Construction of the selfadjoint dilation of a maximal dissipative operator

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Mathematical aspects of the physics with non-self-adjoint operators
Luminy
June 9, 2017

joint work with
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Dissipative operators

Definition

H Hilbert space. A densely defined linear operator A with domain $D(A)$ in H is called *dissipative* if $\Im \langle Au, u \rangle \geq 0$ for all $u \in D(A)$. A is called *anti-dissipative* if $(-A)$ is dissipative.

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A is MDO iff A is dissipative and $\mathbb{C}^- \subset \rho(A)$.

Dilations and minimality

Proposition (Sz.-Nagy)

For any MDO A on a Hilbert space H there exists a selfadjoint operator $\mathcal{L}$ on a Hilbert space $\mathcal{H} \supseteq H$ such that

$$e^{itA} = P_H e^{it\mathcal{L}} P_H, \; t \geq 0 \quad \text{or} \quad (A-\lambda)^{-1} = P_H(\mathcal{L}-\lambda)^{-1}P_H, \quad \lambda \in \mathbb{C}^-.$$

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This allows the use of tools of the theory of self-adjoint operators to study MDOs, e.g. one obtains a functional calculus via

$$\psi(A) = P_H \psi(\mathcal{L}) P_H \text{ for any } \psi \in H^\infty(\mathbb{C}_+)$$

and $\|\psi(A)\| = \|P_H\psi(\mathcal{L})\| \leq \|\psi(\mathcal{L})\| \leq \sup_{\lambda \in \mathbb{C}^+} |\psi(\lambda)|$.

Lagrange identity

Lemma

Let A be a maximally dissipative operator on a Hilbert space H. Then there exists a Hilbert space E and an operator $\Gamma : D(A) \to E$ such that

$$\|\Gamma u\|_E \leq \|u\|_H + \|Au\|_H ,$$

i.e. Γ is bounded in the graph norm of A, Γ has dense range in E and such that for all $u, v \in D(A)$ we have

$$\langle Au, v\rangle_H - \langle u, Av\rangle_H = i \langle \Gamma u, \Gamma v\rangle_E .$$

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Similarly, there exists a Hilbert space E_* and an operator $\Gamma_* : D(A^*) \to E_*$ which is bounded in the graph norm of A^*, has dense range in E_* and such that for all $u, v \in D(A^*)$ we have

$$\langle A^* u, v\rangle_H - \langle u, A^* v\rangle_H = -i \langle \Gamma_* u, \Gamma_* v\rangle_{E_*} .$$

Example: Schrödinger operator

On $H = L^2(\mathbb{R}^+)$, let $(Af)(x) = -f''(x) + q(x)f(x)$, $q \in L^\infty(\mathbb{R}^+)$ with $\Im q(x) \geq 0$ for a.e. $x \in \mathbb{R}^+$ and

$$D(A) := \{f \in H^2(\mathbb{R}^+) : f'(0) = hf(0)\}$$

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$$\begin{aligned} \langle Au, v\rangle - \langle u, Av\rangle &= u'(0)\overline{v(0)} - u(0)\overline{v'(0)} + 2i\int_0^\infty \Im q(x)\ u(x)\overline{v(x)}\ dx \\ &= 2i\left(\Im h\ u(0)\overline{v(0)} + \int_0^\infty \Im q(x)\ u(x)\overline{v(x)}\ dx\right). \end{aligned}$$

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Let $\Omega = \{x \in \mathbb{R}^+ : \Im q(x) > 0\}$, set $E = \mathbb{C} \oplus L^2(\Omega)$ and

$$\Gamma u = \begin{pmatrix} \sqrt{2\Im h}\ u(0) \\ \sqrt{2\Im q}\ u|_\Omega \end{pmatrix}, \quad u \in D(A).$$

Then
$$\langle Au, v\rangle_H - \langle u, Av\rangle_H = i\,\langle \Gamma u, \Gamma v\rangle_E\,.$$

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Then $$\langle Au, v\rangle_H - \langle u, Av\rangle_H = i \langle \Gamma u, \Gamma v\rangle_E .$$

Here, $E_* = E$ and Γ_* acts as Γ, but has a different domain.

Štraus characteristic function

Lemma

For all $u \in D(A)$ and $z \in \mathbb{C}^+$ we have

$$\|\Gamma_*(A^* - z)^{-1}(A - z)u\|^2 = \|\Gamma u\|^2 - 2\Im(z)\|(A^* - z)^{-1}(A - z)u - u\|^2.$$

Hence,

$$\|\Gamma_*(A^* - z)^{-1}(A - z)u\| \leq \|\Gamma u\|,$$

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Hence,

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and there exists a unique contraction $S(z) : E \to E_*$*, analytic in the upper half-plane, such that*

$$S(z)\Gamma u = \Gamma_*(A^* - z)^{-1}(A - z)u \text{ for all } u \in D(A).$$

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$$S(z)\Gamma u = \Gamma_*(A^* - z)^{-1}(A - z)u \text{ for all } u \in D(A).$$

Correspondingly, there exists a contraction $S_*(z) : E_* \to E$*, analytic in the lower half plane, such that*

$$S_*(z)\Gamma_* u = \Gamma(A - z)^{-1}(A^* - z)u \text{ for all } u \in D(A^*).$$

Example: Schrödinger operator

Let $H = L^2(\mathbb{R}^+)$ and $(Af)(x) = -f''(x) + q(x)f(x)$, where $q \in L^\infty(\mathbb{R}^+)$ with $\Im q \geq 0$,

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where $\Im h > 0$, and

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Now, let φ_* and ψ_* be the fundamental solutions of $-y'' + \bar{q}y = \lambda y$ and let m_* denote the Weyl-Titchmarsh function associated with $-y'' + \bar{q}y$, i.e. $m_*(\lambda)\varphi_*(\lambda) + \psi_*(\lambda)$ is the (unique) L^2-solution to $-y'' + \bar{q}y = \lambda y$.

Example: Schrödinger operator

Then $S(z)$ is given by

$$\begin{pmatrix} \frac{h-m_*(z)}{\bar{h}-m_*(z)} & i\sqrt{2\Im h}\ \int_0^\infty \frac{m_*(z)\varphi_*(y)+\psi_*(y)}{\bar{h}-m_*(z)}\sqrt{2\Im q(y)}\ \cdot(y)\ dy \\ i\frac{\sqrt{2\Im h}\ \sqrt{2\Im q}}{\bar{h}-m_*(z)}(m_*(z)\varphi_*(x)+\psi_*(x)) & I+i\sqrt{2\Im q}\ (A^*-z)^{-1}\sqrt{2\Im q} \end{pmatrix}.$$

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Remarks:

- The top left entry coincides with the well-known formula by Pavlov for the case of real q.
- The bottom right entry agrees with the Livšic characteristic function for the case with a selfadjoint boundary condition.
- This formula shows the connection between the m-function and the characteristic function for this example.

Domain of the selfadjoint dilation

Let $\mu \in \mathbb{C}^-$ and $\lambda \in \mathbb{C}^+$. Define $\mathcal{H} = L^2(\mathbb{R}_-, E_*) \oplus H \oplus L^2(\mathbb{R}_+, E)$ and

$$D(\mathcal{L}) = \left\{ U = \begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix} : u \in H,\ v_+ \in H^1(\mathbb{R}_+, E),\ v_- \in H^1(\mathbb{R}_-, E_*), \right.$$

$$\text{(i) } u + (\Gamma_*(A^* + \mu)^{-1})^* v_-(0) \in D(A),$$

$$\text{(ii) } u + (\Gamma(A + \lambda)^{-1})^* v_+(0) \in D(A^*),$$

$$\text{(I) } v_+(0) = S^*(-\mu) v_-(0) + i\Gamma\left(u + (\Gamma_*(A^* + \mu)^{-1})^* v_-(0)\right),$$

$$\left. \text{(II) } v_-(0) = S(-\bar{\lambda}) v_+(0) - i\Gamma_*\left(u + (\Gamma(A + \lambda)^{-1})^* v_+(0)\right) \right\}.$$

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Remarks:

- The set is independent of μ and λ in the appropriate half-planes.
- (i) and (ii) are equivalent.
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Domain of the selfadjoint dilation II

Let $\mu \in \mathbb{C}^-$ and $\lambda \in \mathbb{C}^+$. Define $\mathcal{H} = L^2(\mathbb{R}_-, E_*) \oplus H \oplus L^2(\mathbb{R}_+, E)$ and

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&\text{(I) } v_+(0) = S^*(-\mu)v_-(0) + i\Gamma\left(u + (\Gamma_*(A^* + \mu)^{-1})^* v_-(0)\right), \\
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\end{aligned}$$

Remarks:

- The independence of (II) from λ allows us to take the limit as $\lambda \to \infty$. Whenever the characteristic function $S(-\bar{\lambda})$ has a limit, this gives a more explicit connection between $v_+(0)$ and $v_-(0)$.

Definition of the dilation

Definition

Let $\mu \in \mathbb{C}^-$ and $\lambda \in \mathbb{C}^+$. For $U \in D(\mathcal{L})$, define

$$TU := A^*(u + (\Gamma(A+\lambda)^{-1})^* v_+(0)) + \bar{\lambda}(\Gamma(A+\lambda)^{-1})^* v_+(0),$$

$$T_*U := A(u + (\Gamma_*(A^*+\mu)^{-1})^* v_-(0)) + \bar{\mu}(\Gamma_*(A^*+\mu)^{-1})^* v_-(0).$$

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Definition

We define the operator $\mathcal{L}$ on $\mathcal{H} = L^2(\mathbb{R}_-, E_*) \oplus H \oplus L^2(\mathbb{R}_+, E)$ with domain $D(\mathcal{L})$ by

$$\mathcal{L}U = \mathcal{L}\begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix} = \begin{pmatrix} iv'_- \\ TU \\ iv'_+ \end{pmatrix}.$$

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$\mathcal{L}$ is symmetric and for $\lambda \in \mathbb{C}^-$, we have

$$(\mathcal{L}-\lambda)^{-1}\begin{pmatrix} f \\ w \\ g \end{pmatrix} = \begin{pmatrix} -i\int_{-\infty}^{x} e^{i\lambda(t-x)}f(t)\ dt \\ (A-\lambda)^{-1}w + i(\Gamma_*(A^*-\overline{\lambda})^{-1})^* \int_{-\infty}^{0} e^{i\lambda t}f(t)\ dt \\ v_+(0)e^{i\lambda x} - i\int_0^x e^{i\lambda(t-x)}g(t)\ dt \end{pmatrix}$$

where

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In particular, we have

$$P_H (\mathcal{L}-\lambda)^{-1} P_H = \begin{cases} (A-\lambda)^{-1} & \lambda \in \mathbb{C}^-, \\ (A^*-\lambda)^{-1} & \lambda \in \mathbb{C}^+, \end{cases}$$

so $\mathcal{L}$ is a selfadjoint dilation of A. Moreover, $\mathcal{L}$ is minimal.

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$$\begin{aligned} u + (\Gamma_*(A^* + \mu)^{-1})^* v_-(0) &= u + \sqrt{2\Im h}\ \overline{G_*(0, y, -\mu)}(v_-(0))_1 \\ &\quad + (A - \mu)^{-1}\sqrt{2\Im q}\ (v_-(0))_2. \end{aligned}$$

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Then $u + (\Gamma_*(A^* + \mu)^{-1})^* v_-(0) \in D(A)$ gives

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Thank you for your attention!