_j=z_{i+j}+(1-q)z_{i+j-1}.$$

Formal power series

Returning to the general framework of a quasi-shuffle algebra $(k\langle A\rangle, *)$, consider a formal power series

$$f = c_1 t + c_2 t^2 + c_3 t^3 + \cdots \in tk[[t]].$$

Then f induces a linear function Ψ_f from $k\langle A\rangle$ to itself as follows. For $w = a_1 a_2 \cdots a_n$ a word in $k\langle A\rangle$ and $I = (i_1, \ldots, i_k)$ a composition of n (i.e., a sequence of positive integers whose sum is n), let

$$I[w] = (a_1 \diamond \cdots \diamond a_{i_1})(a_{i_1+1} \diamond \cdots \diamond a_{i_1+i_2}) \cdots (a_{n-i_k+1} \diamond \cdots \diamond a_n).$$

Now define

$$\Psi_f(w) = \sum_{\text{compositions } I=(i_1,\ldots,i_k) \text{ of } n} c_{i_1} \cdots c_{i_k} I[w]$$

Formal power series cont'd

Given two formal power series

$$f = c_1 t + c_2 t^2 + \cdots \quad \text{and} \quad g = d_1 t + d_2 t^2 + \cdots$$

in $tk[[t]]$, there is a "functional composition"

$$\begin{aligned} f \circ g &= c_1(d_1 t + d_2 t^2 + \cdots) + c_2(d_1 t + d_2 t^2 + \cdots)^2 + \cdots \\ &= c_1 d_1 t + (c_1 d_2 + c_2 d_1^2) t^2 + \cdots \in tk[[t]]. \end{aligned}$$

The following result, proved in a special case in my 2000 paper, has been a key feature of my joint work with Ihara.

Theorem (Composition)

For $f, g \in tk[[t]]$, $\Psi_{f \circ g} = \Psi_f \Psi_g$.

Functions Ψ_f

The functions Ψ_f need not preserve the algebra structures $(k\langle A\rangle, *)$ or $(k\langle A\rangle, \star)$, but we will shortly see some examples that do. In our joint work Ihara and I proved the following.

Theorem (Ψ_f of a geometric series)

If $f = c_1 t + c_2 t^2 + \cdots \in tk[[t]]$, then

$$\Psi_f\left(\frac{1}{1-tz}\right) = \frac{1}{1-f_\diamond(tz)}$$

for any $z \in kA[[t]]$.

Here $f_\diamond(tz)$ means

$$tc_1 z + t^2 c_2 z \diamond z + t^3 c_3 z \diamond z \diamond z + \cdots$$

Specific examples

Evidently $\Psi_t = \mathrm{id}$. The function $T = \Psi_{-t}$ sends a word w to $(-1)^{\deg w} w$. In fact we have the following result.

Proposition

$T : (k\langle A\rangle, \star) \to (k\langle A\rangle, *)$ *is an algebra homomorphism, and so is* $T : (k\langle A\rangle, *) \to (k\langle A\rangle, \star)$.

Clearly $T^2 = \mathrm{id}$, so T is an isomorphism. Here are two more functions: $\exp = \Psi_{e^t - 1}$, with inverse $\log = \Psi_{\log(1+t)}$. In my 2000 paper I proved the following.

Theorem

$\exp : (k\langle A\rangle, ш) \to (k\langle A\rangle, *)$ *is an isomorphism of algebras, where* ш *is the usual shuffle product.*

This allows one to deduce the algebra structure of $(k\langle A\rangle, *)$ from known results about $(k\langle A\rangle, ш)$.

The isomorphism Σ

But perhaps the most interesting function is $\Sigma = \Psi_{\frac{t}{1-t}}$. Using the composition theorem above it is easy to show that $T\Sigma T = \Sigma^{-1}$ and $\Sigma = \exp T \log T$. Since $T \log T$ is an algebra isomorphism from $(k\langle A\rangle, \star)$ to $(k\langle A\rangle, ⧢)$ and exp is an algebra isomorphism from $(k\langle A\rangle, ⧢)$ to $(k\langle A\rangle, *)$, we have the following.

Theorem

$\Sigma : (k\langle A\rangle, \star) \to (k\langle A\rangle, *)$ *is an algebra isomorphism that factors through* $(k\langle A\rangle, ⧢)$*.*

But things get even more interesting when we specialize to the case $k\langle A\rangle = \mathfrak{H}^1$.

Σ takes MZVs to MZSVs

We have already noted that $\zeta(\Sigma(w)) = \zeta^\star(w)$ for $w \in \mathfrak{H}^0$. By the "Ψ_f of a geometric series" theorem,

$$\Sigma\left(\frac{1}{1-tz_i}\right) = \frac{1}{1-(tz_i + t^2 z_{2i} + \cdots)}. \tag{1}$$

Now one can show that

$$\sum_{n\geq 0} \zeta(z_k^n)t^n = \exp\left(\sum_{i\geq 1} \frac{(-1)^{i-1}\zeta(ik)}{i} t^i\right) \tag{2}$$

Call this generating function $Z_k(t)$.

Σ takes MZVs to MZSVs cont'd

Eq. (1) and the relation $\zeta^\star = \zeta\Sigma$ imply

$$\sum_{n\geq 0} \zeta^\star(z_k^n)t^n = \zeta\left(\Sigma\left(\frac{1}{1-tz_k}\right)\right) = \frac{1}{Z_k(-t)}.$$

For example, from the well-known result

$$\zeta(z_2^n) = \frac{\pi^{2n}}{(2n+1)!}, \quad \text{i.e.,} \quad Z_2(t) = \frac{\sinh \pi\sqrt{t}}{\pi\sqrt{t}}$$

we get

$$\sum_{n\geq 0} \zeta^\star(z_2^n)t^n = \frac{\pi\sqrt{t}}{\sin \pi\sqrt{t}},$$

and thus

$$\zeta^\star(z_2^n) = \frac{(-1)^n 2(2^{2n-1}-1)B_{2n}\pi^{2n}}{(2n)!}.$$

Σ takes MZVs to MZSVs cont'd

But one can do much more. There is the following identity in $k\langle A\rangle[[t]]$.

Theorem (Ihara-Kajikawa-Ohno-Okuda, H-Ihara)

For $a, b \in A$,

$$\Sigma\left(\frac{1}{1-tab}\right) = \frac{1}{1-tab} * \Sigma\left(\frac{1}{1-ta\diamond b}\right).$$

Applying ζ to the theorem with $a = z_2$, $b = z_1$ gives

$$\sum_{n\geq 0} \zeta^\star((z_2z_1)^n)t^n = \sum_{p\geq 0} \zeta((z_2z_1)^p)t^p \sum_{q\geq 0} \zeta^\star(z_3^q)t^q.$$

Σ takes MZVs to MZSVs cont'd

Using the duality relation $\zeta((z_2z_1)^n) = \zeta(z_3^n)$, this implies

$$\sum_{n\geq 0} \zeta^\star((z_2z_1)^n)t^n = \frac{Z_3(t)}{Z_3(-t)}. \qquad (3)$$

Eq. (3) implies (using Eq. (2)) that

$$\zeta^\star((z_2z_1)^n) = \sum_{i_1+3i_3+5i_5+\cdots=n} \frac{2^{i_1+i_3+i_5+\cdots}\zeta(3)^{i_1}\zeta(9)^{i_3}\zeta(15)^{i_5}\cdots}{1^{i_1}i_1!3^{i_3}i_3!5^{i_5}i_5!\cdots}.$$

Note that this involves zeta values of only *odd* multiples of 3: 3, 9, 15, etc.

Hopf algebras

We have the deconcatenation coproduct Δ defined on $k\langle A\rangle$ by

$$\Delta(a_1 \cdots a_n) = \sum_{j=0}^{n} a_1 \cdots a_j \otimes a_{j+1} \cdots a_n.$$

If $R : k\langle A\rangle \to k\langle A\rangle$ reverses words, i.e.,

$$R(a_1 a_2 \cdots a_n) = a_n a_{n-1} \cdots a_1,$$

then we have the following result.

Theorem

$(k\langle A\rangle, *, \Delta)$ and $(k\langle A\rangle, \star, \Delta)$ are both Hopf algebras, with respective antipodes $S_* = \Sigma T R$ and $S_\star = T\Sigma R$.

A noncommutative algebra

There is a noncommutative product $\diamond$ on $k\langle A\rangle$ defined by $w \diamond 1 = w = 1 \diamond w$ and

$$a_1 \cdots a_n \diamond b_1 \cdots b_m = a_1 \cdots a_{n-1}(a_n \diamond b_1)b_2 \cdots b_m$$

for $m, n \geq 1$. If we let $\tilde{\Delta}$ be the reduced coproduct, i.e., $\tilde{\Delta}(1) = 0$ and $\tilde{\Delta}(w) = \Delta(w) - w \otimes 1 - 1 \otimes w$, then $(k\langle A\rangle, \diamond)$ has a canonical derivation $D = \diamond\tilde{\Delta}$, i.e., $D(1) = D(a) = 0$ for all $a \in A$, and

$$D(a_1 a_2 \cdots a_n) = \sum_{i=1}^{n-1} a_1 \cdots a_i \diamond a_{i+1} \cdots a_n$$

for $n \geq 2$. Note that $D^n(w) = 0$ when n is greater than or equal to the length of the word w.

An infinitesimal Hopf algebra

In fact we have the following result.

Theorem

$(k\langle A\rangle, \diamond, \tilde{\Delta})$ *is an infinitesimal Hopf algebra, with antipode* $S_\diamond = -\Sigma^{-1}$.

This means that

$$\tilde{\Delta}(w \diamond v) = \sum_v (w \diamond v_{(1)}) \otimes v_{(2)} + \sum_w w_{(1)} \otimes (w_{(2)} \diamond v),$$

where

$$\tilde{\Delta}(w) = \sum_w w_{(1)} \otimes w_{(2)}, \quad \tilde{\Delta}(v) = \sum_v v_{(1)} \otimes v_{(2)},$$

and also

$$\sum_w S_\diamond(w_{(1)}) \diamond w_{(2)} + S_\diamond(w) + w = 0 = \sum_w w_{(1)} \diamond S_\diamond(w_{(2)}) + S_\diamond(w) + w.$$

An infinitesimal Hopf algera cont'd

Since D is nilpotent on any given word,

$$e^D = \sum_{n=0}^{\infty} \frac{D^n}{n!}$$

makes sense as an element of $\mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$. From the general theory of infinitestimal Hopf algebras it follows that $\Sigma^{-1} = -S_\diamond = e^{-D}$, and this can easily be improved to $\Sigma^r = e^{rD}$ for any $r \in k$. It follows (since the exponential of a derivation is an automorphism) that Σ^r is an automorphism of $(k\langle A\rangle, \diamond)$ for all $r \in k$.

Interpolated MZVs

Shuji Yamamoto (*J. Algebra*, 2013) defined interpolated multiple zeta values. It is easiest to describe them by an example: the interpolated MZV $\zeta^r(2,3,1)$ is

$$\zeta(2,3,1) + r\zeta(5,1) + r\zeta(2,4) + r^2\zeta(7),$$

where each term comes from combining adjacent integers in the string, and the power of r tells how many combinations have occurred. Evidently $\zeta^0(i_1,\ldots,i_k) = \zeta(i_1,\ldots,i_k)$ and $\zeta^1(i_1,\ldots,i_k) = \zeta^\star(i_1,\ldots,i_k)$. This definition allows one to give unified proofs of results on MZVs and MZSVs, but there is more to say here.

Interpolated MZVs cont'd

Yamamoto proved that the interpolated MZVs have a product $\overset{r}{*}$, in the sense that $\zeta^r(u \overset{r}{*} v) = \zeta^r(u)\zeta^r(v)$ for $u, v \in \mathfrak{H}^0$. Actually we can define $\overset{r}{*}$ for any set A with a product $\diamond$ as above. First define, for $a \in A$ and $w \in k\langle A\rangle$,

$$a \diamond w = \begin{cases} 0, & \text{if } w = 1; \\ (a \diamond b)w', & \text{if } w = bw' \text{ for } b \in A. \end{cases}$$

Now set $w \overset{r}{*} 1 = 1 \overset{r}{*} w$ for $w \in k\langle A\rangle$, and

$$au \overset{r}{*} bv = a(u \overset{r}{*} bv) + b(au \overset{r}{*} v) + (1-2r)(a \diamond b)(u \overset{r}{*} v) \\ + (r^2 - r)(a \diamond b) \diamond (u \overset{r}{*} v)$$

for $a, b \in A$, $u, v \in k\langle A\rangle$.

Interpolated MZVs cont'd

Note that the fourth term is zero for $r = 0, 1$, and indeed that $\overset{r}{*}$ is $*$ for $r = 0$ and $\star$ for $r = 1$. Another exceptional case is $r = \frac{1}{2}$, where the third term vanishes; we will see more about this later. To see how the product operates, note that in the case $A = \{z_1, z_2, \dots\}$ with $z_i \diamond z_j = z_{i+j}$,

$$z_i \overset{r}{*} z_j = z_i z_j + z_j z_i + (1 - 2r) z_{i+j}$$

Interpolated MZVs cont'd

Note that the fourth term is zero for $r = 0, 1$, and indeed that $\overset{r}{*}$ is $*$ for $r = 0$ and $\star$ for $r = 1$. Another exceptional case is $r = \frac{1}{2}$, where the third term vanishes; we will see more about this later. To see how the product operates, note that in the case $A = \{z_1, z_2, \dots\}$ with $z_i \diamond z_j = z_{i+j}$,

$$z_i \overset{r}{*} z_j = z_i z_j + z_j z_i + (1 - 2r) z_{i+j}$$

and

$$\begin{aligned} z_i \overset{r}{*} z_j \overset{r}{*} z_k = z_i z_j z_k + z_i z_k z_j + z_j z_i z_k + z_j z_k z_i + z_k z_i z_j + z_k z_j z_i \\ (1 - 2r)(z_i z_{j+k} + z_{j+k} z_i + z_j z_{i+k} + z_{i+k} z_j + z_k z_{i+j} + z_{i+j} z_k) \\ (1 - 5r - 5r^2) z_{i+j+k}. \end{aligned}$$

Σ^r and Yamamoto's product

For any $r \in k$ we can define Σ^r as $\Psi_{\frac{t}{1-rt}}$. The composition theorem then gives us the following.

Proposition

For any $r, s \in k$, $\Sigma^r \Sigma^s = \Sigma^{r+s}$.

Ihara and I got some interesting results involving fractional Σ^r already in 2012, e.g.,

$$\Sigma^r \left(\frac{1}{1 - tz_i} \right) * \Sigma^{1-r} \left(\frac{1}{1 + tz_i} \right) = 1$$

for any rational r, but the key result is the relation to Yamamoto's product $\overset{r}{*}$.

Theorem

For all monomials $u, v \in k\langle A \rangle$, $\Sigma^r(u \overset{r}{*} v) = \Sigma^r(u) * \Sigma^r(v)$.

MZVs of repeated arguments

Any MZV of repeated arguments $(k, k, \ldots, k)$ is a rational polynomial in the $\zeta(ik)$. As above, we write $Z_k(t)$ for the generating function

$$1 + \sum_{i=1}^{\infty} \zeta(z_k^i) t^i = \exp\left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} t^i \zeta(ik) \right).$$

We have already seen that $Z_2(t) = \sinh(\pi\sqrt{t})/(\pi\sqrt{t})$, and there are formulas for other $Z_k(t)$, k even (see Broadhurst, Borwein and Bradley, *Electron. J. of Combin.* 1997).

Interpolated MZVs of repeated arguments

To relate these to interpolated MZVs, we need the following identity in the quasi-shuffle algebra $(k\langle A\rangle, *)$, which comes from Ihara's and my joint work.

Theorem (Σ^r of a geometric series)

For $z \in kA[[t]]$,

$$\Sigma^r\left(\frac{1}{1-tz}\right) * \frac{1}{1+rtz} = \frac{1}{1-(1-r)tz}.$$

Taking $z = z_k$ in this result and then applying ζ to both sides gives the following formula for interpolated MZVs of repeated arguments.

Interpolated MZVs of repeated arguments cont'd

Corollary

For integers $k \geq 2$,

$$\sum_{n=0}^{\infty} \zeta^r(z_k^n)t^n = \frac{Z_k((1-r)t)}{Z_k(-rt)}.$$

In particular, taking $r = \frac{1}{2}$ and $k = 2$ gives

$$\sum_{n=0}^{\infty} \zeta^{\frac{1}{2}}(z_2^n)t^n = \frac{Z_2\left(\frac{t}{2}\right)}{Z_2\left(-\frac{t}{2}\right)} = \frac{\sinh(\pi\sqrt{\frac{t}{2}})}{\sin(\pi\sqrt{\frac{t}{2}})} =$$

$$1 + \frac{\pi^2}{6}t + \frac{\pi^4}{72}t^2 + \frac{13\pi^6}{15120}t^3 + \cdots$$

Interpolated MZVs of repeated arguments cont'd

Another case of the corollary is $r = \frac{1}{2}$ and $k = 3$:

$$\sum_{n=0}^{\infty} \zeta^{\frac{1}{2}}(z_3^n) t^n = \frac{Z_3\left(\frac{t}{2}\right)}{Z_3\left(-\frac{t}{2}\right)}$$

Comparing this with Eq. (3) above, we have

$$\zeta^{\frac{1}{2}}(z_3^n) = \frac{1}{2^n} \zeta^{\star}((z_2 z_1)^n).$$

This is a special case of the "two-one formula" relating $\zeta^{\frac{1}{2}}$ of odd arguments to $\zeta^{\star}$, a proof of which has recently been published by J. Zhao.

MZVs of even arguments

If for $k \le n$ we let $E(2n, k)$ be the sum of all MZVs of even arguments having depth k and weight $2n$, then I showed (*Int. J. Number Theory, 2017*) that

$$E(2n,k) = \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^j \pi^{2j} \zeta(2n-2j)}{2^{2n-2j-2}(2j+1)!} \binom{2k-2j-1}{k}$$

as follows from the explicit generating function

$$F(t,s) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} E(2n,k) t^n s^k = \frac{\sin(\pi\sqrt{(1-s)t})}{\sqrt{1-s}\sin(\pi\sqrt{t})} =$$

$$1 + \frac{\pi^2 t}{6} s + \frac{\pi^4 t^2}{360}(4s + 3s^2) + \frac{\pi^6 t^3}{15120}(16s + 12s^2 + 3s^3) + \cdots$$

Generating functions

The generating function $F(t,s)$ is the image under an appropriate homomorphism of the generating function

$$\mathcal{F}(t,s) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} N_{n,k} t^n s^k \in \mathrm{QSym}[[t,s]],$$

where $N_{n,k}$ is the sum of all monomial symmetric functions corresponding of partitions of n with k parts. In fact

$$F(t,s) = \zeta \mathcal{P}_2 \mathcal{F}(t,s).$$

where $\mathcal{P}_2 : \mathrm{QSym} \to \mathrm{QSym}$ sends each z_i to z_{2i}. The formula on the preceding slide then follows from

$$\mathcal{F}(t,s) = H(t)E((s-1)t),$$

where $H(t)$ and $E(t)$ are respectively the generating functions for the complete and elementary symmetric functions.

Interpolated MZVs of even arguments

Now we find the corresponding genenerating function for interpolated MZVs. In any quasi-shuffle algebra $(k\langle A\rangle, *)$, the "Σ^r of a geometric series" theorem gives, for $p \in k$, $z \in kA[[t]]$,

$$\Sigma^p\left(\frac{1}{1-tz}\right) = \frac{1}{1-(1-p)tz} * \left(\frac{1}{1+ptz}\right)^{-*}.$$

Specialize to $\mathfrak{H}^1$ and let $z = z_1$ to get

$$\Sigma^p E(t) = E((1-p)t)E(-pt)^{-1} = E((1-p)t)H(pt)$$

and hence

$$\Sigma^r \mathcal{F}(t,s) = \Sigma^r(E((s-1)t)H(t)) = \Sigma^r \Sigma^{\frac{1}{s}} E(st)$$
$$= \Sigma^{r+\frac{1}{s}} E(st) = E((s-rs-1)t)H((1+rs)t)$$

Interpolated MZVs of even arguments cont'd

Now if we let $E^r(2n,k)$ be just like $E(2n,k)$, but with ζ^r replacing ζ, then

$$F^r(t,s) = 1 + \sum_{n\ge k\ge 1} E^r(2n,k)t^n s^k$$

is $\zeta^r \mathcal{P}_2 \mathcal{F}(t,s) = \zeta \Sigma^r \mathcal{P}_2 \mathcal{F}(t,s)$. On the last slide we showed $\Sigma^r \mathcal{F}(t,s) = E((s - rs - 1)t)H((1+rs)t)$, and since $\mathcal{P}_2$ commutes with Σ^r this gives

$$F^r(t,s) = \zeta\mathcal{P}_2 E((s - rs - 1)t)\zeta\mathcal{P}_2(H((1+rs)t)$$
$$= \frac{\sinh(\pi\sqrt{(s-rs-1)t})\sqrt{(1+rs)t}}{\sqrt{(s-rs-1)t}\,\sin(\pi\sqrt{(1+rs)t})} = \frac{F(t,(1-r)s)}{F(t,-rs)}.$$

Interpolated MZVs of even arguments cont'd

In particular,

$$F^1(t,s) = \frac{F(t,0)}{F(t,-s)} = \frac{1}{F(t,-s)},$$

a result I had already proved by other methods, and

$$F^{\frac{1}{2}}(t,s) = \frac{F(t,\frac{s}{2})}{F(t,-\frac{s}{2})} = \frac{\sqrt{1+\frac{s}{2}}\sin\left(\pi\sqrt{(1-\frac{s}{2})t}\right)}{\sqrt{1-\frac{s}{2}}\sin\left(\pi\sqrt{(1+\frac{s}{2})t}\right)}$$

$$= 1 + \frac{\pi^2 t}{6}s + \frac{\pi^4 t^2}{360}(4s+5s^2) + \frac{\pi^6 t^3}{15120}(16s+28s^2+13s^3) + \cdots$$

so, e.g.,

$$\zeta^{\frac{1}{2}}(2,4) + \zeta^{\frac{1}{2}}(4,2) = \frac{28\pi^6}{15120} = \frac{\pi^6}{540} \quad \text{and} \quad \zeta^{\frac{1}{2}}(2,2,2) = \frac{13\pi^6}{15120}.$$

Symmetric sums of MZVs

Finally, I want to talk about some recent results of mine for interpolated MZVs. I begin with symmetric sums, which might be called an extension of some of my oldest results to interpolated MZVs. If we sum $\zeta(i_1, \dots, i_k)$ over all permutations of $i_1, \dots, i_k$ (necessarily each $i_j \geq 2$), we always get a rational polynomial in ordinary zeta values: for example

$$\zeta(i_1, i_2) + \zeta(i_2, i_1) = \zeta(i_1)\zeta(i_2) - \zeta(i_1 + i_2)$$

and

$$\zeta(i_1, i_2, i_3) + \zeta(i_1, i_3, i_2) + \zeta(i_2, i_1, i_3) + \zeta(i_2, i_3, i_1) + \zeta(i_3, i_1, i_2) + \zeta(i_3, i_2, i_1) = \zeta(i_1)\zeta(i_2)\zeta(i_3) - \zeta(i_1)\zeta(i_2 + i_3) - \zeta(i_2)\zeta(i_1 + i_3) - \zeta(i_3)\zeta(i_1 + i_2) + 2\zeta(i_1 + i_2 + i_3).$$

Symmetric sums of MZVs cont'd

I gave the general result in my 1992 *Pacific Journal* paper. Let S_k be the symmetric group on $\{1, \ldots, k\}$, Π_k the set of partitions of $\{1, \ldots, k\}$. For $B = \{B_1, \ldots, B_l\} \in \Pi_k$, define

$$c(B) = (-1)^{k-l}(\text{card } B_1 - 1)! \cdots (\text{card } B_l - 1)!$$

Theorem (Symmetric sums of MZVs)

For integers $i_1, \ldots, i_k \geq 2$,

$$\sum_{\sigma \in S_k} \zeta(i_{\sigma(1)}, \ldots, i_{\sigma(k)}) = \sum_{B=\{B_1,\ldots,B_l\} \in \Pi_k} c(B)\zeta(\sum_{j \in B_1} i_j) \cdots \zeta(\sum_{j \in B_l} i_j).$$

Symmetric sums of MZSVs

Not much change is required if ζ is replaced by $\zeta^\star$. In my 1992 paper I also proved the following.

Theorem (Symmetric sums of MZSVs)

For integers $i_1, \dots, i_k \geq 2$,

$$\sum_{\sigma \in S_k} \zeta^\star(i_{\sigma(1)}, \dots, i_{\sigma(k)}) = \sum_{B=\{B_1,\dots,B_l\}\in\Pi_k} \bar{c}(B)\zeta(\sum_{j\in B_1} i_j)\cdots\zeta(\sum_{j\in B_l} i_j),$$

where for $B = \{B_1, \dots, B_l\} \in \Pi_k$,

$$\bar{c}(B) = (\text{card}\, B_1 - 1)! \cdots (\text{card}\, B_l - 1)!$$

Symmetric sums in $\mathfrak{H}^1$

In fact, these results follow from identities in the quasi-shuffle algebras $(\mathfrak{H}^1, *)$ and $(\mathfrak{H}^1, \star)$. If, for a block B_a of a partition B of $\{1, \dots, k\}$ we define $z_{B_a} = z_i$ for $i = \sum_{j \in B_a} j$, then the theorem about symmetric sums of MZVs follows from

$$\sum_{\sigma \in S_k} z_{i_{\sigma(1)}} z_{i_{\sigma(2)}} \cdots z_{i_{\sigma(k)}} = \sum_{B=\{B_1,\dots,B_l\} \in \Pi_k} c(B) z_{B_1} * \cdots * z_{B_l}$$

and the one about symmetric sums of MZSVs follows from

$$\sum_{\sigma \in S_k} z_{i_{\sigma(1)}} z_{i_{\sigma(2)}} \cdots z_{i_{\sigma(k)}} = \sum_{B=\{B_1,\dots,B_l\} \in \Pi_k} \bar{c}(B) z_{B_1} \star \cdots \star z_{B_l}.$$

Symmetric sums of interpolated MZVs

Indeed this generalizes to interpolated MZVs. For rational r and $B = \{B_1, \dots, B_l\} \in \Pi_k$, define

$$c_r(B) = (-1)^{k-l} \prod_{j=1}^{l} (\text{card}\, B_j - 1)! p_{\text{card}\, B_j}(r),$$

where

$$p_m(r) = \sum_{j=0}^{m-1} \binom{m}{j} (-r)^j = (1-r)^m - (-r)^m.$$

Then I have proved that

$$\sum_{\sigma \in S_k} z_{i_{\sigma(1)}} z_{i_{\sigma(2)}} \cdots z_{i_{\sigma(k)}} = \sum_{B=\{B_1,\dots,B_l\}\in\Pi_k} c_r(B) z_{B_1} \overset{r}{*} \cdots \overset{r}{*} z_{B_l}.$$

Symmetric sums of interpolated MZVs cont'd

By applying $\zeta^r = \zeta\Sigma^r$ to this we obtain the following result.

Theorem (Symmetric sums of interpolated MZVs)

For integers $i_1, \dots, i_k \geq 2$,

$$\sum_{\sigma \in S_k} \zeta^r(i_{\sigma(1)}, \dots, i_{\sigma(k)}) = \sum_{B=\{B_1,\dots,B_l\}\in\Pi_k} c_r(B)\zeta(\sum_{j\in B_1} i_j)\cdots\zeta(\sum_{j\in B_l} i_j).$$

Since $c_0(B) = c(B)$ and $c_1(B) = \bar{c}(B)$, this result generalizes the results for symmetric sums of MZVs and MZSVs. But the case $r = \frac{1}{2}$ is also interesting.

The case $r = \frac{1}{2}$

We note that

$$p_m\left(\frac{1}{2}\right) = \begin{cases} 0, & \text{if } m \text{ even}, \\ 2^{1-m}, & \text{if } m \text{ odd}. \end{cases}$$

Thus

$$c_{\frac{1}{2}}(B) = \begin{cases} (\frac{1}{2})^{k-l} \prod_{j=1}^{l} (\text{card } B_j - 1)!, & \text{if card } B_j \text{ odd for all } j, \\ 0, & \text{otherwise}, \end{cases}$$

so when $r = \frac{1}{2}$ we need only sum over partitions in which every block has odd cardinality, e.g.,

$$\zeta^{\frac{1}{2}}(4,3,2) + \zeta^{\frac{1}{2}}(4,2,3) + \zeta^{\frac{1}{2}}(3,4,2) + \zeta^{\frac{1}{2}}(3,2,4) + \zeta^{\frac{1}{2}}(2,3,4) + \zeta^{\frac{1}{2}}(2,4,3) = \zeta(4)\zeta(3)\zeta(2) + \frac{1}{2}\zeta(9).$$

Interpolated MZVs of repeated arguments

From the theorem on symmetric sums of interpolated MZVs above we obtain the following corollary, which complements the generating function result we gave earlier.

Corollary

For $i \geq 2$ and positive integer n,

$$\zeta^r(z_i^n) = \sum_{\substack{\text{partitions } \lambda = \\ (\lambda_1,\dots,\lambda_l) \text{ of } n}} \frac{(-1)^{n-l}}{1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \cdots} \prod_{j=1}^{l} p_{\lambda_j}(r) \zeta(i\lambda_j),$$

where $m_j(\lambda)$ is the multiplicity of j in λ.

Interpolated MZVs of repeated arguments cont'd

In the case $r = \frac{1}{2}$ this is

$$\zeta^{\frac{1}{2}}(z_i^n) =$$

$$\sum_{\substack{\text{partitions } \lambda = \\ (\lambda_1,\dots,\lambda_l) \text{ of } n \\ \text{all } \lambda_i \text{ odd}}} \frac{2^{l-n}}{1^{m_1(\lambda)} m_1(\lambda)! 3^{m_3(\lambda)} m_3(\lambda)! \cdots} \prod_{j=1}^{l} \zeta(i\lambda_j).$$

For example,

$$\zeta^{\frac{1}{2}}(2,2,2) = \frac{2^{-2}}{3}\zeta(6) + \frac{1}{6}\zeta(2)^3 = \frac{13\pi^6}{15120}.$$

Totally odd sum theorem for $\frac{1}{2}$-MZVs

By combining the two-one theorem of Zhao with the cyclic sum theorem of Ohno and Wakabayashi, I can prove the following "totally odd sum theorem" for interpolated MZVs with $r = \frac{1}{2}$.

Theorem

Let $n \geq 2$, and let $l < n$ be a positive integer of the same parity. Then

$$\sum_{\substack{a_1+\cdots+a_l=n \\ a_i \text{ odd},\ a_1>1}} \zeta^{\frac{1}{2}}(a_1, \dots, a_l) = \frac{n-1}{s-1}\binom{s-1}{l-1}\frac{\zeta(n)}{2^l},$$

where $s = \frac{n+l}{2}$.

Totally odd sum theorem for $\frac{1}{2}$-MZVs cont'd

Since

$$\zeta^{\frac{1}{2}}(a,b,c)=\zeta(a,b,c)+\frac{1}{2}(\zeta(a+b,c)+\zeta(a,b+c))+\frac{1}{4}\zeta(a+b+c),$$

this has the following corollary.

Corollary

If n is odd, then

$$\sum_{\substack{a_1+a_2+a_3=n\\ a_i\ odd,\ a_1>1}} \zeta(a_1,a_2,a_3)$$

is a polynomial in ordinary zeta values with rational coefficients.

(Recall that the parity theorem says that $\zeta(w)$ is a rational polynomial in multiple zeta values of lesser depth when the weight and depth of w have opposite parity.)