Perturbation of eigenvalues of Klein-Gordon operators

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Abstract Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - V\right)^2 \psi - U^2\psi = 0$$

U selfadjoint positive definite, V symmetric in Hilbert space $\mathcal{X}$. 'Linearisation'

$$\psi_1 = U^{1/2}\psi, \quad \psi_1 = U^{-1/2}\left(i\frac{\partial}{\partial t} - V\right)\psi$$

leads to

$$i\frac{\partial}{\partial t}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = H\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}, \quad H = \begin{bmatrix} U^{1/2}VU^{-1/2} & U \\ U & U^{-1/2}VU^{1/2}\end{bmatrix}$$

H is *(abstract) Klein-Gordon Hamiltonian*. Apparently non-selfadjoint, except —

— except for $V = 0$ (free Hamiltonian)

$$H_0 = H_0^* = \begin{bmatrix} 0 & U \\ U & 0 \end{bmatrix}.$$

Note

$$\operatorname{sign}(H_0) = J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \sigma(H_0) = -\sigma(U) \cup \sigma(U),$$

$$G_0 = |H_0| = JH_0 = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$$

pos. definite.

Aim: bounds for the spectrum of general H as function of V.

Constructing the KG-Operator by forms

$$G = JH = \begin{bmatrix} U & U^{-1/2}VU^{1/2} \\ U^{1/2}VU^{-1/2} & U \end{bmatrix}$$

formally symmetric. Under the fundamental assumption

$$b = \|VU^{-1}\| < 1$$

G_0 is given by the form g_0. Set $g = g_0 + v$:

$$g_0(\psi, \phi) = (U^{1/2}\psi_1, U^{1/2}\phi_1) + (U^{1/2}\psi_2, U^{1/2}\phi_2),$$

$$v(\psi, \phi) = (U^{1/2}\psi_2, VU^{-1/2}\phi_1) + (VU^{-1/2}\psi_1, U^{1/2}\phi_2),$$

$$|(v(\psi, \psi)| \le bg_0(\psi, \psi).$$

Theorem (spectrality). Let G_0 be selfadj. and pos. definite, J a symmetry commuting with G_0 and v a symmetric form with $|(v(\psi,\psi)| \leq bg_0(\psi,\psi), \quad b < 1$. Then

- $G = G_0 + v$ is selfadj. pos. definite
- $H = JG$ is similar to selfadjoint:

  $$S^{-1}HS \text{ is selfadjoint, } S, S^{-1} \in \mathcal{B}(\mathcal{X}).$$

  with $S = \sqrt{J \operatorname{sign} H}$ selfadjoint and pos. definite.
- $\sigma(H)$ has a gap at zero.
- In other words, H is selfadjoint in the scalar product $(\cdot,\cdot)_v = (S^2\cdot,\cdot)$ (same topology).

(K.V. 1969, 1970, 1972, etc, H. Langer et al. 2006)

Formulated within the Krein space theory: An operator $H = JG$, with J a symmetry and G selfadjoint can be called just selfadjoint in the Krein space defined by the definite scalar product

$$(\psi, \phi) = (\psi_1, \phi_1) + (\psi_2, \phi_2)$$

and the indefinite one

$$[\psi, \phi] = (J\psi, \phi) = (\psi_2, \phi_1) + (\psi_1, \phi_2).$$

So, our H would be called 'definitizable with regular points at 0 and ∞'.

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This geometry is canonical: The selfadjointness scalar product $(\cdot, \cdot)_V$ varies with H but it coincides with the indefinite one $(J\cdot, \cdot)$ on the positive spectral subspace of H (same with negative spectrum and the opposite sign).

The established properties of the Hamiltonian H secure

- correct quantum mechanical interpretation in one-particle theory (as far as possible)
- construction of the corresponding second-quantised theory with an external field (under standard conditions).

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Perturbations

- General case: $H = JG$ as in the spectrality theorem. Perturbed into $H' = JG'$ by forms

$$g' = g + \delta g,$$

  with δg small with respect to g.
  - One sided bounds or inclusions for both standard and essential spectrum
  - Two sided bounds for discrete eigenvalues, counted with multiplicity
- Application to the Klein-Gordon operator with the potential V perturbed into $V + \delta V$.

Spectral inclusion

Theorem. Let $G, J, H = JG$ as before and H be perturbed into $H' = JG'$ by forms

$$g' = g + \delta g, \quad |\delta g| \leq \kappa g, \quad \kappa < 1.$$

Then G' is again pos. definite and H' similar to selfadjoint.
(B. Curgus, B. Najman 1995)
Let (λ_-, λ_+) be a positive spectral gap of H. Then

$$((1+\kappa)\lambda_-, (1-\kappa)\lambda_+) \subseteq \rho(H')$$

(similarly for negative gaps). For the central gap at zero better bound

$$((1-\kappa)\lambda_-, (1-\kappa)\lambda_+) \subseteq \rho(H')$$

(the latter is never void!). Analogously for essential spectra.
(Some previous work in M. Langer and C. Tretter 2006.)

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Two sided bounds

Theorem Perturb $H = JG$ into $H' = JG'$ as before. Let the positive bottom of $\sigma(H)$ be given by the discrete eigenvalues

$$\lambda_1^+ \le \lambda_2^+ \le \cdots < \min \sigma_{ess}^+(H)$$

with

$$(1+\kappa)\lambda_n^+ < (1-\kappa)\min \sigma_{ess}^+(H) \quad \text{for some } n.$$

Then the lower positive part of $\sigma(H')$ begins with the sequence of discrete eigenvalues

$$\lambda_1'^+ \le \lambda_2'^+ \le \cdots \le \lambda_n'^+$$

(always with multiplicities) such that

$$(1-\kappa)\lambda_k^+ \le \lambda_k'^+ \le (1+\kappa)\lambda_k^+.$$

Perturbing the potential

The potential V is perturbed into $V + \delta V$. Connect the perturbations δV and δg. Put the expressions

$$G = \begin{bmatrix} U & U^{-1/2}VU^{1/2} \\ U^{1/2}VU^{-1/2} & U \end{bmatrix}$$

$$\delta G = \begin{bmatrix} 0 & U^{-1/2}\delta VU^{1/2} \\ U^{1/2}\delta VVU^{-1/2} & 0 \end{bmatrix}.$$

into forms. Crucial technical step: use the Schur-complement decomposition of the matrix operator G. The final result surprisingly simple:

With

$$\|VU^{-1}\| = b, \quad \|\delta VU^{-1}\| = \eta$$

we obtain e.g.

$$|\lambda_k'^+ - \lambda_k^+| \le \frac{\eta}{1-b}\lambda_k^+,$$

under the condition

$$\eta + b < 1.$$

In particular, if δV is bounded

$$|\lambda_k'^+ - \lambda_k^+| \le \alpha\|\delta V\|,$$

$$1 \le \alpha = \frac{1}{1-b}\,\frac{\lambda_k^+}{\min\sigma(U)} \le \frac{1}{1-b},$$

— penalty for non-selfadjointness.

Simple idea:

$$H = JG = G^{-1/2}G^{1/2}JG^{1/2}G^{1/2}$$

suggests similarity between H and $G^{1/2}JG^{1/2}$ (but G unbounded...).
Still true because we know that H is similar to *some* selfadjoint, (use cleverly a polar decomposition). So, study $G^{1/2}JG^{1/2}$ — just a general selfadjoint but in *factorised form*.