From divergence to convergence

O Costin, R D Costin, G Dunne

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Definitions, notations,...

Series occurring in many interesting (as well as uninteresting) problems, such as perturbation theory, are divergent, understood as zero radius of convergence. E.g. $\sum_k k! z^k$. Écalle found that, when their origin is "natural" they are resurgent.

A resurgent function (in Borel plane, or in the convolutive model) is analytic except for a discrete set of singularities on any Riemann sheet, and has at most exponential growth at infinity. Usually zero is assumed to be a point of analyticity. The singularities of a resurgent functions satisfy a rich set of mutual relations, the object of alien calculus.

A resurgent series $\tilde{f}$ is the asymptotic series (in $1/x$) of $f = \mathcal{L}F = \int_0^{\infty} e^{-xp} F(p) dp$ of a resurgent function F. If F is resurgent, f is also called resurgent (in the geometric model, or physical domain).

Resurgent series can be convergent, iff F is entire. Otherwise their domain of convergence is empty: if we for instance integrate by parts, we get that $\mathcal{L}F \sim \tilde{f} = F(0)/x + F'(0)/x^2 + \cdots + F^{(n)}(0)/x^n + \cdots$, the Taylor series stripped of the $1/n!$, the Laplace transform of the series of F (Watson's Lemma).

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Stokes Phenomena

$F(p) = (1-p)^{-1}$ is perhaps the simplest nontrivial resurgent function, giving by Laplace transform,

$$f(x) = \int_0^{\infty - 0i} e^{-xp}(1-p)^{-1}dp =: e^{-x}\mathrm{Ei}^{+}(x)$$

For large x, $e^{-x}\mathrm{Ei}^{+}(x) \sim \mathcal{L}\sum_{k=0}^{\infty} p^k = \sum_k k!x^{-k-1} = \tilde{f}$.

Rotating the contour up gives the **Stokes transition**

$$e^{-x}\mathrm{Ei}^{+}(x) := \int_0^{\infty+0i} e^{-xp}(1-p)^{-1}dp - 2\pi i e^{-x} =: e^{-x}\mathrm{Ei}^{-}(x) - 2\pi i e^{-x}$$

a jump in Laplace representation across the **Stokes line** $\mathbb{R}^+$.

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Further rotation shows that for $x \in i\mathbb{R}^+$, the **antistokes line**,

$$e^{-x}\mathrm{Ei}^+(x) \sim -2\pi i e^{-i|x|} + \tilde{f}$$

to leading order f now oscillates. A qualitative change of asymptotic behavior of f, the **Stokes phenomenon**, would be impossible had $\tilde{f}$ converged to f (Dyson's argument).

Likewise, as the Gamma function has poles at all negative integers, the power series in the Stirling formula

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 - \frac{1}{12x} - \frac{1}{144x^2} \cdots\right)$$

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In particular, resurgent series are Borel summable (or generalized summable, after Écalle medianization). Starting from the divergent series, Borel summation consists of formal inverse Laplace transform, analytic continuation on $\mathbb{R}^+$ and Laplace transform. Of these, analytic continuation is the most problematic, from a practical standpoint.

Plan. • We develop tools to extract maximal information from **resurgent** series, perhaps "numerically" given, and to provide efficient ways of analytic continuation.

• We find simple convergent, while global and asymptotic rational expansions for resurgent functions. (Convergent expansions can only exist in terms of singular, or nontrivial, functions, as discussed.)

• Combining the above we show how divergent resurgent series can be reexpanded convergently.

• We illustrate this approach, which is otherwise general, on Painlevé P1, an important equation for which there is still need for efficient methods to calculate the values of the so-called tronquée solutions in their analyticity sectors.

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An analog of Mittag-Leffler for resurgent functions: the resurgent function decomposition

The classical Mittag-Leffler theorem: any meromorphic function is a convergent sum of $c_{k,l}/(z - z_k)^{p_l} + \text{Polynom}_{kl}$ plus an entire function (generalizing the usual partial fraction decomposition).

There is a similar decomposition for resurgent functions, regardless of the singularity types.

On the first Riemann sheet, any resurgent function is an entire function plus a geometrically convergent sum of resurgent elements: resurgent functions with only one singularity and algebraic decay at infinity.

$$\frac{\pi}{\sqrt{1-p^2}} = -\frac{2i \arccos\left(\frac{\sqrt{1-p}}{\sqrt{2}}\right)}{\sqrt{1-p^2}} + \frac{2i \operatorname{arccosh}\left(\frac{\sqrt{p+1}}{\sqrt{2}}\right)}{\sqrt{1-p^2}} \tag{1}$$

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Theorem (OC, R Costin)

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Lemma

Let F be resurgent, $e^{\nu|p|}$ its exponential bound and ω a singular point. Define

$$F_\omega(p) = \frac{\exp(\mu_\omega p)}{2\pi i} \int_{\mathcal{C}} \frac{F(s)\exp(-\mu_\omega s)}{s-p} ds \tag{2}$$

where $\mu_\omega > \nu$ and C a surrounds the singularity as shown. Then $F_\omega - F$ is analytic at ω and on the first Riemann sheet, F_ω is only singular at ω. $F_\omega = O(1/p)$ for large p.

Note

If $y_k = \mathcal{L}Y_k$ are the series in the transseries of y_0, if the singularities of Y_k are L^1 and $Y_k(p) = O(1/p)$ for large p, then $2\pi i F_\omega = \int_0^\infty (\omega_k + s - p)^{-1} Y_k(s) ds$.

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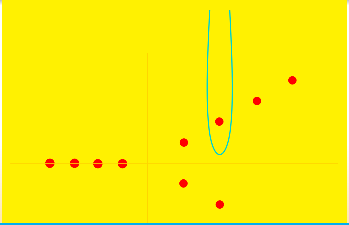

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The function F_ω is manifestly analytic outside or inside the contour (the integrand is analytic in p), which can be deformed if $p \not\to \omega$. But the difference between the integral with p outside and inside is the residue, $F(p)$. So $F - F_\omega$ is analytic at ω.

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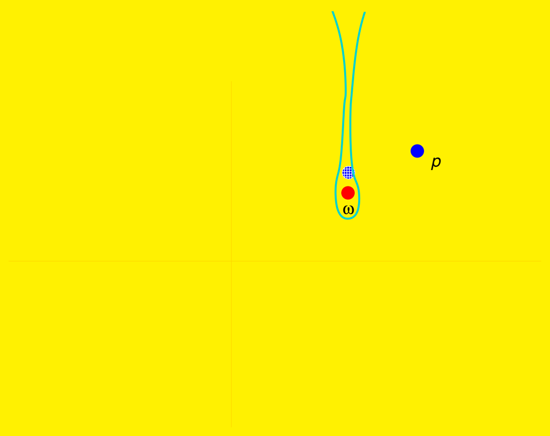

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Lemma

There are (concrete) choices of μ_ω which make the sum $S(p) = \sum_\omega F_\omega$ convergent. The function $F - S$ is entire.

The proof of the first part is technical. The second part is immediate since $F - S$ is singularity-free on the first (thus only) Riemann sheet.

Dyadic expansions: convergent rational functions re-expansions of divergent series

$$F_\omega(p) = \int_{\mathcal{C}} \frac{F(s)}{s-p}\,ds \tag{5}$$

To find rapidly convergent representations of Laplace transforms of resurgent functions in terms of rational functions (ultimately in terms of $a_i/(x-x_i)$).

Since

$$a_i/(x-x_i) = a_i\mathcal{L}[e^{x_i p}]$$

In Borel plane we need rapidly convergent representations of resurgent functions **in terms of exponentials** valid in a half plane, to ensure a 2π sector in the geometric model.

By (5) we only need to obtain exponential expansions for $1/(s-p)$.

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Proposition (A curious identity)

$\forall p \in \mathbb{C}$ (all singularities are removable)

$$\frac{1}{p} + \frac{1}{e^{-p} - 1} + \sum_{k=1}^{\infty} \frac{2^{-k}}{e^{-2^{-k}p} + 1} = 0$$

By changes of variable, $\forall a \in \mathbb{C}$ useful in choosing the domain of convergence

$$\frac{1}{a} \; \frac{1}{s - p} = \frac{e^{as}}{e^{as} - e^{ap}} - \sum_{k=1}^{\infty} \frac{2^{-k} e^{a2^{-k}s}}{e^{a2^{-k}s} + e^{a2^{-k}p}}$$

Proof.

The partial fraction decomposition of $1/(1 - x^{2^n})$ with $x = e^{-p/2^n}$ gives

$$\frac{1}{2^n(1 - e^{-\frac{p}{2^n}})} = \frac{1}{1 - e^{-p}} - \sum_{k=1}^{n} \frac{2^{-k}}{e^{-\frac{p}{2^k}} + 1} \tag{6}$$

The equality follows by passing to the limit $n \to \infty$. □

An even more elementary proof relies on the fact that any integer is even or odd... The fact that any integer is $3k, 3k+1$ or $3k+2$ yields an expansion in base 3 etc.

The convergence is obviously geometric and uniform on compact sets, and remains geometric convergent after Laplace transform.

We then write

$$\frac{e^{as}}{e^{as}-e^{ap}} = \frac{e^{as}}{e^{as}-1+(1-e^{ap})} = -ae^{as}\sum_{k=0}^{\infty}\frac{(1-e^{ap})^k}{(1-e^{as})^{k+1}}$$

(same for the 2^{-k} terms), integrate in s and note that

$$\mathcal{L}(1-e^{ap})^k = \frac{k!}{x(x+a)\cdots(x+ka)} =: \frac{k!}{a^k(x/a)_{k+1}} = O(a^{-k}k^{a-x/a})$$

where $(x)_{k+1} := x(x+1)\cdots(x+k)$.

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Theorem

If F is resurgent, then $f = \mathcal{L}\mathcal{F}$ has a generalized Mittag-Leffler decomposition $\sum_k \frac{c_k}{x - x_k}$, where all x_k lie on one ray (where they are typically dense), and the convergence is geometric in the sector where f has transseries representation(s).

Examples

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Unlike classical factorial series, these dyadic expansions converge geometrically, and in a sector of width 2π rather than π (the partial fraction decomposition of usual factorial series does not converge).

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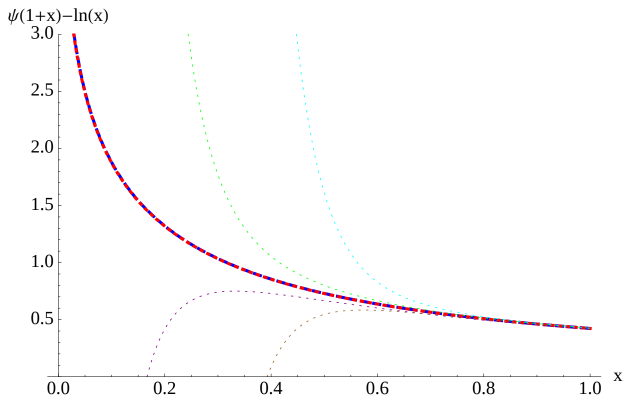

Figure: Exact $(\psi(1+x) - \ln x)$ and large x dyadic expansion (dashed red and blue curves), contrasted with the first 4 partial sums of the standard asymptotic expansion (dotted curves). Even at very small x, the dyadic expansion is very accurate, while the asymptotic expansions show the typical behavior of breaking down at small x.

Global information from the divergent series (with G Dunne)

This important in the many problems where the exact solution is unavailable, or worse the origin problem is too complicated to be rigorously analyzed.

I'll take a particularly important solution of the Painlevé equation $P1$,

$$f'' = 6f^2 + z$$

In normalized, Boutroux variables, the equation becomes

$$y''(x) + \frac{y'(x)}{x} - \left(1 + \frac{4}{25x^2}\right) y(x) - \frac{1}{2} y(x)^2 - \frac{4}{25x^2} = 0$$

All solutions have poles in some sectors. Modulo symmetries, there is a unique one, the **tritronquée** which is pole free in a sector of width $8\pi/5$. It has a (divergent) asymptotic series in the sector of analyticity,

$$y(x) \sim -\frac{4}{25x^2} - \frac{392}{625x^4} - \frac{6272}{625x^6} - \cdots$$

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We took 200 nonzero terms of the divergent series (~ 200 terms are available in a number of interesting models in physics), with the purpose of devising efficient ways to obtain accurate information about (1) the structure of singularities in Borel plane; (2) obtaining higher terms in the transseries (giving thus an approximation for all the **tronquée** solutions, those which are pole-free in **some** sector); (3) finding connection formulae: the Stokes constant (known) and the central connection (infinity to zero), not known exactly.

The methods we are seeking should be provable with concrete estimates for resurgent functions and be general.

The radius of convergence of the Borel plane $\tilde{Y}_0(p)$ series is one. How can we see what the function does past the unit disk? Analytic continuation by series re-expansion is known as very inefficient. Padé approximants are much better and often used, but quite suboptimal for resurgent functions.

Padé$[n, m]$ of a series $\mathcal{S}$ is the unique rational function $R_{nm} = P_n/Q_m$ such that $\mathcal{S} - R_{nm} = o(p^{m+n})$. Clearly R_{nm} converge wherever $\mathcal{S}$ does, and often beyond. For a meromorphic function f, convergence holds in $\mathbb{C} \setminus \{\text{the poles of } f\}$.

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Poles of the Padé approximation of $P1$, p plane

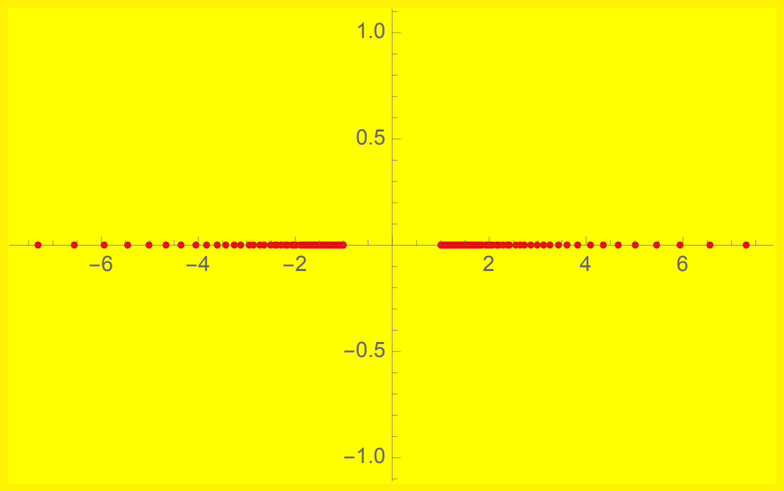

Figure: Placement of the poles of the Padé[98, 98] of the tritronquée in Borel plane, Y_0. The actual singularities of Y_0 are, however, exactly the nonzero integers.

Improvement by conformal maps

The Taylor series of Y_0, as such, does not give any information past the disk of convergence; Padé only shows one singularity per ray.

$Y_0 = \mathcal{L}^{-1}\tilde{y}_0$ is analytic in the simply connected domain $D := \mathbb{C} \setminus \pm[1, \infty)$. This is known, and can be determined based on the series.

$\varphi(z) = \dfrac{2z}{1+z^2}$ maps the unit disk $\mathbb{D}$ conformally onto D. Thus the Taylor series of $Y_0(\varphi(z))$ also has radius of convergence one. With $\varphi^{-1}(p) = \dfrac{p}{\sqrt{1-p^2}+1}$, the series $\tilde{Y}_0(\varphi)(\varphi^{-1}(p))$, a function series in powers of $\dfrac{p}{\sqrt{1-p^2}+1}$ now converges in the whole of D! It is known that z^n is the best polynomial basis in $\mathbb{D}$, so this is the best approximation in powers of a function. Conformal-Taylor expansions (our ad-hoc name) are better than Padé, especially near singularities.

It turned out that the idea of conformal-Taylor was known, and in fact introduced for Borel plane analysis (D.I. Kazakov, O.V. Tarasov, D.V. Shirkov, (1979)).

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The resulting accuracy is quite surprising, at least to us.

Experimentally, if the error of Pade or conformal-Taylor is ε, then conformal-Padé gives roughly ε^2. On the singular line conformal-Padé works very well, Padé fails, and conformal-Taylor would only see the position of the singularities and their rough type.

To summarize conformal-Padé:

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The conformal-Padé method for y_0 of P_1, 200 terms

Some features of conformal-Padé, $P1$:

The accuracy conformal-Padé on a circle of radius 55 in p is at least $\sim 10^{-18}$ at ± 55 and $\sim 10^{-33}$ at $\pm 55i$, farthest away from the singularities.

Upon Laplace transform, this yields 34 accurate digits for the tritronquée when the un-normalized variable $|z| = 1$.

Curiously perhaps, conformal-Padé works on the second Riemann sheet of Y_0, on a distance less than the separation of the singularities (1 here).

From the given truncation, conformal-Padé gives an approximate Stokes constant with 96 digits. If the resurgent element decomposition is used **together with** the asymptotics of $y_1, \ldots, y_{10}$ (formally obtainable from P1) up to a multiplicative constant, **and we use their resurgence relations** the accuracy improves to 716 digits.

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Poles of the Padé $[98, 98]$ approximation of Y_0

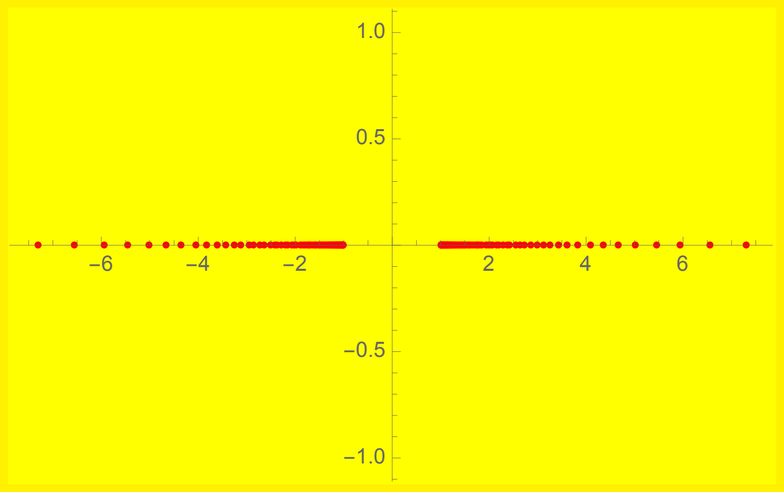
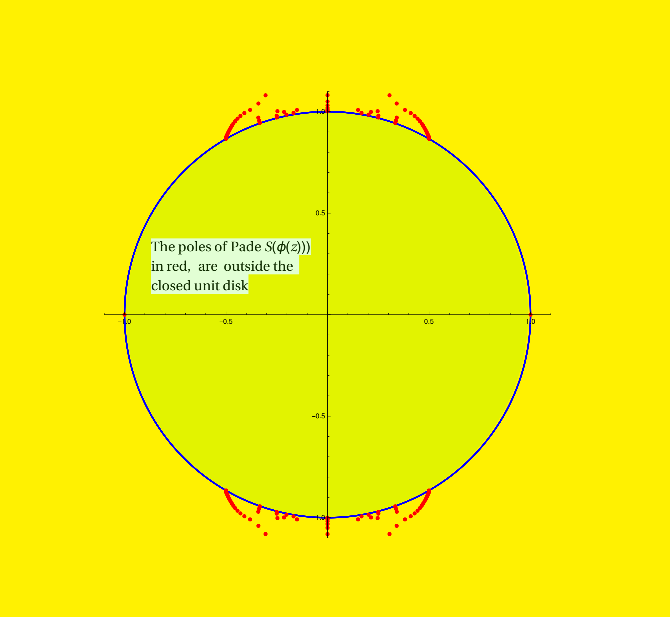
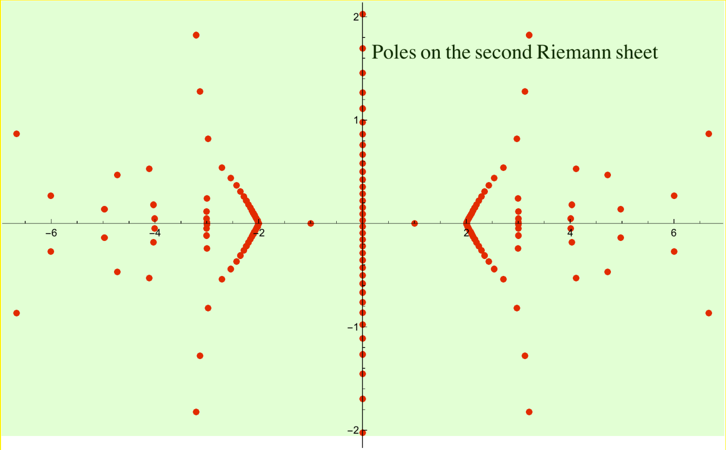


Figure: Placement of the poles of the Padé$[98, 98]$ of the tritronquée in Borel plane, Y_0. The actual singularities of Y_0 are, however, exactly the nonzero integers.

The [98,98] Padé of $Y_0(\varphi(z))$, $P \circ S \circ \varphi$



Poles, in p plane. If not on $\mathbb{R}$, they are on the 2*nd* Riemann sheet



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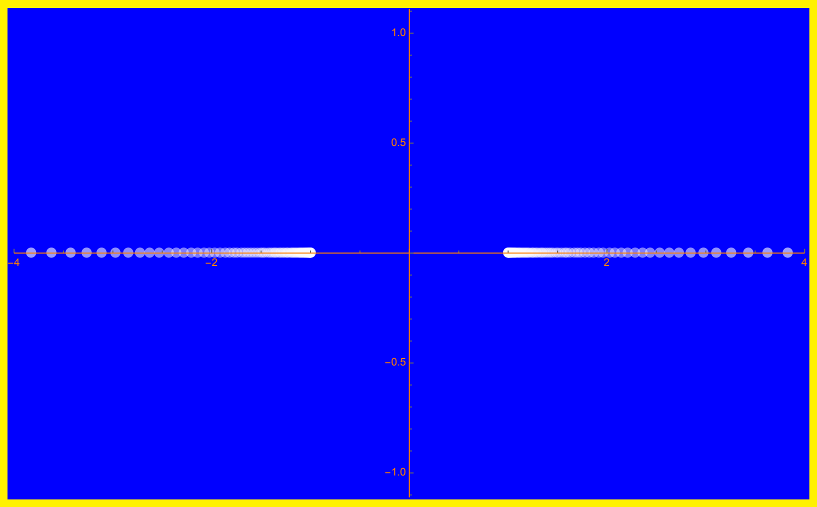

Figure: Placement of the poles of the Padé$[98, 98]$ of the tritronquée in Borel plane, Y_0. The "brilliance" is proportional to the residue, and it is additive.

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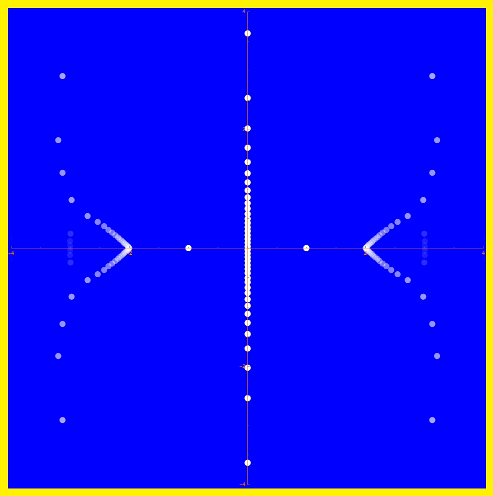

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The Stokes constant and the series for Y_1

Y_0 is now approximated by $\tilde{Y}_0 := (P \circ S \circ \varphi)(\varphi^{-1})$. The leading term in the Puiseux series of $\tilde{Y}_0$ at 1 is $C(1-p)^{-1/2}$ where $C = (-i/\sqrt{2\pi})S$. We get

$$C = 0.24656177762459992 22 \cdots$$

which differs from the correct value $S_1 = \pi^{-1}\sqrt{3/5}$ by $3.3 \cdot 10^{-96}$.

The rest of the series at 1 is of the form $A(1-p) + \sqrt{1-p}B(1-p)$, A, B analytic. The singular part is the expansion of $S_1 Y_1$ and A is the analytic part of Y_0. We calculated the first 60 terms, and the accuracy, with respect to the known Y_1 is 10^{-94} for the first one down to 10^{-31} for the $60th$. Higher terms can be calculated by other means (I'll describe them later.) Using the resurgent element approach, with the current method and 200 coefficients, S can be calculated with about 710 digits.

Resonance method for determining the Stokes constant S

For resurgent functions such as the tritronquée, the Y_k are calculable algorithmically, up to multiplicative constants. All these constants are linearly related through alien calculus to $S = \pi^{-1}\sqrt{3/5}$, the multiplicative constant of Y_1 (pretend it's unknown).

The Stokes constant S is the unique one with the property that subtracting from Y_0 the resurgent elements corresponding to S, Y_0 becomes entire. In P1, subtracting out $Y_1, \dots Y_{10}$ with the correct constant results in a drop of the highest order coefficient of the series of Y_0 by a factor of 10^{716}.

The high order Taylor coefficients of the resurgent elements,

$$c_{km} = \frac{1}{2\pi i} \oint \frac{Y_k(s)}{(k+s)^{m+1}} ds$$

can be, in turn, calculated with very high accuracy: the integrand becomes a simple rational function and the integral is explicit, if we change variables to $\mathbb{D}$ and expand the $\mathbb{D}-$ Padé approximant by partial fractions.

Next: pictures.

Resonance method for determining the Stokes constant S

For resurgent functions such as the tritronquée, the Y_k are calculable algorithmically, up to multiplicative constants. All these constants are linearly related through alien calculus to $S = \pi^{-1}\sqrt{3/5}$, the multiplicative constant of Y_1 (pretend it's unknown).

The Stokes constant S is the unique one with the property that subtracting from Y_0 the resurgent elements corresponding to S, Y_0 becomes entire. In P1, subtracting out $Y_1, \ldots Y_{10}$ with the correct constant results in a drop of the highest order coefficient of the series of Y_0 by a factor of 10^{716}.

The high order Taylor coefficients of the resurgent elements,

$$c_{km} = \frac{1}{2\pi i} \oint \frac{Y_k(s)}{(k+s)^{m+1}} ds$$

can be, in turn, calculated with very high accuracy: the integrand becomes a simple rational function and the integral is explicit, if we change variables to $\mathbb{D}$ and expand the $\mathbb{D}-$ Padé approximant by partial fractions.

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Plot of Y_0 on the imaginary axis

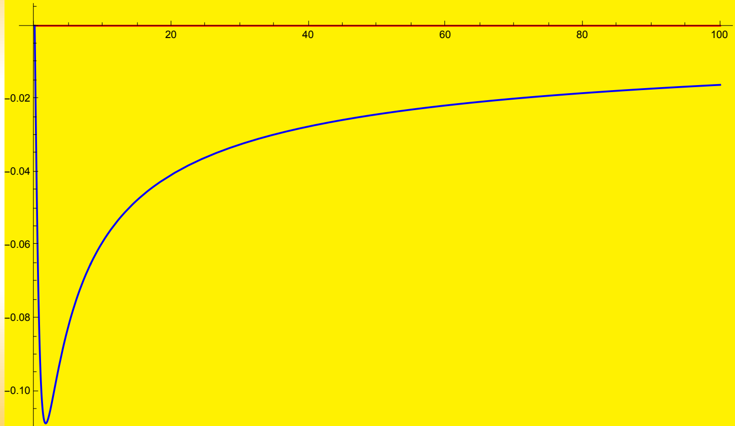


Figure: Y_0 on the imaginary line (red=real part, blue=imaginary part; it looks essentially the same in all directions except for the Stokes ray.

Plot of Y_0 on the singularity line

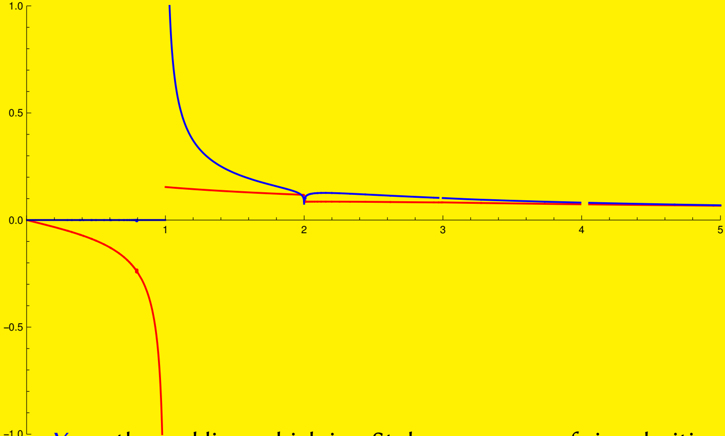


Figure: Y_0 on the real line, which is a Stokes ray, a ray of singularities (red=real part, blue=imaginary part. Note: conformal-Padé is calculated on the very singular line.

Plot of y_0 in the domain of analyticity

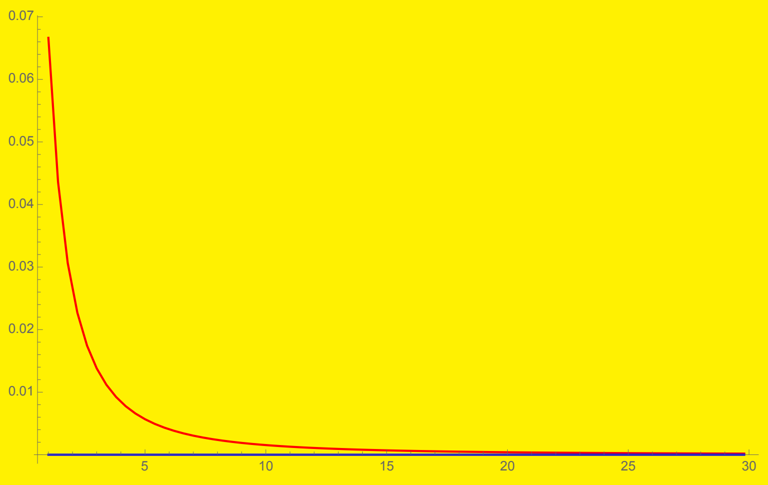


Figure: y_0 on $-i\mathbb{R}^+$; it looks essentially the same inside the analyticity sector.

The Stokes transition: $\operatorname{Im} e^{x}\sqrt{x}(y_0^+ - y_0^-)$ on the Stokes line $\mathbb{R}^+$

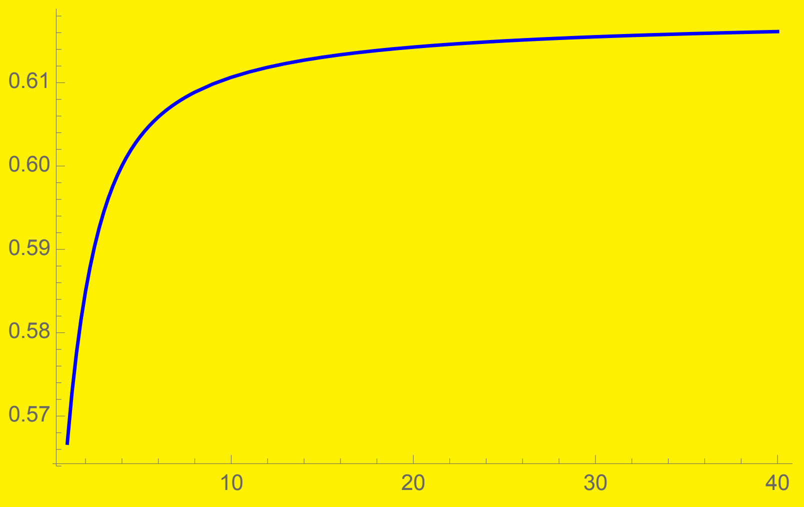
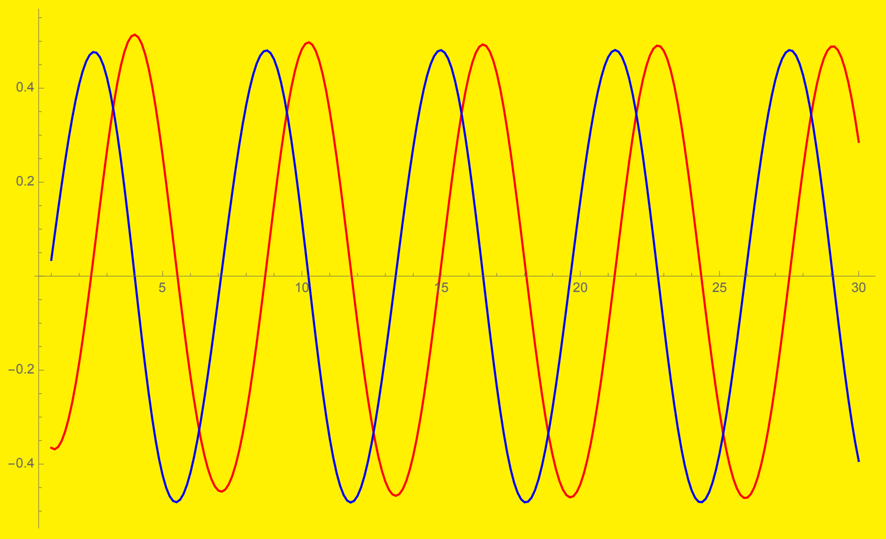


Figure: At $x = 110$ one gets S with 3 digits, where the real part is about 10^{50}.

Plot of y_0 on the edge of the sector of analyticity, an antistokes line



Thank you!