Spectral decomposition of linear operators

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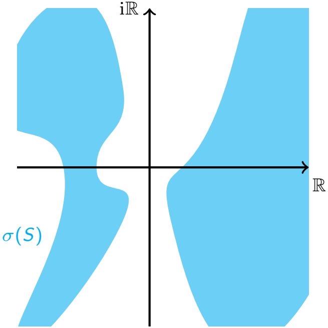
Mathematical aspects of the physics with non-self-adjoint operators

CIRM, June 6, 2017

Question: Spectral decomposition?

Let S be a closed operator on a Banach space X with $\mathrm{i}\mathbb{R} \subseteq \varrho(S)$.



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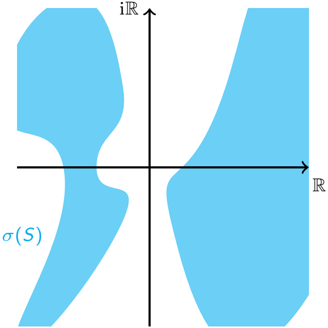

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If in addition

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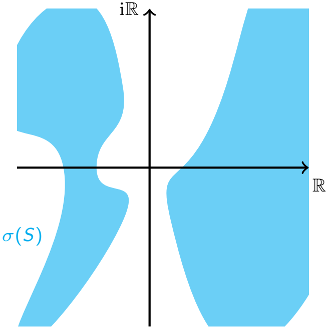
then S is called **strictly dichotomous**.



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If S is dichotomous with respect to $X = X_+ \oplus X_-$, then S has a diagonal block operator matrix representation:

$$S = \begin{pmatrix} S|X_+ & 0 \\ 0 & S|X_- \end{pmatrix}.$$

Examples

3. Let $X = \mathbb{C}^3$, $S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\begin{aligned} X_+ &= \operatorname{span}\{e_1,\ e_2\}, \\ X_- &= \operatorname{span}\{e_3\}. \end{aligned}$

   - Clearly, $\sigma(S) = \{-1, 1\}$, S is dichotomous for $X = X_+ \oplus X_-$ and
   $$(S - \lambda)^{-1} = \begin{pmatrix} (1-\lambda)^{-1} & -(1-\lambda)^{-2} & 0 \\ 0 & (1-\lambda)^{-1} & 0 \\ 0 & 0 & -(1+\lambda)^{-1} \end{pmatrix} \quad \text{for } \lambda \neq \pm 1.$$

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- $(S-\lambda)^{-1}$ cannot be extended analytically to either $\mathbb{C}_+$ or $\mathbb{C}_-$.

- **But:**
  - for $x_+ \in X_+$, the **vectorvalued** function $\lambda \mapsto (S-\lambda)^{-1}x_+$ has a bounded analytic extension to $\mathbb{C}_-$
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     - ⋆ for $x_- \in X_-$, the **vectorvalued** function $\lambda \mapsto (S-\lambda)^{-1}x_-$has a bounded analytic extension to $\mathbb{C}_+$
   - Actually:
$$X_+ = \{x \in X : (S-\lambda)^{-1}x \ \text{ has a bounded analytic extension to } \ \mathbb{C}_-\},$$
$$X_- = \{x \in X : (S-\lambda)^{-1}x \ \text{ has a bounded analytic extension to } \ \mathbb{C}_+\}.$$

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Then: $X_\pm$ are S-invariant and $\sigma(S|X_\pm) = \pm\mathbb{N}$, but $X \neq X_+ \oplus X_-$.

Reason: The projections of $X_+ \oplus X_-$ onto $X_\pm$ along $X_\mp$ are unbounded because $P_\pm = \operatorname{diag}(P_1^\pm, P_2^\pm, \dots)$ with

$$P_n^+ = \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix}, \qquad P_n^- = \begin{pmatrix} 0 & -n \\ 0 & 1 \end{pmatrix},$$

therefore only

$$X = \overline{X_+ \oplus X_-} \neq X_+ \oplus X_-.$$

Hence S is **not** dichotomous.

$\Longrightarrow$ $\mathrm{i}\mathbb{R} \subset \varrho(S)$ alone is not enough to decompose X into spectral subspaces of S.

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   In general:

   - if S is dichotomous, then the corresponding decomposition of X may not be unique,
   - if S is **strictly** dichotomous, then the corresponding decomposition of X is always unique because . . .

Uniqueness of decomposition of $X = X_+ \oplus X_-$

Lemma

$S(X \to X)$ with $\mathrm{i}\mathbb{R} \in \varrho(S)$. Let

$$G_\pm = \{x \in X : (S-\lambda)^{-1}x \text{ has a bounded analytic extension to } \overline{\mathbb{C}_\mp}\}.$$

Then:

1. $G_+ \cap G_- = \{0\}$.
2. *If S is strictly dichotomous for $X = X_+ \oplus X_-$, then $X_\pm = G_\pm$.*

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How can we compute $X_\pm$?

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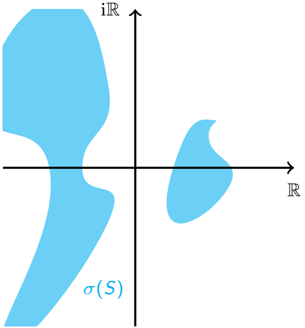

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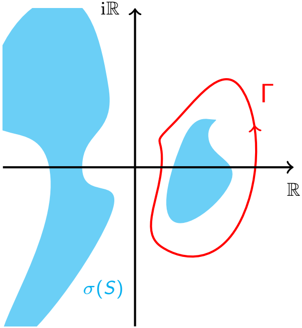

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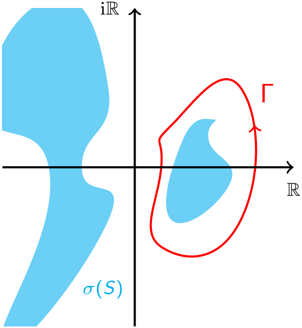

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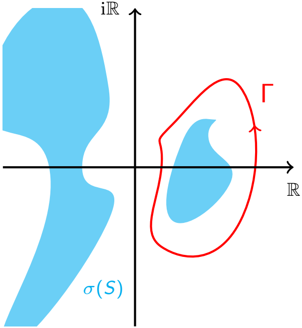

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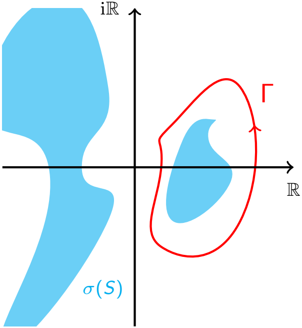

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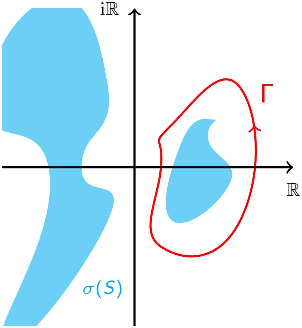

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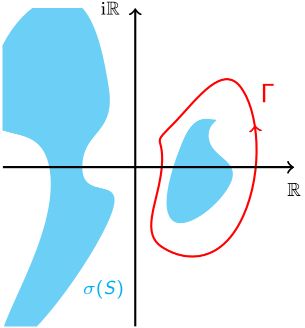

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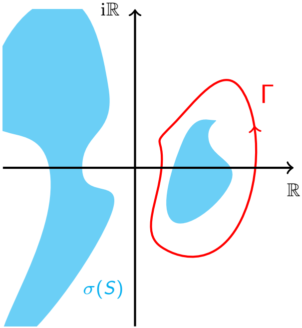

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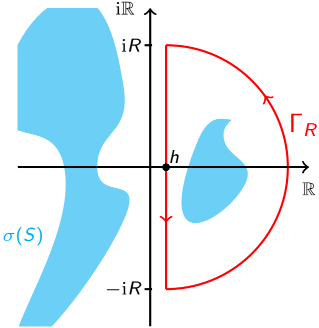
Idea: Deform contour Γ and modify the integral.

Heuristic derivation of formula for $P_\pm$

Assume S is bounded and deform path Γ to path Γ_R

$$P_+ x = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_R} (\lambda - S)^{-1} x \, \mathrm{d}\lambda$$



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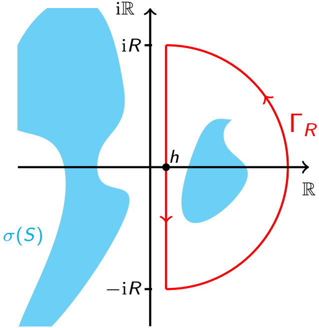

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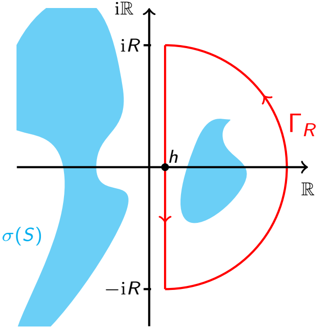

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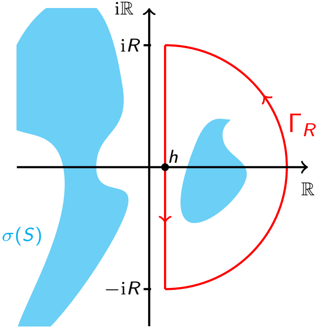

Take limit $R \to \infty$:

$$P_+ = \frac{1}{2\pi \mathrm{i}}\, S^2 \int_{h-\mathrm{i}\infty}^{h+\mathrm{i}\infty} \lambda^{-2} (S - \lambda)^{-1} \,\mathrm{d}\lambda.$$

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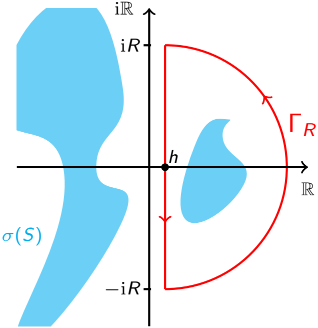

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Problems for unbounded S: Does the integral converge? Does the integral map to $\mathcal{D}(S^2)$? ...

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$\Longrightarrow$

- $\{\lambda \in \mathbb{C} : |\operatorname{Re}(\lambda)| \leq h\} \subset \varrho(S)$ for some $h > 0$,
- $\sup_{|\operatorname{Re}\lambda| \leq h} \|(S - \lambda)^{-1}\| < \infty$.

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Additional assumption: $(S-\lambda)^{-1}$ is uniformly bounded on the imaginary axis.

$\Longrightarrow$
- $\{\lambda \in \mathbb{C} : |\operatorname{Re}(\lambda)| \le h\} \subset \varrho(S)$ for some $h > 0$,
- $\sup_{|\operatorname{Re}\lambda| \le h} \|(S-\lambda)^{-1}\| < \infty$.

$\Longrightarrow \quad A_{\pm} = \dfrac{\pm 1}{2\pi \mathrm{i}} \displaystyle\int_{\pm h - \mathrm{i}\infty}^{\pm h + \mathrm{i}\infty} \lambda^{-2}(S-\lambda)^{-1}\,\mathrm{d}\lambda$ well-defined bounded operators!

Spectral splitting along imaginary axis

Theorem

Let $S(X \to X)$ densely defined, with $\mathrm{i}\mathbb{R} \subset \varrho(S)$ and $(S - \lambda)^{-1}$ uniformly bounded on $\mathrm{i}\mathbb{R}$.

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1. *$P_\pm$ are complementary projections with $\mathcal{D}(P_+) = \mathcal{D}(P_-) = G_+ \oplus G_-$ and*

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2. *$G_\pm$ are S- and $(S-\lambda)^{-1}$-invariant closed subspaces and*
$$\sigma(S|G_\pm) = \sigma(S) \cap \mathbb{C}_\pm,$$
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3. *$\mathcal{D}(S^2) \subset \mathcal{D}(P_\pm)$ and*

$$P_\pm x = \frac{\pm 1}{2\pi\mathrm{i}} \int_{\pm h - \mathrm{i}\infty}^{\pm h + \mathrm{i}\infty} \lambda^{-2}(S-\lambda)^{-1} S^2 x \,\mathrm{d}\lambda, \qquad x \in \mathcal{D}(S^2).$$

In particular, $P_\pm$ are densely defined.

By the previous theorem: uniform boundedness of $(S-\lambda)^{-1}$ on $\mathrm{i}\mathbb{R}$ is sufficient for existence of S-invariant subspaces $G_\pm$ with

- $\sigma(S|G_\pm) = \sigma(S) \cap \mathbb{C}_\pm$,
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Corollary

Let S as above. Then the following is equivalent:

1. *S is strictly dichotomous.*
2. $X = G_+ \oplus G_-$.
3. *$P_\pm$ is bounded.*

In this case, $X = G_+ \oplus G_-$ is the corresponding unique spectral decomposition.

- The formula $P_+ = \dfrac{1}{2\pi\mathrm{i}} \displaystyle\int_{h-\mathrm{i}\infty}^{h+\mathrm{i}\infty} \lambda^{-2}(S-\lambda)^{-1}S^2\,\mathrm{d}\lambda$ appears already in Bart, Gohberg, Kaashoek (1986). They proved:
  - If P_+ is bounded on $\mathcal{D}(S^2)$, then S is dichotomous and P_+ is projection on X_+.
  - $G_\pm \subset \ker(A_\mp) = \mathrm{Rg}(P_\pm)$.
- The similar integral formula $P_+x - P_-x = \dfrac{1}{\pi\mathrm{i}} \displaystyle\int_{\mathrm{i}\infty}^{\mathrm{i}\infty}{}' (S-\lambda)^{-1}x\,\mathrm{d}\lambda$ was proved in Langer, Tretter (2001) under the assumption that $\lim_{|t|\to\infty} \|(S-\mathrm{i}t)^{-1}\| = 0$ and the additional assumption that the integral exists for every $x \in X$.

Bisectorial operators

An operator $S(X \to X)$ is called **bisectorial** if $\mathrm{i}\mathbb{R} \setminus \{0\} \subset \varrho(S)$ and

$$\|(S-\lambda)^{-1}\| \le \frac{M}{|\lambda|}, \qquad \lambda \in \mathrm{i}\mathbb{R} \setminus \{0\}. \qquad (*)$$

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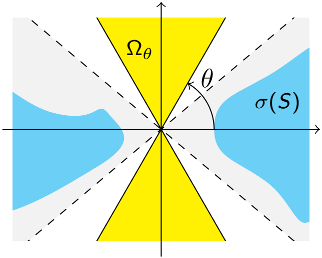
If S is bisectorial, then there exists $0 < \theta < \pi/2$ such that the bisector

$$\begin{aligned}\Omega_\theta &= \mathbb{C} \setminus \big(\Sigma_\theta \cup (-\Sigma_\theta)\big)\\ &= \{\lambda \in \mathbb{C} : \theta < |\arg \lambda| < \pi - \theta\}\end{aligned}$$

belongs to $\varrho(S)$ and $(*)$ holds on Ω_θ.



Bisectorial operators

For bisectorial operators S with $0 \in \varrho(S)$ the formula for the spectral projections simplify:

$$P_+ = \frac{1}{2\pi \mathrm{i}}\ S^{2} \int_{h-\mathrm{i}\infty}^{h+\mathrm{i}\infty} \lambda^{-2}(S-\lambda)^{-1}\,\mathrm{d}\lambda$$
$$= \frac{1}{2\pi \mathrm{i}}\ S^{1} \int_{h-\mathrm{i}\infty}^{h+\mathrm{i}\infty} \lambda^{-1}(S-\lambda)^{-1}\,\mathrm{d}\lambda$$

because due to the decay of $\|(S-\lambda)^{-1}\|$ the power -1 of λ is sufficient to guarantee existence of the integral.

Set $B_\pm = \frac{1}{2\pi \mathrm{i}} \int_{\pm h-\mathrm{i}\infty}^{\pm h+\mathrm{i}\infty} \lambda^{-1}(S-\lambda)^{-1}\,\mathrm{d}\lambda$. Then we obtain the following theorem.

Bisectorial operators

Theorem

Let $S(X \to X)$ be bisectorial with $0 \in \varrho(S)$ and $P_\pm$ as in the theorem on spectral splitting. Then:

1. $P_\pm = SB_\pm$, $\mathcal{D}(S) \subset \mathcal{D}(P_\pm)$ *and*

$$P_\pm x = \frac{\pm 1}{2\pi \mathrm{i}} \int_{\pm h - \mathrm{i}\infty}^{\pm h + \mathrm{i}\infty} \frac{1}{\lambda} (S - \lambda)^{-1} Sx \, d\lambda, \quad x \in \mathcal{D}(S).$$

2. *Let S be bisectorial with $0 \in \varrho(S)$ and θ as before. Then $\pm S|G_\pm$ are sectorial with angle θ and unchanged constant M.*

As before: S is strictly dichotomous if and only if the projections $P_\pm$ are bounded.

Perturbation Results

Theorem

Let $S(X \to X)$ be densely defined and strictly dichotomous.
Let $T(X \to X)$ densely defined such that there exist $h > 0$, $\epsilon > 0$ with:

1. $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| \le h\} \subset \varrho(S) \cap \varrho(T)$;
2. $\sup_{|\operatorname{Re}\lambda|\le h} |\lambda|^{1+\epsilon} \|(S-\lambda)^{-1} - (T-\lambda)^{-1}\| < \infty$;
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Let $S(X \to X)$ be densely defined, bisectorial and strictly dichotomous.
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Then T is also strictly dichotomous and bisectorial.

Thank you for the attention!

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