Dichotomy of Hamiltonian operator matrices from systems theory

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Hamiltonian operator matrix

Hamiltonian operator matrix from mathematical systems theory:

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}$$

Setting:

- A closed, densely defined operator on Hilbert space H,
- $B : U \to H$, $C : H \to Y$ bounded,
- U, Y Hilbert spaces.

Then

- $BB^*, C^*C : H \to H$ bounded,
- T closed, densely defined on $H \times H$.

Hamiltonian and Riccati equation

Hamiltonian

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}.$$

Operator Riccati equation associated with Hamiltonian:

$$A^*X + XA - XBB^*X + C^*C = 0$$

Connection (formal)

Linear operator X is solution of Riccati equation if and only if its graph subspace $\Gamma(X) = \left\{ \begin{pmatrix} v \\ Xv \end{pmatrix} \,\middle|\, v \in \mathcal{D}(X) \right\}$ is invariant under T.

Systems theory: interested in bounded selfadjoint nonnegative solution X.

The role of A, B, C in systems theory

Linear system described by operators A, B, C:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0,$$
$$y(t) = Cx(t)$$

- A generator of a strongly continuous semigroup
- B control or input operator
- C observation or output operator

General properties of the Hamiltonian

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}.$$

Note: T is (in general) not selfadjoint or normal.

But T has a symmetry:
Consider indefinite inner product $[\cdot,\cdot]$ on $H \times H$:

$$[\cdot,\cdot] = (J\cdot,\cdot), \qquad J = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}.$$

T is J-skew-selfadjoint, $T^{[*]} = -T$.

Consequence: $\sigma(T)$ symmetric w.r.t. imaginary axis.

Existence of invariant graph subspaces

Theorem (Langer, Ran, van de Rotten (2002))

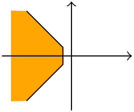
Let A be sectorial, $0 \in \varrho(A)$. Let B, C bounded as above. Then T is bisectorial and dichotomous, in particular

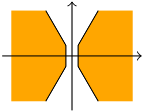
$$V = V_+ \oplus V_- \text{ with } V_\pm \ T\text{-invariant}, \ \sigma(T|_{V_\pm}) \subset \mathbb{C}_\pm.$$

If moreover

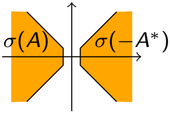
$$\bigcap_{\lambda \in i\mathbb{R}} \ker B^*(A^* - \lambda)^{-1} = \{0\}, \tag{ac}$$

then $V_\pm = \Gamma(X_\pm)$ where X_+ selfadjoint nonpositive, X_- bounded selfadjoint nonnegative.

$\sigma(A)$: 

$\sigma(T)$: 

Idea of the proof

- $T = S + R = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} + \begin{pmatrix} 0 & -BB^* \\ -C^*C & 0 \end{pmatrix}$
- S is bisectorial and dichotomous $\qquad \sigma(S)$: 

- structure of T: $i\mathbb{R} \subset \varrho(T)$
- perturbation (R bounded): T is bisectorial and dichotomous
- T J-skew-selfadjoint & cond. (ac) $\Rightarrow$ $V_\pm = \Gamma(X_\pm)$,
  $X_\pm$ selfadjoint
- T $\widetilde{J}$-dissipative $\Rightarrow$ X_+ nonpositive, X_- nonnegative
- X_- bounded ...

Generalisation 1: BB^*, C^*C unbounded on H

Tretter, W. (2013): Generalisation of theorem to

$$T = \begin{pmatrix} A & -Q_1 \\ -Q_2 & A^* \end{pmatrix}$$

where

- Q_1, Q_2 symmetric nonnegative operators on H,
- Q_1, Q_2 p-subordinate to A^*, A, respectively, with $0 \leq p < 1$; e.g.
$$\|Q_1 x\| \leq \beta \|x\|^{1-p} \|A^* x\|^p, \quad x \in \mathcal{D}(A^*).$$

However: Setting does not allow for systems with boundary control or observation.

Generalisation 2: Extrapolation spaces

Setting:

- A closed, densely defined operator on Hilbert space H,
- To simplify the notation: A normal
- $B : U \to H_{-r}$, $C : H_s \to Y$ bounded, $r + s \leq 1$

Here

$$H_1 \subset H_t \subset H \subset H_{-t} \subset H_{-1}, \qquad 0 < t < 1,$$

scale of Hilbert spaces defined by

$$H_t = \mathcal{D}(|A|^t), \quad H_{-t} = \text{completion of } H \text{ w.r.t. } \|(I + |A|^t)^{-1} \cdot \|.$$

Duality:

- $(H_t)' \cong H_{-t}$ via inner product of H
- $B^* : H_r \to U$, $C^* : Y \to H_{-s}$

Example: heat equation with bndry. control & observation

$$\begin{aligned} \partial_t v &= \Delta v && \text{on } \Omega \subset \mathbb{R}^d \text{ bounded,} \\ \partial_n v &= u && \text{on } \partial\Omega \text{ smooth,} && (u \text{ control}) \\ y &= v|_{\partial\Omega} && \text{on } \partial\Omega. && (y \text{ observation}) \end{aligned}$$

Reformulation as linear system:

- $H = L^2(\Omega)$, $U = Y = L^2(\partial\Omega)$
- $A = \Delta$, $\mathcal{D}(A) = W^{2,2}(\Omega) +$ Neumann b. c.
- $B^*, C : W^{\frac{1}{2},2}(\Omega) \to L^2(\partial\Omega)$ Dirichlet trace

Then

$$H_1 = \mathcal{D}(A) \subset W^{2,2}(\Omega) \quad \Rightarrow \quad H_{1/4} \subset W^{\frac{1}{2},2}(\Omega)$$
$$\Rightarrow \quad B^*, C : H_{1/4} \to L^2(\partial\Omega)$$

Remark: B^*, C not closable as unbounded operators on $L^2(\Omega)$.

Hamiltonian in extrapolation setting

- A normal on H,
- $B : U \to H_{-r}$, $C : H_s \to Y$ bounded, $r + s \leq 1$.

Then

- $B^* : H_r \to U$, $C^* : Y \to H_{-s}$
- $BB^* : H_r \to H_{-r}$, $C^*C : H_s \to H_{-s}$
- extensions $A : H_{1-r} \to H_{-r}$, $A^* : H_{1-s} \to H_{-s}$
- $H_{1-r} \subset H_s$, $H_{1-s} \subset H_r$

We obtain (under mild add. assumptions):

$$T_0 = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix} : H_{1-r} \times H_{1-s} \to H_{-r} \times H_{-s} \quad \text{well defined,}$$

$T = T_0|_{H \times H}$ (part of T_0 in $H \times H$),

T is J-skew-selfadjoint, $\sigma(T)$ symmetric to $i\mathbb{R}$.

The case $r, s < \frac{1}{2}$

Theorem

Let A normal, sectorial, $0 \in \varrho(A)$. Let $B : U \to H_{-r}$, $C : H_s \to Y$ bounded with $r, s < \frac{1}{2}$. Then T is bisectorial and dichotomous. If in addition

$$\bigcap_{\lambda \in i\mathbb{R}} \ker B^*(A^* - \lambda)^{-1} \cap H = \{0\}, \qquad \text{(ac)}$$

then $V_\pm = \Gamma(X_\pm)$ where X_+ selfadjoint nonpositive, X_- selfadjoint nonnegative.

Heat equation example: theorem applies.

The case $r + s < 1$

Theorem

Let A normal, sectorial, $0 \in \varrho(A)$. Let $B : U \to H_{-r}$, $C : H_s \to Y$ bounded with $r + s < 1$. Then T is almost bisectorial, i.e. $i\mathbb{R} \subset \varrho(T)$ and

$$\|(T - \lambda)^{-1}\| \leq M/|\lambda|^{\beta}, \quad \lambda \in i\mathbb{R},$$

with some $0 < \beta < 1$, $M > 0$. In particular, there exist $V_{\pm}$ closed, T-invariant,

$$V_+ \oplus V_- \subset V, \qquad \sigma(T|_{V_\pm}) \subset \mathbb{C}_{\pm}.$$

If in addition (ac) holds, then $V_{\pm} = \Gamma(X_{\pm})$ and exist $X_{0\pm} \subset X_{\pm}$ where X_{0+} symmetric nonpositive, X_{0-} symmetric nonnegative, and $X_{0\pm}^* = X_{\pm}$.

Idea of the proof

$$T_0 = S_0 + R = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} + \begin{pmatrix} 0 & -BB^* \\ -C^*C & 0 \end{pmatrix} \text{ on } H_{-r} \times H_{-s}$$

- S_0 bisectorial on $H_{-r} \times H_{-s}$
- $r, s < \frac{1}{2}$: $BB^* : H_r \to H_{-r}$, $C^*C : H_s \to H_{-s}$ “less unbounded” than A, A^* $\rightsquigarrow$ T bisectorial
- $r + s < 1$ and (e.g.) $r \geq \frac{1}{2}$: BB^* “more unbounded” than A, A^* $\rightsquigarrow$ T almost bisectorial

Closing remarks

Generalisations:

- A normal is not needed
- If A has compact resolvent: condition $0 \in \sigma(A)$ can be relaxed, spectra of A and $-A^*$ may touch

Open:

- X_- bounded ?
- Case $r + s < 1$: role of $X_{0\pm}$?